Thermodynamic description of the relaxation of two-dimensional turbulence using Tsallis statistics

Bruce M. Boghosian
Center for Computational Science, Boston University, 3 Cummington Street, Boston, Massachusetts 02215
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Two-dimensional Euler turbulence and drift turbulence in a pure-electron plasma column have been experimentally observed to relax to metaequilibrium states that do not maximize the Boltzmann entropy, but rather seem to minimize enstrophy. We show that a recent generalization of thermodynamics and statistics due to Tsallis [Phys. Lett. A **195**, 329 (1994); J. Stat. Phys. **52**, 479 (1988)] is capable of explaining this phenomenon in a natural way. In particular, the maximization of the generalized entropy S_q with $q = \frac{1}{2}$ for the pure-electron plasma column leads to precisely the same profiles predicted by the restricted minimum enstrophy theory of Huang and Driscoll [Phys. Rev. Lett. **72**, 2187 (1994)]. These observations make possible the construction of a comprehensive thermodynamic description of two-dimensional turbulence.

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I. INTRODUCTION

The kinetic and field equations for the drift motion of a pure-electron plasma column in a strong magnetic field are isomorphic to the equations of motion of a two-dimensional Euler fluid [1]. The density of the plasma corresponds to the vorticity of the fluid and the electrostatic potential corresponds to the stream function. This observation has made it possible to use pure-electron plasmas to study Euler turbulence in the laboratory [2]. Such experiments have followed the relaxation of Euler turbulence through several identifiable stages [3]: An initially hollow vorticity profile develops a linear diochotron instability that saturates with the creation of long-lived vortex patches. These patches move about for hundreds of diochotron periods, shedding filiaments, and eventually mixing and inwardly transporting. This process gives rise to an axisymmetric metaequilibrium state, whose density decreases monotonically with radius, which then persists for tens of thousands of diochotron periods. The eventual decay of this state is due only to three-dimensional effects that destroy the idealization of the two-dimensional Euler fluid [1,4].

The shape of the radial vorticity profile of the metaequilibrium state is an interesting and fundamental problem. One would expect that it could be described by a variational principle, but the most natural principle of this sort—the maximization of the Boltzmann entropy under the constraints of constant mass, energy, and angular momentum—has been found to yield profiles that are substantially flatter than those observed in experiments [4,5]. On the other hand, an alternative variational principle, in which the enstrophy (the integral of the square of the vorticity) is minimized, has been found to yield results in excellent agreement with experiment [4,5]. To date, however, there has existed no satisfactory theoretical explanation for this unusual variational principle.

In this paper we show that the failure of the Boltzmann entropy to predict the radial density profile of the metaequilibrium state can be understood as but one example of a systemic breakdown of Boltzmann-Gibbs statistics for systems with long-range interactions, long-time memory [6], or fractal space-time structure [7]. Moreover, we show that a recent generalization of statistics and thermodynamics due to Tsallis [8] is capable of explaining this phenomenon much more naturally: The maximization of the Tsallis entropy S_q, with $q = \frac{1}{2}$, leads to precisely the same profiles predicted by the restricted minimum enstrophy theory of Huang and Driscoll [5] for the pure-electron plasma. This observation makes it possible to develop a consistent thermodynamic description of such systems and to associate this phenomenon with a wide body of research on generalized statistics and thermodynamics.

The outline of this paper is as follows. In Sec. II we describe the dynamical equations of the pure-electron plasma column (or, equivalently, of the two-dimensional Euler fluid), cast them in Hamiltonian format, and present the constants of the motion. We also review the experimental results for this system and describe previous attempts to explain the metaequilibrium density profile by a variational principle. In Sec. III we describe Tsallis's generalization of thermodynamics and in Sec. IV we review the application of Tsallis's formalism to the problem of stellar polytropes, which are static solutions to the Poisson-Vlasov equations. This problem was considered by Plastino and Plastino [9], who applied Tsallis's methods to a linear energy functional. Noting that the energy functional of the Poisson-Vlasov system is, strictly speaking, quadratic [10], we redo this analysis. Finally, in Sec. V we return to the problem of the metaequilibrium state of the pure-electron plasma and we show that Tsallis's generalized thermodynamics may be used to explain the observed density profiles.

II. NON-NEUTRAL PLASMA PROFILES AND EULER TURBULENCE

A. Dynamical equations

Strongly magnetized pure-electron plasmas in "Penning traps" with cylindrical geometry and electrostatic axial confinement have been studied for some time now [2]. Such plasmas typically have a gyrofrequency that is much greater than the bounce frequency, which in turn is much greater than the drift frequency. That being the case, we can average over the gyro and bounce time scales, and describe the system by the two-dimensional drift motion of guiding centers, perpendicular to the magnetic field.

Since the magnetic field is uniform, the dominant drift mechanism is the $\mathbf{E}\times\mathbf{B}$ drift, given by

$$\mathbf{v}_{\mathbf{E}\times\mathbf{B}} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2},$$

where $\mathbf{B}$ is the applied magnetic field, $\mathbf{E}$ is the self-consistent electric field, and c is the speed of light. Since this drift velocity is independent of particle thermal velocity, it is possible to project out the velocity degrees of freedom in phase space and thereby write a Vlasov equation directly for the guiding-center density $n(\mathbf{r},t)$,

$$0 = \frac{\partial n(\mathbf{r},t)}{\partial t} + \mathbf{v}_{\mathbf{E}\times\mathbf{B}}\cdot\nabla n(\mathbf{r},t).$$

Writing $\mathbf{E} = -\nabla\Phi$ and adopting dimensionless units with a magnetic field of unit magnitude, this can be written

$$0 = \frac{\partial n(\mathbf{r},t)}{\partial t} + \hat{\mathbf{b}}\cdot[\nabla\Phi\times\nabla n(\mathbf{r},t)], \qquad (1)$$

where $\hat{\mathbf{b}}$ is a unit vector in the direction of the magnetic field and we have used the vector "triple-product" identity. The self-consistent electrostatic potential is then given by the Poisson equation

$$\nabla^2\Phi(\mathbf{r},t) = 4\pi e n(\mathbf{r},t),$$

where $-e$ is the electronic charge. (Henceforth, we set $e=1$.)

If we identify n as the vorticity and Φ as the stream function, we note that these equations are isomorphic to Euler's equations of inviscid fluid dynamics in two dimensions. Likewise, Dirichlet boundary conditions, for which the wall is an equipotential, correspond to the condition that the normal velocity of the Euler fluid vanishes at the wall. Indeed, simulations of pure-electron plasma columns provide an important experimental tool for the study of two-dimensional Euler turbulence. Henceforth, in discussing the *conservative* dynamics of these systems, we interchangibly refer to their physical embodiment as a pure-electron plasma column or as two-dimensional Euler turbulence. It should be noted, however, that the dissipation mechanisms for these two systems may be quite different; we shall return to this point later.

B. Hamiltonian structure

In spite of the fact that we have projected from phase space to configuration space, we note that the Vlasov equation, Eq. (1), has a symplectic Hamiltonian form in two dimensions. Specifically, taking the magnetic field in the z direction, so that $\hat{\mathbf{b}} = \hat{\mathbf{z}}$, the Vlasov equation has the form

$$\begin{aligned}\frac{\partial n(\mathbf{r},t)}{\partial t} &= \frac{\partial n(\mathbf{r},t)}{\partial x}\frac{\partial \Phi(\mathbf{r},t)}{\partial y} - \frac{\partial n(\mathbf{r},t)}{\partial y}\frac{\partial \Phi(\mathbf{r},t)}{\partial x}\\ &= -\left[n(\mathbf{r},t),h(\mathbf{r},t)\right],\end{aligned}$$

where the single-guiding-center Hamiltonian is

$$h(\mathbf{r},t) = -\Phi(\mathbf{r},t),$$

and we have defined the corresponding Poisson bracket

$$[a(\mathbf{r},t),b(\mathbf{r},t)] = \frac{\partial a(\mathbf{r},t)}{\partial x}\frac{\partial b(\mathbf{r},t)}{\partial y} - \frac{\partial a(\mathbf{r},t)}{\partial y}\frac{\partial b(\mathbf{r},t)}{\partial x},$$

so that x and y are canonically conjugate variables. The configuration space can thus be regarded as a phase space and the dynamics of the plasma are then a symplectomorphism in configuration space.

The corresponding Hamiltonian field structure is noncanonical and Lie-Poisson in form [10]. That is, the equation of motion is

$$\frac{\partial n(\mathbf{r},t)}{\partial t} = \{n(\mathbf{r},t),H[n]\},$$

where the field Lie-Poisson bracket of two functionals of n is

$$\{A,B\} = \int d^2r\; n(\mathbf{r},t)\left[\frac{\delta A}{\delta n(\mathbf{r},t)},\frac{\delta B}{\delta n(\mathbf{r},t)}\right]$$

and the field Hamiltonian functional is given by

$$\begin{aligned}H[n] &= \frac{1}{2}\int d^2r\; n(\mathbf{r},t)h(\mathbf{r},t)\\ &= -\frac{1}{2}\int d^2r\; n(\mathbf{r},t)\Phi(\mathbf{r},t)\\ &= -\frac{1}{2}\int d^2r\; n(\mathbf{r},t)\int d^2r'\; n(\mathbf{r}',t)G(\mathbf{r},\mathbf{r}'), \qquad (2)\end{aligned}$$

where, in turn, $G(\mathbf{r},\mathbf{r}')$ denotes the Green's function of the Poisson problem

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = 4\pi\delta(\mathbf{r}-\mathbf{r}').$$

The factor of $\frac{1}{2}$ in the Hamiltonian prevents double counting of the energy.

The Lie-Poisson bracket admits the infinite set of Casimir functionals [10],

$$\mathcal{Z}_\psi[n] \equiv \int d^2r\; \psi(n(\mathbf{r},t)),$$

where ψ is any function of its argument. These Casimir functionals commute with any other functional, including the Hamiltonian, and hence they are constants of the motion. We may span the set of Casimir functionals with analytic ψ by the set

$$\mathcal{Z}_j[n] \equiv \frac{1}{j} \int d^2r \; n^j(\mathbf{r},t),$$

indexed by the integers $j \geq 1$. The Casimir functional $\mathcal{Z}_2$ is of special importance; it is called the *enstrophy*, since its analog for the Euler fluid is the integral of the square of the vorticity. If we further suppose that the Penning trap is cylindrically symmetric, with a grounded outer wall, then the Hamiltonian is invariant under rotation and time translation, so that the angular momentum

$$L[n] = \int d^2r \; r^2 n(\mathbf{r},t)$$

and the total energy $H[n]$ are also good invariants.

C. Variational descriptions of the metaequilibrium state

In spite of the elegance of this Hamiltonian structure, both laboratory and numerical experiments indicate that some of these theoretical invariants are broken, presumably due to collisional effects that are, of course, ignored in a Hamiltonian formulation. Such collisional effects are, at least for the case of the pure-electron plasma column, poorly understood at present. Moreover, as has been noted, the details of such dissipation mechanisms for the pure-electron plasma column and the Euler fluid may be quite different, and both of these may be likewise different from that of any numerical simulation of these systems (e.g., a particle-in-cell simulation).

Stable equilibria, both axisymmetric and nonaxisymmetric, have been observed for the pure-electron plasma column [11]. If the plasma is initialized with a hollow density profile, however, the spatial gradients will excite diochotron (Kelvin-Helmholtz-like) instabilities on a short time scale, which will, in turn, give rise to much longer-lived vortex patches. As these patches move about and collide, they shed filiaments of particles that erode the vortex patches further, until a *metaequilibrium* state with a characteristic profile shape is eventually reached. This metaequilibrium state can persist for tens of thousands of diochotron periods, until it is finally destroyed by three-dimensional effects, which are outside the scope of this paper [1,4]. Here we focus on the metaequilibria of initially axisymmetric configurations.

In the course of the above-described evolution, the total mass $\mathcal{Z}_1$ and the angular momentum L are well conserved. The energy H is reasonably well conserved. The enstrophy $\mathcal{Z}_2$ tends to decrease in more-or-less monotonic fashion and other Casimir invariants, such as the Boltzmann entropy

$$S[n] = -\int d^2r \; n(\mathbf{r}) \ln\left[n(\mathbf{r})\right],$$

are badly broken. For this reason, $\mathcal{Z}_1$, L, and H are often referred to as *robust* or *rugged* invariants, while the $\mathcal{Z}_j$ with $j \geq 2$ are termed *fragile* or *dissipated* invariants [4].

It is tempting to try to derive the shape of the final profile from a variational principle. Most work has centered on maximizing the Boltzmann entropy, under the condition that the robust invariants are fixed [12]. Using the Boltzmann entropy, one can demand

$$0 = \delta(S - \alpha\mathcal{Z}_1 - \beta H - \lambda L),$$

which yields the relationship

$$-1 - \ln\left[n(\mathbf{r})\right] + \beta\Phi(\mathbf{r}) = \alpha + \lambda r^2. \tag{3}$$

Taking the Laplacian of both sides, we arrive at an equation for the density profile

$$-\nabla^2\left[\ln n(\mathbf{r})\right] + 4\pi\beta n(\mathbf{r}) = 4\lambda. \tag{4}$$

Unfortunately, the observed metaequilibrium density profiles are significantly more peaked than the solutions to this equation [4,5].

Matthaeus and Montgomery [13] have suggested that turbulent relaxation follows a *selective decay hypothesis*, according to which the approach to equilibrium is governed by the most slowly decaying fragile invariant. In this case, because the enstrophy seems to be the most slowly decaying of all the fragile invariants, it has been proposed that non-neutral plasmas [5] and Euler turbulence [14] tend to minimize enstrophy, rather than maximize the Boltzmann entropy, while still respecting the robust invariants. Indeed, if we replace the above variational principle with

$$0 = \delta(\mathcal{Z}_2 - \alpha\mathcal{Z}_1 - \beta H - \lambda L),$$

then we are quickly led to the relationship

$$n(\mathbf{r}) + \beta\Phi(\mathbf{r}) = \alpha + \lambda r^2. \tag{5}$$

Taking the Laplacian of both sides, we arrive at the linear Helmholtz equation for the density profile

$$\nabla^2 n(\mathbf{r}) + 4\pi\beta n(\mathbf{r}) = 4\lambda. \tag{6}$$

The well behaved cylindrically symmetric solutions to this equation are of the form

$$n(r) = \nu J_0(\kappa r) + \mu, \tag{7}$$

where J_0 is the Bessel function and the constants μ, ν, and κ, which have replaced β, λ, and the constant of integration, are fully determined by the constrained quantities $\mathcal{Z}_1$, H, and L.

Because the solutions to the above variational problem often predict a negative density near the wall, Huang and Driscoll also introduced a *restricted minimum enstrophy* (RME) model [5] in which the above profile is replaced by the cutoff form

$$n(r) = \begin{cases} \nu\left[J_0(\kappa r) - J_0(\kappa r_0)\right] & \text{for } 0 \leq r \leq r_0 \\ 0 & \text{for } r_0 \leq r \leq r_w, \end{cases} \tag{8}$$

where r_w is the wall radius (which, in dimensionless units, can be set to unity). The constant r_0 has replaced the constant μ; all three constants are still fully determined by the three constraints. This form is justified only by the observation that a negative density near the wall cannot be physical and that nonmonotonic profiles are typically subject to diochotron instabilities.

Huang and Driscoll then carefully compared [5] experimental data for the pure-electron plasma column with the profiles generated by Eqs. (4), (6), and (8). They found that the data clearly ruled out the maximum Boltzmann entropy profile of Eq. (4). The minimum enstrophy profile of Eq. (6) was much better. Best of all was the RME profile of Eq. (8). The experimental data were clearly consistent with the truncated Bessel function profiles.

To date, a completely satisfactory explanation of this tendency to minimize enstrophy, rather than maximize entropy, does not exist. In the remainder of this paper, we shall show that this phenomenon is consistent with a generalization of thermodynamics and statistical physics recently proposed by Tsallis [8]. Though this is still not an explanation *per se*, it certainly makes possible the association of this phenomenon with a much larger—and growing—body of research.

III. GENERALIZED THERMODYNAMICS

Tsallis [8] has proposed a generalization of thermodynamics and statistical physics to describe systems with long-range interactions, or with long-time memory. For a system with W microscopic state probabilities $p_i \geq 0$, which are normalized according to

$$1 = \sum_i^W p_i, \tag{9}$$

Tsallis bases his formalism upon the following two axioms.

Axiom 1. The entropy of the system is given by

$$S_q = k\frac{1-\sum_i^W p_i^q}{q-1} = \frac{k}{q-1}\sum_i^W (p_i - p_i^q),$$

where k and q are real constants.

Axiom 2. An experimental measurement of an observable O, whose value in state i is o_i, yields the q-expectation value,

$$O_q = \sum_i^W p_i^q o_i$$

of the observable O.

It is to be emphasized that these statements are taken as *axioms*. As such, their validity is to be decided solely by the conclusions to which they lead and ultimately by comparison with experiment. From a more intuitive standpoint, however, we note that the behavior of some physical systems can be dominated by very rare events, while that for other physical systems can be dominated only by the most frequent events. The easiest way of introducing this sort of influence bias into a statistical physical description is to raise the corresponding probabilities to a power q. Accordingly, systems with $q < 1$ give more weight to rare events, while systems with $q > 1$ give more weight to frequent events [15].

We first note that in the limit as q approaches unity we recover the familiar expressions

$$S_1 = -k\sum_i p_i \ln p_i$$

and

$$O_1 = \sum_i p_i o_i,$$

whence we may identify k with Boltzmann's constant k_B. More generally, it has been noted [16] that k may be q-dependent, and need only coincide with Boltzmann's constant for $q = 1$; for the purposes of this paper, however, we disregard that possibility and henceforth adopt units so that $k = k_B = 1$. In any case, it is clear that Tsallis's thermodynamics contain the more orthodox variety as a special case.

The success of thermodynamics and statistical physics depends crucially upon certain properties of the entropy and energy, and much effort has been devoted to showing that many of these are valid for arbitrary q and to finding appropriate generalizations of the rest. Following Tsallis's presentation [17], it is straightforward to verify the following properties.

Property 1. The generalized entropy is positive.

That is, we have $S_q \geq 0$, where equality holds for pure states ($\exists i$, $p_i = 1$) and for $q > 0$.

Property 2. The microcanonical ensemble has equiprobability.

To see this, we extremize the generalized entropy under the constraint of normalized probabilities Eq. (9). Introducing the Lagrange multiplier λ, we set

$$0 = \frac{\partial}{\partial p_i}\left(S_q - \lambda\sum_i^W p_i\right) = -\frac{q}{q-1}p_i^{q-1} - \lambda.$$

It follows that

$$p_i = \left[\frac{\lambda(1-q)}{q}\right]^{1/(q-1)}.$$

Since this is independent of i, imposition of the constraint Eq. (9) immediately yields $p_i = 1/W$.

Property 3. The entropy is concave for $q > 0$ and convex for $q < 0$.

This follows immediately from the Hessian matrix

$$\frac{\partial^2}{\partial p_i \partial p_j}\left(S_q - \lambda\sum_i^W p_i\right) = -qp_i^{q-2}\delta_{ij},$$

which is clearly negative (positive) definite for $q > 0$ ($q < 0$). Thus the generalized entropy is maximized for $q > 0$ and minimized for $q < 0$.

Next, we consider the canonical ensemble. If we define a state energy ε_i, so that the generalized internal energy is given by

$$U_q = \sum_i^W p_i^q \varepsilon_i, \tag{10}$$

then we can extremize S_q under the constraint that probability is conserved and that the energy is fixed. We find the following.

Property 4. The canonical ensemble probability distribution is

$$p_i = \begin{cases} \frac{1}{Z_q}\left[1-(1-q)\beta\varepsilon_i\right]^{1/(1-q)} \\ \text{if } 1-(1-q)\beta\varepsilon_i \geq 0 \\ 0 \text{ otherwise,} \end{cases} \tag{11}$$

where we have defined the generalized partition function

$$Z_q \equiv \sum_i^W \left[1-(1-q)\beta\varepsilon_i\right]^{1/(1-q)}, \tag{12}$$

and the inverse temperature $\beta \equiv 1/T$.

We note that, in the limit as q approaches unity, we recover the familiar expressions

$$p_i = \frac{e^{-\beta\varepsilon_i}}{Z_1} \tag{13}$$

and

$$Z_1 \equiv \sum_i^W e^{-\beta\varepsilon_i}.$$

For $q \neq 1$, we note that the absolute value of the energy may matter—an additive constant in the energy spectrum can produce physical effects. Moreover, we note that, for generic real values of q, the above expression for p_i must cut off to zero when $1-(1-q)\beta\varepsilon_i < 0$. This is because, in addition to the equality constraint Eq. (10) that is enforced in the derivation of the canonical ensemble distribution function, there are also the—usually implicit—inequality constraints that $p_i \in \Re$ and $p_i \geq 0$. These inequality constraints are not accorded much attention in Boltzmann-Gibbs thermostatistics because they are manifestly satisfied by the distribution Eq. (13). In the generalized thermostatistics, on the other hand, their enforcement requires the truncation specified in Eq. (11). When $1-(1-q)\beta\varepsilon_i < 0$, state i is said to be *thermally forbidden* and it is excluded from the sum defining the generalized partition function Eq. (12). For a positive energy spectrum that is unbounded above and assuming that $\beta > 0$, this will happen for sufficiently high ε_i if $q < 1$. Thus, in this situation, the Tsallis distribution has a natural cutoff in energy for $q < 1$. This cutoff is illustrated for various values of q in Fig. 1. More

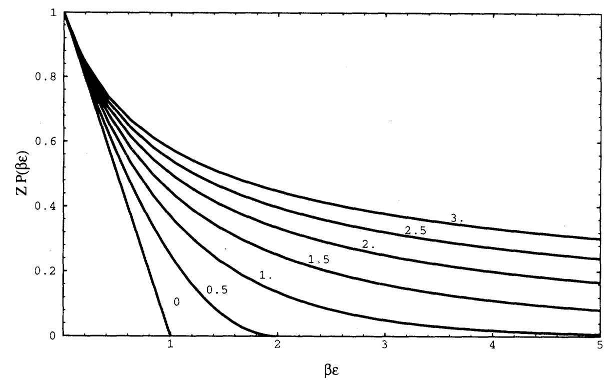

FIG. 1. $Z_q p(\beta\varepsilon)$ versus $\beta\varepsilon$ for several values of q.

significantly, we note the following.

Property 5. The Legendre-transform structure of thermodynamics is invariant for all q.

To see this, we first note that

$$-\frac{\partial}{\partial\beta}\left(\frac{Z_q^{1-q}-1}{1-q}\right) = U_q,$$

whence we identify the free energy

$$F_q = -\frac{1}{\beta}\left(\frac{Z_q^{1-q}-1}{1-q}\right).$$

It is then possible to verify that

$$F_q = U_q - TS_q$$

and it follows that

$$\frac{\partial S_q}{\partial U_q} = \frac{1}{T}.$$

As Tsallis points out [18], these equations lie at the very heart of thermodynamics and the fact that they are invariant under q is significant. The grand canonical ensemble has also been treated [19].

The most striking and significant differences between the generalized thermodynamics and the more usual variety have to do with the extensivity of the state variables. If we partition the microscopic states of the system into two disjoint subsets $L = \{1, \ldots, V\}$ and $R = \{V+1, \ldots, W\}$, with respective probabilities

$$p_L \equiv \sum_{i=1}^V p_i$$

and

$$p_R \equiv \sum_{i=V+1}^W p_i,$$

then it is straightforward to verify the following.

Property 6. The generalized entropy obeys the following generalization of the Shannon additivity property:

$$S_q(p_1,\dots,p_W) = p_L^q S_q\left(\frac{p_1}{p_L},\dots,\frac{p_V}{p_L}\right) + p_R^q S_q\left(\frac{p_{V+1}}{p_R},\dots,\frac{p_W}{p_R}\right) + S_q(p_L,p_R).$$

Alternatively, we can consider the total entropy of two completely independent subsystems A and B. Since the subsystems are independent, the probability that their union $A \cup B$ has subsystem A in state i and subsystem B in state j is given by

$$p_{ij}^{A\cup B} = p_i^A p_j^B.$$

After a bit of algebra, we find the following.

Property 7. The generalized entropy obeys the additivity rule

$$S_q(A\cup B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B),$$

and is thus superadditive (entropy of whole is greater than the sum of its parts) for $q < 1$ and subadditive for $q > 1$.

Likewise, we find the following.

Property 8. The generalized expectation value of an observable O obeys the additivity rule

$$O_q(A\cup B) = O_q(A) + O_q(B) + (1-q)\left[O_q(A)S_q(B) + O_q(B)S_q(A)\right].$$

Note that, in both cases, extensivity is recovered only when $q = 1$.

It is believed—but not proven at the time of this writing—that the Tsallis entropy is the only one for which all of the above properties hold. Moreover, generalized versions of the Boltzmann H theorem [20], fluctuation-dissipation theorem [21], and Onsager reciprocity theorem [22] exist for all q. The formalism is thus an important generalization of most of the principal results of thermodynamics and statistical physics.

Of course, to verify that this generalization is useful, it is necessary to show that it holds for certain physical systems with values of q that are different from unity. In the past two years, much work has been done along these lines and the method has been applied with great benefit to astrophysical problems such as stellar polytropes [9], Lévy flights [23], the specific heat of the hydrogen atom [24], and numerous other physical systems [25,26]. For some of these systems, strict inequalities have been proven, demonstrating that q must be different from unity in order to obtain a consistent thermodynamic description [9].

IV. STELLAR POLYTROPES

A. Hydrostatic equilibria

One of the first problems to which Tsallis's thermodynamics was applied [9] was that of stellar polytropes, first studied by Kelvin [27] and treated in detail by Chandrasekhar [28]. Because stellar polytropes are equilibria of the Poisson-Vlasov equations, they are highly relevant to the present study. Therefore, in this section, we shall review the previous application of Tsallis's formalism to this problem by Plastino and Plastino [9]. We note that they used a energy functional that was linear in the distribution function, whereas the full Poisson-Vlasov energy functional is quadratic. In this treatment, we use the full quadratic functional and compare our analysis to theirs. One of the side benefits of this treatment is that it shows the extension of Tsallis's second axiom to observables that are quadratic functionals of the distribution.

A polytropic process has the equation of state

$$P = K\rho^\gamma,$$

where P is the pressure, ρ is the density, and γ is a constant that can be related to the specific heats. If $\Phi(\mathbf{r})$ denotes the gravitational potential, then the hydrostatic equilibrium is given by

$$\mathbf{0} = -\nabla P - \rho\nabla\Phi.$$

It follows that

$$\mathbf{0} = \nabla\left(\rho^{\gamma-1} + \frac{\gamma-1}{K\gamma}\Phi\right).$$

We now seek solutions with compact support in domain $\mathcal{D}$. If we require that the density ρ vanish on the boundary $\partial\mathcal{D}$, then we must have the following relationship between ρ and Φ:

$$\rho = \left[\frac{\gamma-1}{K\gamma}\left(\Phi^{(0)} - \Phi\right)\right]^{1/(\gamma-1)},$$

where $\Phi^{(0)}$ is the potential on the boundary. The nonlinear Poisson equation for the gravitational potential is then

$$\nabla^2\Psi = -C\Psi^{1/(\gamma-1)}, \tag{14}$$

where $\Psi \equiv \Phi^{(0)} - \Phi$ and C is a constant. The boundary condition is that $\Psi = 0$ on $\partial\mathcal{D}$. This equation has, for example, spherically symmetric solutions, corresponding to compact spherical configurations of self-gravitating mass, which are called *stellar polytropes.*

B. Kinetic equilibria

As an alternative to the above hydrodynamic description, we can seek polytropic equilibria of the Vlasov equation for the mass distribution function $f(z,t)$, where

$z = (\mathbf{r}, \mathbf{v})$ coordinatizes the phase space of the system. As is well known, the equilibria of the Vlasov equations are functions of the constants of the motion. We denote the (negative of the) total energy by

$$\mathcal{E}(z) \equiv \Psi(\mathbf{r}) - \frac{m}{2}v^2,$$

so that a marginally confined particle on $\partial\mathcal{D}$ with zero velocity has $\mathcal{E} = 0$ and a confined particle has $\mathcal{E} > 0$. Noting that any function of $\mathcal{E}$ is a solution of the Vlasov equation, we examine solutions of the form

$$f = \begin{cases} \theta\mathcal{E}^{n-3/2} & \text{for } \mathcal{E} > 0 \\ 0 & \text{for } \mathcal{E} \leq 0, \end{cases}$$

where θ is a constant. The mass density of a spherically symmetric configuration is then given by

$$\begin{aligned}\rho(\mathbf{r}) &= \int dz'\ f(z')\delta(\mathbf{r} - \mathbf{r}') \\ &= \int d^3v'\ f(\mathbf{r}, \mathbf{v}') \\ &= \theta \int_0^{\sqrt{2\Psi}} dv\ 4\pi v^2 \left[\Psi(\mathbf{r}) - \frac{v^2}{2}\right]^{n-3/2}. \end{aligned} \tag{15}$$

The integral gives rise to a β function, which can be expressed in terms of γ functions to finally yield

$$\rho(\mathbf{r}) = (2\pi)^{3/2}\theta\,\frac{\Gamma(n - \frac{1}{2})}{\Gamma(n+1)}\Psi^n(\mathbf{r}). \tag{16}$$

Comparing this to Eq. (14), we can identify

$$n = \frac{1}{\gamma - 1}$$

or

$$\gamma = 1 + \frac{1}{n}.$$

C. Variational principle with a linear energy functional

Note that the stellar polytropes comprise a one-parameter family of equilibria, where the parameter is γ (or, equivalently, n). We now examine the question of whether or not these polytropic equilibria are thermodynamically stable, in the sense that they can be obtained from an entropic variational principle, and, if so, for what values of the parameter γ (or n) this is possible.

Plastino and Plastino have addressed this question [9] by extremizing Tsallis's entropy for this problem

$$S_q[f] = \frac{1}{q-1}\int dz\ \left[f(z) - f^q(z)\right],$$

under the constraints of fixed mass and energy expectation values

$$M_q[f] = \int dz\ f^q(z),$$

$$U_q[f] = \int dz\ f^q(z)\left(\frac{v^2}{2} + \Phi(\mathbf{r})\right). \tag{17}$$

In fact, they used M_1 and U_1 in their work because this was before Tsallis had advanced his second axiom about expectation values. This issue was subsequently rectified in a paper by Nobre and Tsallis [29], and we present only the corrected version here.

Introducing Lagrange multipliers, the variational problem

$$0 = \delta\left(S_q + \alpha M_q + \beta U_q\right)$$

yields the equilibrium distribution

$$\begin{aligned} f(z) &= \left\{q\left[1 - (q-1)\alpha - (q-1)\beta\left(\frac{v^2}{2} + \Phi(\mathbf{r})\right)\right]\right\}^{1/(1-q)} \\ &= \left\{q\left[1 - (q-1)\left(\alpha + \beta\Phi^{(0)}\right) + (q-1)\beta\mathcal{E}(z)\right]\right\}^{1/(1-q)}. \end{aligned} \tag{18}$$

For this to be a power law in $\mathcal{E}$, we select α so that $1-(q-1)(\alpha + \beta\Phi^{(0)}) = 0$, so we can write $f(z) = D\mathcal{E}^{1/(1-q)}(z)$, where D is a constant. The density measured at a point $\mathbf{r}$ is then given by the q-expectation value of the spatial δ function,

$$\rho_q(\mathbf{r}) = \int dz'\ f^q(z')\delta(\mathbf{r} - \mathbf{r}') = D^q\int d^3v'\ \mathcal{E}^{q/(1-q)}(\mathbf{r}, \mathbf{v}').$$

We see that this corresponds to Eq. (15) if we identify $n - \frac{3}{2} = q/(1-q)$ or

$$n = \frac{3}{2} + \frac{q}{1-q}.$$

As pointed out by Plastino and Plastino [9], it is known that n must exceed $\frac{1}{2}$ in order to avoid the singularity in the γ function in Eq. (16), but that values in excess of 5 give rise to unnormalizable mass distributions and are therefore unphysical. This means that $q \in (-\infty, \frac{7}{9})$. Thus stellar polytropes cannot be described thermodynamically unless q values less than $\frac{7}{9}$ are used Finally, note that the aforementioned cutoff of the distribution with energy—a generic feature of Tsallis distributions with $q < 1$—naturally gives rise to the spatial cutoff of the mass distribution and hence the compact nature of the stellar polytrope.

D. Variational principle with a quadratic energy functional

Note that the energy functional in Eq. (17) was regarded as linear in $f^q(z)$ in the above analysis. Specifically, in deriving Eq. (18), we wrote the functional derivative of U_q with respect to $f(z)$ as

$$\frac{\delta U_q}{\delta f(z)} = q f^{q-1}(z) \left[\frac{v^2}{2} + \Phi(\mathbf{r}) \right].$$

Strictly speaking, this is not correct because the potential $\Phi(\mathbf{r})$ depends on $f(z)$ and we did not account for this in the above variation. This is precisely the problem of self-consistency of the field—a crucial feature of the Poisson-Vlasov system [10]. To correct this problem, it is best to write the energy as a quadratic functional of the distribution, just as in Eq. (2)—only now q-expectation values should be used throughout. Thus

$$U_q^Q[f] = \int dz\, f^q(z) \frac{v^2}{2} + \frac{1}{2} \int dz\, f^q(z) \int dz'\, f^q(z') G(\mathbf{r}, \mathbf{r}'),$$

where the superscript Q denotes *quadratic*, the factor of $\frac{1}{2}$ in front of the potential prevents double counting of the energy, and $G(\mathbf{r}, \mathbf{r}')$ is the Green's function for Poisson's equation which satisfies

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = 4\pi \delta(\mathbf{r} - \mathbf{r}').$$

Note that the functional derivative of U_q^Q with respect to $f(z)$ is now

$$\frac{\delta U_q^Q}{\delta f(z)} = q f^{q-1}(z) \left[\frac{v^2}{2} + \Phi_q(\mathbf{r}) \right],$$

where we have defined

$$\Phi_q(\mathbf{r}) \equiv \int dz'\, f^q(z') G(\mathbf{r}, \mathbf{r}'),$$

which in turn satisfies

$$\nabla^2 \Phi_q(\mathbf{r}) = 4\pi \rho_q(\mathbf{r}). \tag{19}$$

The variational principle thus results in an equation very similar to Eq. (18), except with $\mathcal{E}$ replaced by

$$\mathcal{E}_q \equiv \Psi_q(\mathbf{r}) - \frac{v^2}{2},$$

where in turn

$$\Psi_q(\mathbf{r}) \equiv \Phi_q^{(0)} - \Phi_q(\mathbf{r}).$$

The resulting expression for ρ is then a power law in Ψ_q, rather than in Ψ. We still conclude that

$$n = \frac{3}{2} + \frac{q}{1-q}$$

and so the upper bound on q of $\frac{7}{9}$ still holds. Note that the nonlinear Poisson equation for the gravitational potential

$$\nabla^2 \Psi_q = -C \Psi_q^n$$

is now satisfied by Ψ_q rather than by Ψ_1.

This exercise might be dismissed as demonstrating little more than the fact that the potential used in Sec. IV C should be interpreted as the q-expectation value of the Green's function rather than as the usual one. This objection notwithstanding, the derivation using the quadratic energy functional has the following virtues.

(i) It more clearly shows that the conclusions reached by this method are valid for the system of particles in their own *self-consistent* field.

(ii) It demonstrates that Tsallis's second axiom extends in a natural way to quadratic functionals of distributions.

(iii) It yields the natural generalization of the potential and the density and shows that the form of Poisson's equation, Eq. (20), relating them is q invariant.

(iv) It is generally more consistent with the flavor and spirit of Tsallis's formalism than previous derivations.

V. GENERALIZED THERMODYNAMIC DESCRIPTION OF PURE-ELECTRON PLASMA DRIFT TURBULENCE

We now return to the problem of deriving the metaequilibrium profiles of relaxed drift turbulence of the pure-electron plasma column. We redo the calculation of Sec. II C, using the Tsallis prescriptions for the entropy and the robustly conserved quantities. The entropy is thus

$$S_q[n] = \frac{1}{q-1} \int d^2r\ [n(\mathbf{r}) - n^q(\mathbf{r})]$$

and the constraints, expressed in terms of q-expectation values, are

$$\mathcal{Z}_q[n] = \frac{1}{q} \int d^2r\, n^q(\mathbf{r}),$$

$$H_q[n] = -\frac{1}{2} \int d^2r\, n^q(\mathbf{r}) \int d^2r'\, n^q(\mathbf{r}') G(\mathbf{r}, \mathbf{r}'),$$

and

$$L_q[n] = \int d^2r\, r^2 n^q(\mathbf{r}).$$

Setting

$$\delta\left(S_q - \alpha \mathcal{Z}_q - \beta H_q - \lambda L_q\right) = 0,$$

we find

$$\frac{n^{1-q}(\mathbf{r}) - q}{q(q-1)} + \beta \Phi_q(\mathbf{r}) = \alpha + \lambda r^2, \tag{20}$$

where

$$\Phi_q(\mathbf{r}) \equiv \int d^2r'\ n^q(\mathbf{r}')G(\mathbf{r},\mathbf{r}')$$

satisfies

$$\nabla^2\Phi_q(\mathbf{r}) = 4\pi n^q(\mathbf{r}). \tag{21}$$

Applying the Laplacian to Eq. (21), we obtain the nonlinear Helmholtz equation

$$\frac{1}{q(q-1)}\nabla^2\left[n^{1-q}(\mathbf{r})\right] + 4\pi\beta n^q(\mathbf{r}) = 4\lambda. \tag{22}$$

Now, the *observed* particle density $\rho_q(\mathbf{r})$, i.e., that which is measured in any experiment, is then the q-expectation value of the δ function

$$\rho_q(\mathbf{r}) = \int d^2r'\ n^q(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') = n^q(\mathbf{r}).$$

In terms of this, Eqs. (20)–(22) can be written

$$\frac{\rho_q^{(1-q)/q}(\mathbf{r}) - q}{q(q-1)} + \beta\Phi_q(\mathbf{r}) = \alpha + \lambda r^2, \tag{23}$$

$$\nabla^2\Phi_q(\mathbf{r}) = 4\pi\rho_q(\mathbf{r}),$$

$$\frac{1}{q(q-1)}\nabla^2\left[\rho_q^{(1-q)/q}(\mathbf{r})\right] + 4\pi\beta\rho_q(\mathbf{r}) = 4\lambda.$$

As $q \to 1$, it is seen that this reproduces the results of the maximum Boltzmann entropy relationship, Eqs. (3) and (4). When $q = \frac{1}{2}$, on the other hand, we see that, within trivial redefinitions of the Lagrange multipliers and the use of $\Phi_{1/2}$ instead of Φ, this reproduces the results obtained by minimizing the enstrophy, Eqs. (5) and (6), *but for a completely different reason.* Moreover, just as in the example of the stellar polytrope, the cutoff in density at a finite radius r_0 appears as a *completely natural and generic feature of the Tsallis distribution, since $q < 1$, and does not need ad hoc justification.* Thus we conclude that all prior observations that have indicated that the relaxation of two-dimensional pure-electron plasma drift turbulence tends to follow the RME principle of Huang and Driscoll can now be reinterpreted as rather indicating that it maximizes the Tsallis entropy S_q for $q = \frac{1}{2}$.

We note in passing that there is an easier way to obtain the result that $q = \frac{1}{2}$. Without going through this analysis, we note that the enstrophy itself looks rather like (a linear function of) the Tsallis entropy with $q = 2$. Of course, this is misleading because it is necessary to use q-expectation values in the extremization process. Nevertheless, Nobre and Tsallis [29] have shown that one result of *not* using q-expectation values in the extremization process is to effect the transformation $q \to 1/q$. Hence we are again led immediately to the result $q = \frac{1}{2}$ for this system.

VI. GENERALIZED THERMODYNAMIC DESCRIPTION OF EULER TURBULENCE

As has been noted, the dissipation mechanisms of turbulence in the pure-electron plasma and the Euler fluid may be quite different. Having established that Huang and Driscoll's results are consistent with $q = \frac{1}{2}$ for the pure-electron plasma, it is interesting to inquire whether a similar result holds for the Euler fluid.

A partial answer to this question may be found in the work of Sommeria *et al.* [30], who have studied the decay of turbulence in the two-dimensional incompressible Navier-Stokes fluid (Euler fluid with viscous dissipation) by numerical simulation with a pseudospectral method. Since the boundary conditions used in this study were not cylindrically symmetric, the constraint of constant angular momentum can be discarded; this is most easily done by setting $\lambda = 0$ in the analysis of the preceding section. Equation (24) then indicates the following relationship betwen the vorticity and the stream function:

$$\rho_q \propto C\left(\Phi_{q0} - \Phi_q\right)^{q/(1-q)}, \tag{24}$$

with a cutoff at $\Phi_q = \Phi_{q0}$. Indeed, Fig. 8 in the paper of Sommeria *et al.* [30], which plots vorticity versus stream function at the longest times reported, *clearly indicates a cutoff of ρ_q* (denoted by ω in that reference). Moreover, the concavity of the curves near the cutoff indicates that $\frac{1}{2} < q < 1$. While better measurements are needed to determine this value more accurately, it is clear that the generalized thermostatistics provide a much more natural description of this system than those of Boltzmann and Gibbs.

VII. CONCLUSIONS

The tendency of a pure-electron plasma column to minimize enstrophy, rather than maximize the Boltzmann entropy, during turbulent relaxation to a metaequilibrium state has resisted theoretical explanation to date. In this work, we have shown that density profiles resulting from the restricted minimum enstrophy theory of Huang and Driscoll also maximize the Tsallis entropy S_q with $q = \frac{1}{2}$. We have also noted that the studies of the Euler fluid by Sommeria *et al.* are consistent with $\frac{1}{2} < q < 1$. We have thereby provided an alternative way to understand this phenomenon—one in which the density (vorticity) cutoff at finite radius emerges quite naturally—and to build a consistent thermodynamical and statistical physical explanation for it. In the course of doing this, we have verified Plastino and Plastino's upper bound of $\frac{7}{9}$ on q for the stellar polytrope problem using the full quadratic energy functional for the Poisson-Vlasov system; we have also demonstrated the q invariance of the Poisson equation for these systems.

While still short of a first-principles *explanation* of the RME model—it would be nice, for example, to have an *a priori* way of knowing *why* q should be equal to $\frac{1}{2}$ for the pure-electron plasma column—the observation that

RME is consistent with Tsallis statistics does effectively associate it with a large and rapidly expanding body of theory [31]. In recent years, generalized thermodynamics has been used to describe numerous, widely disparate physical systems, with long-range interactions, long-time (non-Markovian) memory, or fractal space-time structure, that have resisted previous attempts at a thermodynamic description. It is hoped that this observation will stimulate further research in the use of generalized thermodynamics to describe fluid turbulence.

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