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Correlations in quantum plasmas. II. Algebraic tails

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For a system of point charges that interact through the three-dimensional electrostatic Coulomb potential (without any regularization) and obey the laws of nonrelativistic quantum mechanics with Bose or Fermi statistics, the static correlations between particles are shown to have a $1/r^6$ tail, at least at distances that are large with respect to the length of exponential screening. After a review of previous work, a term-by-term diagrammatic proof is given by using the formalism of paper I, where the quantum particle-particle correlations are expressed in terms of classical-loop distribution functions. The integrable graphs of the resummed Mayer-like diagrammatics for the loop distributions contain bonds between loops that decay either exponentially or algebraically, with a $1/r^3$ leading term analogous to a dipole-dipole interaction. This reflects the fact that the charge-charge or multipole-charge interactions between clusters of particles surrounded by their polarization clouds are exponentially screened, as at a classical level, whereas the multipole-multipole interactions are only partially screened. The correlation between loops decays as $1/r^3$, but the spherical symmetry of the quantum fluctuations makes this power law fall to $1/r^5$, and the harmonicity of the Coulomb potential eventually enforces the correlations between quantum particles to decay only as $1/r^6$. The coefficient of the $1/r^6$ tail at low density is planned to be given in a subsequent paper. Moreover, because of Coulomb screening, the induced charge density, which describes the response to an *external* infinitesimal charge, is shown to fall off as $1/r^8$, while the charge-charge correlation in the medium decreases as $1/r^{10}$. However, in spite of the departure of the *quantum microscopic* correlations from the classical exponential clustering, the *total* induced charge is still essentially determined by the exponentially screened charge-charge interactions, as in *classical macroscopic* electrostatics. [S1063-651X(96)05205-1]

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I. INTRODUCTION

The present paper is the second part of a series about the equilibrium static correlations in matter under usual conditions, i.e., when electrons and nuclei can be seen as nonrelativistic quantum point charges. The first point of the paper is to show that the particle-particle correlation in a multicomponent plasma does have a $1/r^6$ tail, when the quantum statistics is taken into account and the interaction is the pure Coulomb potential (without any regularization). These two aspects of real matter were approximated in previous works about the decay of correlations in multicomponent plasmas [1–5]. Another aim of the paper is to display clearly how the harmonicity of the Coulomb potential and the spherical symmetry of the quantum-fluctuation distribution for one particle enforce this power law. As explained in Sec. VI of paper I, the standard perturbation many-body theory is not very helpful in this respect and we use the loop formalism devised in paper I. Moreover, this formalism also allows one to obtain the following results: the charge-charge correlation decays only as $1/r^{10}$, while the induced charge density and the corresponding total potential, which measure the linear response to a localized infinitesimal external charge, fall off as $1/r^8$ and $1/r^6$, respectively. The reason is that screening mechanisms that are similar to the usual effects in classical macroscopic electrostatics do survive in the plasma at a quantum microscopic level. We notice that, for each announced algebraic tail, we are not able to control the convergence of the whole diagrammatic series, each diagram of which decays algebraically with the exponent mentioned above. However, the calculation of the coefficient of the $1/r^6$ tail at low density [6] strongly suggests that no more cancellation occurs. In the present introduction, the connection between screening and the decay of the correlations is reviewed (Sec. I A) and the insight given by the loop formalism is pointed out (Sec. I B).

A. Historical review about screening and correlations

As recalled in paper I, the harmonicity of the Coulomb potential is responsible for a very special screening that arises in both dielectric and conductive phases of Coulomb plasmas at a classical as well as at a quantum level. The Coulomb screening enforces the local neutrality relation between the densities of charges and, at a more microscopic level, ensures that the net charge of a point charge in the medium together with its polarization cloud is zero. Subsequently, this net charge creates an effective potential that decays at large distances faster than the $1/r$ Coulomb potential. This latter property is linked to the fact that the total induced charge in the presence of an external static charge is finite, as stressed in Sec. IV D of paper I. The charge neutrality of a particle and its polarization cloud also exists if the potential decays more slowly [7] or faster [8] than the Cou-

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lomb interaction, but the harmonicity of the Coulomb potential generates further screening effects.

Some properties of the particle-particle correlation can be investigated by considering the perturbation in the charge distributions that is induced by an *external* infinitesimal charge distribution (this fixed charge is assumed to be classical). The latter perturbations are related exactly to some charge distribution functions through the linear response theory, but the relation is different in the classical and quantum regimes. For instance, in a conductive phase, the total induced charge around an infinitesimal charge distribution $\delta q(\mathbf{r})$ is exactly opposite the total charge $\int d\mathbf{r}\ \delta q(\mathbf{r})$ [see Eq. (1.3) of paper I] and, as recalled in Sec. IV D of paper I, this property implies a sum rule for the second moment of the charge-charge correlation function in the classical case and for the second moment of the zero-frequency component of the time-ordered charge-charge correlation in the quantum case. More generally, information about the decay of the truncated distribution functions can be inferred from the variation in the n-body distribution functions between particles of the medium when an infinitesimal external charge is immersed in the plasma. At a classical level, if the variation decreases exponentially fast far away from the external charge, then the classical charge distributions for the particles of the medium satisfy the so-called multipole sum rules. The multipole sum rules describe a ''perfect'' screening in which the polarization clouds compensate not only the total charge but even all the multipoles of any given configuration of charges. In quantum statistics [9], this perfect screening would be realized if the distribution functions for the particles of the medium decreased faster than any inverse power law, as is the case in classical statistics [7,10] when the correlations in the plasma have a fast clustering. We notice that, for any other integrable power-law potential [11], the falloff of the classical correlations is bounded below by an inverse power and such a perfect screening cannot exist.

The first descriptions of screening in plasmas dealt with the simplest quantities: the charge density that is induced in the plasma when an infinitesimal distribution of charge $\delta q(\mathbf{r})$ is immersed into it and the total potential V^{tot} in the bulk in the presence of the external charge. These two quantities are related by the Poisson equation (6.7) (see paper I). Self-consistent models were solved for classical electrolytes [12–14] (Debye-Hückel theory) and for the fermionic one-component plasma (OCP) in the semiclassical regime at high density [15,16] (Thomas-Fermi theory) as well as in the quantum case [17,18], with the many versions of the so-called random-phase approximation (RPA). (For a brief historical review about the RPA theory, see Ref. [17].) In the classical and semiclassical regimes, the self-consistent models are mean-field approachs in which the correlations between particles are approximated in an indirect way: the classical particles or the quantum quasiparticles do not interact together, but they move in a mean-field potential. The approximation is reasonable for classical plasmas in a weak coupling regime (low density or high temperature) and for fermionic quantum plasmas in a semiclassical regime when the density is sufficiently high so that the interaction energy becomes negligible with respect to the fermionic quantum kinetic energy. Indeed, in this regime and under the semiclassical assumption that the total potential is appreciable only in a finite region and varies slowly over scales larger than the mean interparticle distance, the system can be described by quasiparticles with a chemical potential $\mu_\alpha^0+e_\alpha V_{\mathrm{MF}}^{\mathrm{tot}}(\mathbf{r})$. In the linearized versions of the mean-field models, at large distances, the approximated induced charge density $\Sigma_\alpha e_\alpha \rho_{\alpha,\mathrm{MF}}^{\mathrm{ind}}$ is proportional to the potential $V_{\mathrm{MF}}^{\mathrm{tot}}$,

$$\sum_\alpha e_\alpha \rho_{\mathrm{MF},\alpha}^{\mathrm{ind}}(\mathbf{r}) = -\frac{\kappa_{\mathrm{MF}}^2}{4\pi} V_{\mathrm{MF}}^{\mathrm{tot}}(\mathbf{r}), \tag{1.1}$$

where κ_{MF} is an inverse length that depends on the self-consistent mean-field model. According to the Poisson equation (6.7) of paper I, the corresponding mean-field potential $V_{\mathrm{MF}}^{\mathrm{tot}}$ in the presence of a point charge δq located at $\mathbf{r}=\mathbf{0}$ is a Yukawa-like potential

$$V_{\mathrm{MF}}^{\mathrm{tot}}(\mathbf{r}) = \delta q\ \frac{e^{-\kappa_{\mathrm{MF}} r}}{r} \tag{1.2}$$

and the induced charge density has the same exponential falloff. In the quantum case, a more detailed description of the correlations must be used. In the various versions of the self-consistent RPA model, $\Sigma_\alpha e_\alpha \rho_{\mathrm{RPA},\alpha}^{\mathrm{ind}}(\mathbf{r})$ and $V_{\mathrm{RPA}}^{\mathrm{tot}}(\mathbf{r})$ decay faster than any inverse power law and the leading terms in their asymptotic behaviors are similar to the mean-field Yukawa potential (1.2). Since $\Sigma_\alpha e_\alpha \rho_\alpha^{\mathrm{ind}}(\mathbf{r})$ and $V^{\mathrm{tot}}(\mathbf{r})$ are related indirectly to the particle-particle correlation by the linear response theory, this result proves to be linked to a fast decay of the corresponding approximated correlation between particles [4,5]. We notice that, in this microscopic model, an oscillatory algebraic term, known as the Friedel oscillations [19], occurs at *zero* temperature because of the discontinuity of the Fermi distribution for independent entities in the ground state. This phenomenon has been observed as a broadening of nucleon magnetic resonance lines in dilute alloys (see Ref. [18] and references cited therein). However, at *finite* temperature, the Fermi surface is smeared over a thickness $k_B T$ in energy (where k_B is the Boltzmann constant) and the oscillations are multiplied by an exponentially decaying factor.

Though the picture of Debye screening for the particle-particle correlation is usually taken for granted, very few rigorous results are known about it. For a classical multicomponent plasma (with short-ranged regularization of the Coulomb potential in order to avoid collapse), in the limit of weak coupling (low-density or high-temperature regime), the existence of exponential clustering of the correlation functions was proved first for a lattice Coulomb gas (with the discretized version of $1/r$) by Brydges [20] and for the continuous system by Brydges and Federbush [21]. Next the proof was extended to the OCP by Imbrie [22]. However, in the quantum case, strong doubts about an exponential falloff were raised by Brydges and Federbush [1] and then by Brydges and Seiler [2]. In the latter reference, the authors cast the nonrelativistic quantum Coulomb system into the form of some kind of lattice gauge model and found that an infinite correlation length appears in some Green's functions associated with external charges depending on an imaginary time. They conjectured that the electric field of a static external source should not be screened exponentially; rather it should have a long-ranged tail decaying as an inverse power of the distance. Indeed, in nearly classical regimes, the exponential decay would be very close to the truth for all but extremely large distances. In the language of field theory, the interpretation is that vacuum polarization produces dipoles [23] and dipoles do not screen [24]. In other words, though a charge of the medium and its cloud carry no mean charge and no mean dipole as a whole, nevertheless, it has a nonvanishing instantaneous dipole because of fluctuations and in the quantum case the interactions between dipolar fluctuations are not screened as efficiently as in the classical regime. Henceforth, the correlation between two particles of the system has an algebraic tail that may fall off *a priori* only as a $1/r^3$ dipole-dipole interaction energy. In fact, smaller upper bounds on the decay of the correlations were obtained by Alastuey and Martin [3,4] in a regime of high temperature and low density. In this regime, the correlations are assumed to decrease monotonically with an integer inverse power-law expansion starting with $1/r^3$ and the hierarchy of equilibrium equations for imaginary-time observables (or ''evolution equations'') is supposed to be valid in the thermodynamic limit. Under those assumptions, the particle-particle correlation decreases at least as $1/r^5$ in a multicomponent plasma (and faster than $1/r^6$ in the very special case of the OCP, in which charge and particle observables are proportional to each other). In contrast, the corresponding investigation of the equilibrium equations in the classical case enables exclusion of any monotonic inverse power law [11].

The next steps in questioning the nature of the asymptotic behavior of the quantum correlations were done by perturbative approachs. First, in the approximation of Maxwell-Boltzmann statistics for a modified Coulomb potential, algebraic lower bounds were obtained from an $\hbar$ expansion developed by Alastuey and Martin [4] in a path integral formalism. The approximation of distinguishable particles (classical Maxwell-Boltzmann statistics) is valid only in a regime of high temperature and low density and requires a regularization of the potential at short distances in order to prevent the macroscopic collapse of the plasma (except in the case of the OCP, where there is only one species of point charges). In this semiclassical regime, the exchange effects appear as corrections that vanish exponentially when $\hbar$ goes to zero [25]. In the $\hbar$ expansion of the correlations about their classical values, the correction of order $\hbar^2$ has a fast decay, but each term of order $\hbar^{2n}$, with $n \geqslant 2$, decays algebraically as $1/r^6$. In particular, the particle-particle correlation should not decay faster than $1/r^6$, while the several point correlations should have even the slower decay $1/r^3$, when groups of particles are separated. These bounds are compatible with the upper ones that are deduced from the above nonperturbative approach of the equations of motion for a system with quantum statistics [4]. Moreover, they do not violate the basic sum rules recalled in Eqs. (1.1)–(1.3) of paper I [26]. We notice that a formally similar analysis can be carried out for the classical time-displaced particle-particle correlation at equilibrium; a short-time expansion about the static correlations together with an investigation of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy also leads to an algebraic decay of the classical time-displaced correlations [3]. The reason is that collisions tend to destroy the ''perfect'' organization of the clouds, while inertia prevents classical particles to follow instantaneously the moving charge. In a quantum regime, the above lower algebraic bounds show that the intrinsic quantum fluctuations destroy even the static perfect organization of the polarization cloud around a charge of the medium. The quantum nature of the subsequent ''dipole-dipole-like'' interaction between two point charges is quite different from the van der Waals interaction that arises between objects with an internal structure, such as atoms or molecules, and that comes from the polarizability of quantum bound states. Indeed, the algebraic falloff appears in the OCP, where no bound states can be formed, while the large-distance correlation between two quantum point charges embedded in a classical plasma proves to be exactly a $1/r^6$ decay [4].

At this point a natural question to ask was the following: how can one go beyond the RPA and find algebraic tails in the standard many-body perturbation theory at finite temperature? In this formalism, the Fermi statistics is taken into account, so the results are valid even at low temperatures and the potential dealt with as a perturbation is the true Coulomb potential. The particle-particle correlation is derived from the sum of all chains built with ''proper polarization'' graphs linked by Coulomb interaction lines; the Fourier transform of the total potential $V^{\rm tot}$ in the presence of an infinitesimal external charge distribution is proportional to the zero-frequency component of the ''effective'' potential of the formalism, which is defined as the sum of all chains of bare Coulomb interaction lines linked by proper polarization graphs. As detailed in Sec. V, the RPA theory consists in approximating the proper polarization by its value for noninteracting particles, namely, a loop of fermionic free propagators. The next natural correction to the RPA value for the proper polarization is the graph where two loops of fermionic free propagators are linked by two RPA effective potential lines. It happens that two exact sum rules [27,17] can be deduced for the OCP because in this model the charge density is proportional to the particle density. These rules allow one to show [5] that, when the above correction to the RPA proper polarization is taken into account, $V^{\rm tot}$ decays as $1/r^6$, the induced charge density as $1/r^8$, and the density-density correlation as $1/r^{10}$, whereas the corresponding RPA quantities decrease faster than any inverse power of the distance. (All the ladder graphs with more than two interaction lines in the proper polarization lead to faster algebraic decays.) We notice that, even at zero temperature, diagrams analogous to those exhibited in Ref. [5] are expected to generate a $1/r^6$ tail in some density-density response function of the homogenous electron gas [28,29]. Moreover, they seem to play a crucial role in the existence of fine-scale peaks in the frequency dependence of the dynamic structure factor for large fixed momentum transfers at metallic densities [30]. As a conclusion, the investigation of the standard many-body theory shows that the exchange effects, which are rather important in short-ranged phenomena, prove not to change drastically the results obtained previously [4] for the OCP in the semiclassical Maxwell-Boltzmann regime at the order $\hbar^4$.

However, these results rely on the assumption that the tails induced by some particular diagrams are not canceled by those coming from other proper polarization graphs in the perturbation series. The validity of this assumption is not obvious because the study of the large-distance behavior of the particle-particle correlation is cumbersome in the standard perturbation framework. In the latter, the basic objects are the frequency components of the proper polarization and the small-**k** behavior of the diagrams can be investigated only for isolated and very simple diagrams. Moreover, in the absence of general sum rules, as it is the case for multicomponent plasmas, no conclusion about the particle-particle correlation can be drawn from the investigation of the large-distance behavior of the proper polarization graphs. At this point the formalism of paper I proves to be more efficient.

B. New results

According to paper I, the virial expansions in the loop formalism give the following picture, after resummation of the collective effects due to the interactions between the total charges of the loops. Before averaging over the shapes of the loops, the charge-charge and charge-multipole interactions between loops are exponentially screened, while the multipole-multipole interactions are only partially screened because of the difference between the bare loop potential and the classical electrostatic potential between charged loops. Subsequently, an algebraically decaying resummed bond F_R appears. According to paper I, the contribution to the particle-particle correlation arising directly from the exchange between particles decays faster than any inverse power law (except in a phase similar to a Bose condensation, where it tends to a finite constant value plus fast-decaying corrections).

In the present paper, we show that the part of the particle-particle correlation given by configurations in which the particles at $\mathbf{r}_a$ and $\mathbf{r}_b$ belong to different exchange loops has an algebraic falloff induced by the algebraic F_R bond, the asymptotic behavior of which can be written as a series of $1/r^{\gamma}$ algebraic tails, with $\gamma \geqslant 3$. Each tail of the Mayer bond F_R involves the shapes of *both* its arguments because the residual interactions after resummation of the interactions between the total charges of the loops are essentially multipole-multipole-like. Subsequently, since the interaction bonds between the loops as well as the measure associated with the shapes of the loops are invariant under global rotation of their arguments, a mere dimensional analysis shows that the part of the particle-particle correlation where the two particles are not exchanged within the same cycle decays at least as $1/r^5$, though the loop correlation decays only as $1/r^3$. This $1/r^5$ behavior comes from the $1/r^3$ and $1/r^4$ terms in the algebraic resummed bond and may appear for the diagrams that can be written as convolution chains involving at least one algebraic bond. The harmonicity of the Coulomb potential enforces in fact a slower $1/r^8$ decay for the latter convolution chains. Eventually, the leading asymptotic behavior of the particle-particle correlation originates from a product of two functions, both of which involve at least one dipole-dipole-like interaction, and decays as $1/r^6$. We notice that the mechanism is not as simple as in the case of a classical system of point particles with fundamental $1/r^3$ dipole-dipole interactions, in which the asymptotic behavior of the particle-particle correlation is determined by the fluctuations of interactions and decays as $(1/r^3)^2$ because of rotational invariance. Besides, in the multicomponent plasma, the subleading tails of the quantum particle-particle correlation are $1/r^8$, $1/r^9$, and $1/r^{\gamma}$, with $\gamma \geqslant 10$.

We stress that the reason why the $1/r^5$ falloff allowed by the dimensional analysis proves not to appear lies in the harmonicity of the Coulomb potential. Indeed, the rotational invariance of interactions and quantum fluctuations of one particle is such that some terms with at least two derivatives of the Coulomb potential, which are *a priori* long ranged, prove to be short ranged because they involve $\Delta(1/r) = -4\pi\delta(\mathbf{r})$ once the integration over the shapes of the loops has been performed. In the case of a diagram made of one algebraic F_R bond, the shape of each root point is integrated over with a weight equal to the fast-decaying loop density; then, the $1/r^3$ and $1/r^4$ tails are canceled by rotational invariance, while the square of the Laplacian of $1/r$ is generated from the $1/r^5$ tail and the corresponding contribution decays faster than any inverse power law, as in the following example. If $G(|\mathbf{X}|)$ is invariant under rotations of $\mathbf{X}$ and is a fast-decaying function of the extent of the loop,

$$\int D(\mathbf{X}) G(|\mathbf{X}|) [\mathbf{X}]_{\mu_1} [\mathbf{X}]_{\mu_2} \partial_{\mu_1 \mu_2 \nu_1 \nu_2} \left(\frac{1}{r}\right) = \partial_{\nu_1 \nu_2} \Delta \left(\frac{1}{r}\right) \frac{1}{3} \int D(\mathbf{X}) G(|\mathbf{X}|) \mathbf{X}^2 \qquad (1.3)$$

is short ranged. (The space index μ of the vector components runs from 1 to 3 and ∂_μ denotes a partial derivative with respect to the component $[\mathbf{r}]_\mu$ of $\mathbf{r}$.) Eventually, the diagram considered decays as some kind of squared $1/r^3$ tail. The mechanism for a convolution involving algebraic bonds and algebraically decaying subdiagrams is a little more complicated than for a diagram with a single bond because the property exemplified in (1.3) involves only the leading tail of the convolution and the latter still has an algebraic decay determined by the terms that would have been only subleading without this special property. The basic mechanism is exemplified in the following equation, where a function, which decays as $1/r^4$ according to rotational invariance arguments and dimensional analysis, proves to have a faster decay because of the appearance of the Laplacian of $1/r$. If $G(\mathbf{r},\mathbf{X})$ is invariant under global rotations of $(\mathbf{r},\mathbf{X})$ and falls off at least as $1/r^6$ when r goes to infinity,

$$\int d\mathbf{r}' \int D(\mathbf{X}) G(\mathbf{r}',\mathbf{X}) [\mathbf{X}]_\mu \partial_{\mu\nu} \left(\frac{1}{|\mathbf{r}'-\mathbf{r}|}\right) \underset{r\to\infty}{\sim} \partial_\nu \Delta \left(\frac{1}{r}\right) \frac{1}{3} \int d\mathbf{r}' \int D(\mathbf{X}) G(\mathbf{r}',\mathbf{X}) (\mathbf{r}' \cdot \mathbf{X}) + O\left(\frac{1}{r^6}\right). \qquad (1.4)$$

$[O(1/r^6)$ denotes a term that decays at least as $1/r^6$.$]$ As detailed in Sec. I C, the possible $1/r^5$ asymptotic behavior of the particle-particle correlation allowed by dimensional analysis arises in convolution chains that involve $1/r^3$ of $1/r^4$ tails as first and last algebraic terms and $1/r^3$ intermediate tails (together with functions decaying at least as $1/r^6$ at both ends and in the middle of the convolution.) In Fourier space, the singularities coming from the $1/r^3$ and $1/r^4$ tails of I resummed algebraic bonds in these convolutions are

$$\prod_{i=1}^{I} [k_{\mu_i} k_{\nu_i} v_C(\mathbf{k})], \quad k_{\mu_{1,1}} k_{\mu_{1,2}} k_{\nu_1} v_C(\mathbf{k}) \prod_{i=2}^{I} [k_{\mu_i} k_{\nu_i} v_C(\mathbf{k})]. \tag{1.5}$$

Since the derivatives of the Coulomb potential are contracted with one or two space components of *both* loop shapes $\mathbf{X}_i$ and $\mathbf{X}_j$ in the $1/r^3$ and $1/r^4$ tails of the algebraic resummed bond $F_R(\mathcal{P}_i, \mathcal{P}_j)$, rotational invariance arguments show that, after integration over the shapes of the loops, the singularities (1.5) are contracted with the product of the tensor $k_{\nu_I}\Pi_{i=1}^{I-1}\delta_{\nu_i,\mu_{i+1}}$ and the tensors k_{μ_1} and $\delta_{\mu_{1,1},\mu_{1,2}}$, respectively, with a coefficient independent from $\mathbf{k}$. Subsequently, the singularities (1.5) lead to analytic contributions because $v_C(\mathbf{k}) \propto 1/\mathbf{k}^2$. [The corresponding tails in position space involve derivatives of the Laplacian $\Delta(1/r)$, which is short ranged.] Thus the algebraic decay of one of the above convolution chains is due to subleading tails and the convolution decreases faster than the inverse power that is associated by dimensional analysis with the derivatives of $1/r$ involved in its dominant large-distance behavior.

We point out that, in various formalisms, it can be understood in terms of the chain potentials, which describe the screening of the monopole-monopole interaction (see paper I), how the $1/r^3$ dipole-dipole-like interaction between charges surrounded by their polarization clouds eventually lead not to a $1/r^5$ tail of the particle-particle correlation, as the sole rotational invariance would imply, but to a $1/r^6$ tail enforced by the harmonicity of the Coulomb potential. In particular, in the approximation where only the graph with one chain-potential bond is retained, the corresponding approximated induced charge density and particle-particle correlation both decrease faster than any inverse power law of the distance because the rotational invariance makes the short-ranged Laplacian of the Coulomb potential appear. More generally, in the exact correlation function, the rotational invariance of the quantum fluctuations of the polarization cloud around a quantum charge and the harmonicity of the Coulomb potential cancel the long-ranged part of the chain interaction and only products of two convolutions, each of which involves the algebraic part of the chain potential, remain in the dominant asymptotic behavior of the particle-particle correlation. This is the general mechanism that leads to the $1/r^6$ power law of the particle-particle-correlation decay.

Moreover, the loop formalism allows one to discuss the large-distance behavior of the charge-charge correlation and induced charge density. Since the nonexchange part of the particle-particle correlation $\rho^{(2)TQ}_{\alpha\alpha'}(\mathbf{r})$ decays as $1/r^6$, a mere dimensional analysis of the linear response relation in the loop formalism [see (4.27) of paper I] shows that the induced charge density $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{r})$ in the presence of a localized infinitesimal external charge decays at least as $1/r^4$ and, according to the Poisson equation [see (6.7) of paper I], the corresponding total potential falls off at least as $1/r^2$. However, in a multicomponent plasma, the charge density $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{r})$ is expected to decay faster than the correlation between quantum particles [4]. Though there is no exact sum rule for the nonzero moments of $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{r})$ in multicomponent plasmas, the present loop formalism allows one to show that $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{r})$ rather decays as $1/r^8$ (and the corresponding total potential as $1/r^6$), while the quantum charge-charge structure factor $C^Q(\mathbf{r})$ decays as $1/r^{10}$. On one hand, the difference between the latter two exponents is mainly a consequence of the response relation (4.32) of paper I. Roughly speaking, the change from a $|\mathbf{k}|^7$ singularity in the Fourier transform of $C^Q(\mathbf{k})$ to a $|\mathbf{k}|^5$ singularity in $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{k})$ originates from the fact that $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{k})$ involves a quantity very similar to $C^Q(\mathbf{k})$ divided by $\mathbf{k}^2$. On the other hand, the fall from a $1/r^6$ decay for the particle-particle correlation $\rho^{(2)TQ}_{\alpha\alpha'}(\mathbf{r})$ to a $1/r^{10}$ large-distance behavior for the charge-charge correlation $\Sigma_\alpha e_\alpha \Sigma_{\alpha'} e_{\alpha'} \rho^{(2)TQ}_{\alpha\alpha'}(\mathbf{r})$ is determined by two mechanisms. The first one is very similar to the mechanism that ensures that, in the classical regime, the exact charge density of the system made of a particle and its polarization cloud, as well as its Debye expression, both have Fourier transforms that behave as $|\mathbf{k}|^2$ when $|\mathbf{k}|$ goes to zero. In other words, in both cases, the total induced charge around an infinitesimal external charge distribution is exactly opposite it. [The Debye model amounts to approximating the classical Ursell function $h^{\text{cl}}_{\alpha_a\alpha_b}(\mathbf{k})$ by $-\beta e_{\alpha_a} e_{\alpha_b} \phi_{\text{DH}}(\mathbf{k})$.] This macroscopic classical Debye screening still operates in the quantum system, where it is described by the resummed charge-charge bond F^{cc}. The second mechanism is that the large-distance ''diffraction-like'' quantum effects described by the resummed multipole-charge bonds F^{mc} and F^{cm} also partially screen the quantum algebraic tail. Moreover, the *total* induced charge is completely determined by the diagram with only one charge-charge resummed bond F^{cc}: the screening of macroscopic electrostatics is not changed by the departure of the quantum microscopic correlations from the classical exponential clustering.

C. Contents

The discussion of the exponent of the algebraic decay of the part of the quantum particle-particle correlation coming from the loop correlation is organized as follows. The algebraic tail of the F_R bond is equal to $\exp(W)-1$, where W is a series of algebraic terms W_γ ($\gamma \geq 3$, where γ is an integer), each of which decays as $1/r^\gamma$. In Sec. II A we stress the particularity of the discussion that comes from the nonintegrability of the algebraic resummed bond F_R and the basic peculiar mechanism due to the harmonicity of the Coulomb potential is exemplified on the simplest diagram. In Sec. II B the diagrammatics are reorganized in order to display a contribution H and convolution chains $\mathfrak{C}$ made of bonds W joined directly together or linked by graphs H. In Sec. II C we give the principles of the technical discussion in Fourier space that allow one to determine the exponent of an algebraic decay in position space.

The graph H decays as $1/r_{ab}^6$ (Sec. III A) and a possible slower decay of the particle-particle correlation can arise only from the convolutions involving H and bonds W. However, the latter convolution chains $\mathfrak{C}$ prove to decrease faster than $1/r^6$. Indeed, we show in Sec. III B that, after integration over the shapes of the loops, rotational invariance arguments determine the order of the first two terms in the small-$\mathbf{k}$ expansion of a chain $\mathfrak{C}$ and, because of the harmonicity of the Coulomb potential, the first term proves to be analytic. This mere dimensional analysis shows that $1/r^7$ is an upper bound for the algebraic large-distance behavior of a convolution $\mathfrak{C}$. The translation of the discussion in position space (see Appendix A) is that, after averaging over the shapes of the loops, rotational invariance arguments and dimensional analysis enforce a falloff faster than $1/r^6$ for all chains $\mathfrak{C}$, except those with a bond W_3 or W_4 at both ends and only W_3 intermediate bonds, which may decrease *a priori* only as $1/r^5$. However, by virtue of the specific forms of W_3 and W_4 in terms of the derivatives of the Coulomb potential, they decay at least as $1/r^7$ because $\Delta(1/r)=-4\pi\delta(\mathbf{r})$. The general diagrammatic structure of the $1/r^6$ tail of the particle-particle correlation is given in Sec. III C. In Sec. III D (and Appendix B), we give the general tensorial structure of the algebraic tails of a diagram in H before integration over the shapes of its end points. The allowed exponents of the various tails of H and of the convolutions $\mathfrak{C}$ are discussed in Secs. III E and III F (and Appendix C). H generates $1/r^6, 1/r^8, 1/r^9,\ldots$ tails, while the convolutions $\mathfrak{C}$ only contribute to $1/r^8, 1/r^{10}, 1/r^{11},\ldots$ tails. Eventually, the algebraic tails of the particle-particle correlation are $1/r^6$, $1/r^8$, $1/r^\gamma$ with γ any (even or odd) integer greater than or equal to 9 (and there are no logarithmic corrections).

In Sec. IV A we show that, at a classical level, the fact that the exact total induced charge around an infinitesimal external charge $\delta q(\mathbf{r})$ is equal to $-\delta q(\mathbf{r})$ can be linked to the fact that its Debye approximation also satisfies this property. The link lies in a suitable reorganization of the classical resummed diagrammatics, which exhibits the ''dressing'' of the particle-particle correlation by the Debye polarization cloud around a charge of the medium. (This structure imposes a constraint on the construction of coherent approximations in the diagrammatic framework.) In Sec. IV B we point out the basic mechanisms that operate in the loop system. In Sec. IV C we introduce integral relations for the loop Ursell functions, which exhibit the dressing by some kind of ''Debye'' loop polarization cloud. These relations are used to reorganize the diagrams of the loop Ursell function (Sec. IV D) and a discussion similar to that performed for the particle-particle correlation shows that the charge-charge correlation decays at least as $1/r^{10}$ (see also Appendixes C and D). A slightly different reorganization (Sec. IV E) allows one to show that the induced charge density falls off as $1/r^8$. In Sec. IV F we give a diagrammatic expression for the second moment of the charge-charge correlation (which involves the diagram with only one F^{cc} bond together with other diagrams). However, the F^{cc} bond proves to contribute by itself to the total induced charge and we mention the corresponding constraints that are required to build approximations (in the loop formalism) that satisfy the screening sum rule (1.3) of paper I.

As a conclusion (Sec. V), we compare the screening mechanisms in various models. In Sec. V A, we investigate the fast decay of the quantum particle-particle correlation and induced charge density in the chain approximations and particularly in the version of the RPA model written in the standard perturbation many-body theory. The various screening lengths defined from the large-distance behavior of the induced charge density are compared in Sec. V B. In order to see how the algebraic tail of the chain potential eventually pollutes the correlations in various formalisms, one has to take into account diagrams with at least two chain potentials (Sec. V C). Eventually, we recall a very simple model that exhibits the basic mechanisms involved in papers I and II.

II. ALGEBRAIC DECAY IN THE LOOP FORMALISM

A. Basic mechanisms

As shown in paper I, the resummed Mayer diagrams for the loop-density expansion of the loop correlation involve both exponentially decaying bonds F^{cc}, F^{cm}, and F^{mc} and an algebraic bond F_R. The asymptotic behavior of the F_R bond is given by the expansion of $\exp(W)-1$ and its leading term behaves as $1/r^3$, which is the borderline of integrability. Henceforth, the asymptotic behavior of the loop correlation $\rho^{(2)T}(\mathscr{L},\mathscr{L}')$ is not given by the result for a classical fluid of noncharged particles interacting by a potential Φ with an integrable power-law decay [31]. In the case of such a one-component fluid, the correlation behaves as

$$\rho^{(2)T\ \mathrm{cl}}(r_{ab}) \underset{r_{ab}\to\infty}{\sim} \int d\mathbf{r}_1 \int d\mathbf{r}_2 S^{\mathrm{cl}}(\mathbf{r}_a,\mathbf{r}_1) f_{\mathrm{as}}(\mathbf{r}_1,\mathbf{r}_2) \times S^{\mathrm{cl}}(\mathbf{r}_2,\mathbf{r}_b)$$

$$\underset{r_{ab}\to\infty}{\sim} -\beta\Phi(r_{ab})\left[\frac{\rho^2\chi_T}{\beta}\right]^2, \tag{2.1}$$

where $f_{\mathrm{as}}=-\beta\Phi$ is the asymptotic behavior of the Mayer bond $f=\exp(-\beta\Phi)-1$. The classical structure factor $S^{\mathrm{cl}}(\mathbf{r}_i,\mathbf{r}_j)\equiv\rho\,\delta(\mathbf{r}_i)+\rho^{(2)T\ \mathrm{cl}}(\mathbf{r}_i,\mathbf{r}_j)$, which is equal to the classical density-density response function, obeys the Ornstein-Zernicke relations (2.10) and (2.11) of paper I, which are valid for either quantum or classical quantities. However, since the resummed diagrams of the loop-density expansion are integrable, the loop correlation is expected to decay as $1/r^3$, while the power law of the asymptotic behavior of the quantum correlation may be greater because of the integration over the internal degrees of freedom of the loops. The first algebraic term in the $\hbar$ expansion of the large-distance quantum correlation in a OCP is given by a formula that is analogous to (2.1) [5], but involves the square of some kind of effective potential instead of the bare potential. Moreover, in the special latter case, according to (1.2) of paper I, $\int d\mathbf{r}\, S^{\mathrm{cl}}(\mathbf{r})=0$ and the above term involves in fact derivatives of the squared effective potential, so that the correlation eventually proves to have a $1/r^{10}$ falloff.

As an introduction to the following discussion, we study the asymptotic behavior of the Π diagram with only one F_R bond. The algebraic tail of F_R can be written as the sum of W and $\exp(W)-1-W$. It can be readily shown that, after integration with the weight $D(\mathbf{X}_a)D(\mathbf{X}_b)\rho_{\alpha_a,p_a}(\mathbf{X}_b)\rho_{\alpha_b,p_b}(\mathbf{X}_b)$, W gives short-ranged contributions to the asymptotic behavior of the diagram and the dominant large-distance decay of the latter is in fact given by the $1/r^6$ leading term in the algebraic tail $\frac{1}{2}(W)^2$. Indeed, according to paper I, $\rho_{\alpha,p}(\mathbf{X})$ is a fast-decaying function of $|\mathbf{X}|$, every moment of $\rho_{\alpha,p}(\mathbf{X})$ is finite, and Eqs. (5.33)–(5.35) of paper I allow to write the Fourier transform of the contribution from W to the particle-particle correlation as a series (over p_a, p_b, m, and n; $m\geqslant 1$, $n\geqslant 1$), in which each term is proportional to

$$\int_0^{p_a} d\tau \int_0^{p_b} d\tau' \{\delta([\tau - P(\tau)] - [\tau' - P(\tau')]) - 1\} \times \frac{1}{\mathbf{k}^2} \left(\int D(\mathbf{X}_a) \rho_{\alpha_a, p_a}(\mathbf{X}_a) [\mathbf{k} \cdot \mathbf{X}_a(\tau)]^m \right) \times \left(\int D(\mathbf{X}_b) \rho_{\alpha_b, p_b}(\mathbf{X}_b) [\mathbf{k} \cdot \mathbf{X}_b(\tau')]^n \right). \quad (2.2)$$

Let X_μ be the component of $\mathbf{X}$ with space index μ ($\mu = 1,...,3$). Because of the rotational invariance of $D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})$, the moment $\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X}) X_{\mu_1} \cdots X_{\mu_m}$ vanishes if m is odd; if m is even ($m = 2m' \geqslant 2$) $\int D(\mathbf{X})\rho_{\alpha,p}(\mathbf{X})(\mathbf{k}\cdot\mathbf{X})^{2m'}$ is proportional to $(\mathbf{k}^2)^{m'}$ times a function independent from the orientation of $\mathbf{k}$. Since $m' \geqslant 1$, the singularity $1/\mathbf{k}^2$ is canceled and the term (2.2) is in fact analytic. In position space, this means that powers of the Laplacian of $1/r$ appear in the asymptotic behavior of $\int D(\mathbf{X}_a) \int D(\mathbf{X}_b) \rho_{\alpha_a,p_a}(\mathbf{X}_b) \rho_{\alpha_b,p_b}(\mathbf{X}_b) W(\mathcal{L}_a, \mathcal{L}_b)$ and the corresponding terms are in fact short ranged. The mechanism is similar to what happens to the contribution of the analog of W in the $\hbar$ expansion of the Maxwell-Boltzmann correlations in terms of the classical correlations [4]; in this latter case, all the classical distribution functions are fast-decaying functions and they play the role of the weight $\rho_{\alpha,p}(\mathbf{X})$. Eventually, the dominant asymptotic behavior of the diagram with one F_R bond is given by the leading term in $\exp(W) - 1 - W, \frac{1}{2} W_3^2$, and reads

$$\sum_{p_a} p_a \int D(\mathbf{X}_a) \rho_{\alpha_a,p_a}(\mathbf{X}_a) \sum_{p_b} p_b \int D(\mathbf{X}_b) \rho_{\alpha_b,p_b}(\mathbf{X}_b)\ \tfrac{1}{2}[W_3(\mathbf{r}_{ab}, \mathbf{X}_a, \mathbf{X}_b; \alpha_a, p_a; \alpha_b, p_b)]^2$$
$$= \frac{\beta^2 e_{\alpha_a}^2 e_{\alpha_b}^2}{2} \sum_{\mu,\nu} \left[\partial_{\mu\nu} \left(\frac{1}{r_{ab}} \right) \right]^2 \sum_{p_a} p_a \sum_{p_b} p_b \int_0^{p_a} d\tau_1 \int_0^{p_b} d\tau_1' \{\delta([\tau_1 - P(\tau_1)] - [\tau_1' - P(\tau_1')]) - 1\}$$
$$\times \int_0^{p_a} d\tau_2 \int_0^{p_b} d\tau_2' \{\delta([\tau_2 - P(\tau_2)] - [\tau_2' - P(\tau_2')]) - 1\} \int D(\mathbf{X}_a) \rho_{\alpha_a,p_a}(\mathbf{X}_a) \tfrac{1}{3} [\mathbf{X}_a(\tau_1) \cdot \mathbf{X}_a(\tau_2)]$$
$$\times \int D(\mathbf{X}_b) \rho_{\alpha_b,p_b}(\mathbf{X}_b) \tfrac{1}{3} [\mathbf{X}_b(\tau_1') \cdot \mathbf{X}_b(\tau_2')]. \quad (2.3)$$

B. Topological reorganization of the diagrams

Our aim is to analyze the influence of the nonintegrable algebraic decay F_R on the large-distance behavior of the diagrams after integration over the shapes of the loops and to show that the large-distance contributions that involve only one W or only one convolution involving at least one W decay faster than $1/r^6$. For this purpose, we split F_R into

$$F_R = W + F_{R6}. \quad (2.4)$$

The dominant asymptotic behavior of $W(\mathbf{r}, \chi_i, \chi_j)$ up to order $1/r^6$ is equal to $W_3 + W_4 + W_5 + W_6$ and that of F_{R6} is $\frac{1}{2} W_3^2$. We introduce the $\tilde{\Pi}$ diagrams that have the same topological definition as the Π diagrams, with the only difference that the bond F_R is now replaced by either a W or an F_{R6} bond. The $\tilde{F}$ bonds in the $\tilde{\Pi}$ diagrams are equal to either F^{cc}, F^{cm}, F^{mc}, W, or F_{R6}. Since F_R is just the sum of W and F_{R6}, we get an identity analogous to Eq. (5.9) of paper I,

$$h(\mathcal{L}_a, \mathcal{L}_b) = \sum_{\tilde{\Pi}} \frac{1}{S_{\tilde{\Pi}}} \int \prod_{m=1}^{M} d\mathcal{P}_m \rho(\mathcal{P}_m) \left[\prod \tilde{F} \right]_{\tilde{\Pi}}. \quad (2.5)$$

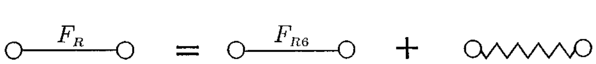

FIG. 1. Diagrammatic representation of the decomposition (2.4) of an F bond into two auxiliary $\tilde{F}$ bonds. The F_{R6} bond is denoted by a solid line with a superscript F_{R6} and the W bond is represented by a serrated line.

For instance, the Π diagram with only one bond F_R leads to two $\tilde{\Pi}$ diagrams, as shown in Fig. 1. The $\tilde{\Pi}$ diagrams are integrable at large distances, as the Π diagrams (see paper I). They may be not integrable at short distances because of products of the $\tilde{F}$ bond. However, this does not matter because this nonintegrability does not interplay with the large-distance behavior of the diagrams and, further, this is an artifact that disappears when the $\tilde{\Pi}$ diagrams are collected together properly, as in the case of Π diagrams [32,33].

The next step of the general discussion is to exhibit the subclass of the so-called $\tilde{\Pi}_{Wc}$ diagrams defined as the $\tilde{\Pi}$ diagrams that remain connected when an insertion W is removed. Let $H(\mathbf{r}_{ab}, \chi_i, \chi_j)$ denote both the sum of the $\tilde{\Pi}_{Wc}$ diagrams and the graph associated with it. According to the topological definition of the $\tilde{\Pi}_{Wc}$ diagrams, $h(\mathcal{L}_a, \mathcal{L}_b)$ can be reexpressed by an exact Dyson equation in terms of convolution chains involving both H and W. Let $\chi = (\alpha, p, \mathbf{X})$ be a global notation for the internal degrees of freedom of a

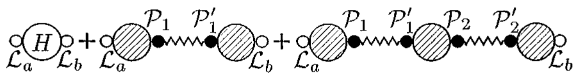

FIG. 2. Diagrammatic representation of the Dyson equation (2.6). The small white disks correspond to $\mathscr{L}_a$ and $\mathscr{L}_b$ and the black points to the internal points χ and χ'. The big white disk stands for H and the hachured disks represent K.

loop, with the associated measure $\int d\chi = \Sigma_{\alpha=1}^{n_s}\Sigma_{p=1}^{\infty}\int D(\mathbf{X})$. In Fourier space, the relation reads

$$h(\mathbf{k},\chi_a,\chi_b) = H(\mathbf{k},\chi_a,\chi_b) + \sum_{I=1}^{\infty}\int d\chi_1\cdots d\chi_I d\chi_1'\cdots d\chi_I' \times K(\mathbf{k},\chi_a,\chi_1)W(\mathbf{k},\chi_1,\chi_1')K(\mathbf{k},\chi_1',\chi_2)\cdots \times W(\mathbf{k},\chi_I,\chi_I')K(\mathbf{k},\chi_I',\chi_b). \tag{2.6}$$

For brevity, we have introduced

$$K(\mathbf{R}_{ij},\chi_i,\chi_j) \equiv \delta(\mathbf{R}_i - \mathbf{R}_j)\,\delta_{\chi_i,\chi_j}\rho(\chi_i) + H(\mathbf{R}_{ij},\chi_i,\chi_j), \tag{2.7}$$

with $\delta_{\chi_i,\chi_j} \equiv \delta_{\alpha_i,\alpha_j}\delta_{p_i,p_j}\delta(\mathbf{X}_i - \mathbf{X}_j)$ and $\rho(\chi_i) \equiv \rho_{\alpha_i,p_i}(\mathbf{X}_i)$. The representation of (2.6) in terms of the graph H and a sum of chains made with graphs K linked by I, W bonds is shown in Fig. 2.

C. Characterization of an algebraic tail

The analysis of the algebraic decay of the graphs is performed according to the following principles. First, only a finite number of the moments of a function with an algebraic asymptotic decay are well defined and the dominant term in the asymptotic behavior of a function $g(\mathbf{r})$ decreases as $1/r^\gamma$ or $(\ln r)^n/r^\gamma$, with γ and n nonzero positive integers, if and only if its Fourier transform $g(\mathbf{k}) \equiv \int d\mathbf{r}\exp(i\mathbf{k}\cdot\mathbf{r})g(\mathbf{r})$ has a small-$\mathbf{k}$ expansion that becomes nonanalytic with respect to the components of the vector $\mathbf{k}$ at the order $|\mathbf{k}|^{\gamma-3}$. The *first* nonanalytic term $S^{(\gamma-3,n)}(\mathbf{k})$ corresponds to the divergent moment of $g(\mathbf{r})$ with the lowest order and is given by the theory of distributions [34]. For instance, first nonanalytic terms such as $k_\mu k_\nu/\mathbf{k}^2$ and $\ln|\mathbf{k}|$ correspond to $1/r^3$ decays, a dominant term $(\ln|\mathbf{k}|)^2$ signals a $(\ln r)/r^3$ falloff, while first nonanalytic terms such as $|\mathbf{k}|$ and $|\mathbf{k}|^2\ln|\mathbf{k}|$ originate from $1/r^4$ and $1/r^5$ decays, respectively. Second, the large-distance behavior of the convolution of several functions with algebraic decays is more conveniently studied in Fourier representation than in position space. The reason is that a convolution of various functions $g(\mathbf{r})$ is changed into a product in Fourier space and, according to the previous argument, its dominant large-distance behavior is merely deduced from the first nonanalytic term in the expansion of the product of the small-$\mathbf{k}$ behaviors of the Fourier transforms $g(\mathbf{k})$. These principles are further detailed in the present section.

In the generic case, according to the theory of distributions, in a position space of dimension 3, if a function, which is regular at short distances, decays algebraically at large distances with a *leading* term $g_{\rm as}(\mathbf{r}) = A(\ln r)^n/r^\gamma$, with $n \geqslant 0$, the small-$\mathbf{k}$ expansion of its Fourier transform contains a leading singular term $S^{(\gamma-3,n)}(\mathbf{k})$ of order $|\mathbf{k}|^{\gamma-3}(\ln|\mathbf{k}|)^{n+1}$ if $\gamma-3$ is even and $|\mathbf{k}|^{\gamma-3}(\ln|\mathbf{k}|)^n$ if $\gamma-3$ is odd. [As discussed below, if $g_{as}(\mathbf{r})$ is a partial derivative of $1/r$, $S^{(\gamma-3,n=0)}(\mathbf{k})$ is a nonanalytic term of order $|\mathbf{k}|^{\gamma-3}$ without any logarithm, even if $\gamma-3$ is even.] The singular terms of greater order than $S^{(\gamma-3,n)}(\mathbf{k})$ are nonanalytic terms $S^{(\gamma-3,n')}(\mathbf{k})$ with $n' < n$ and singular terms of higher order in $|\mathbf{k}|$ that all depend on the subleading terms in the large-distance behavior of $g(\mathbf{r}) - g_{as}(\mathbf{r})$. More precisely, the dominant behavior $g_{as}(r) = A(\ln r)^n/r^\gamma$ is a purely algebraic function and, in a position space of dimension 3, this function is not integrable at the origin if $\gamma \geqslant 3$. Thus we have to consider the corresponding distribution $g_{\rm alg,reg}$ that is regularized at short distances and the Fourier transform of the latter reads

$$g_{\rm as,reg}(\mathbf{k}) = \alpha_{\rm reg}^{(0)}|\mathbf{k}|^{2m} + \sum_{n'=1}^{n}\alpha_{\rm reg}^{(n')}|\mathbf{k}|^{2m}(\ln|\mathbf{k}|)^{n'} + A \times c_{3,2m,n}|\mathbf{k}|^{2m}(\ln|\mathbf{k}|)^{n+1} \tag{2.8}$$

if $\gamma-3=2m$, while

$$g_{\rm as,reg}(\mathbf{k}) = \widetilde{\alpha}_{\rm reg}^{(0)}|\mathbf{k}|^{2m+1} + \sum_{n'=1}^{n-1}\widetilde{\alpha}_{\rm reg}^{(n')}|\mathbf{k}|^{2m+1}(\ln|\mathbf{k}|)^{n'} + A \times \widetilde{c}_{3,2m+1,n}|\mathbf{k}|^{2m+1}(\ln|\mathbf{k}|)^{n} \tag{2.9}$$

if $\gamma-3=2m+1$. In these cases, $S^{(\gamma-3,n)}(\mathbf{k})$ is equal to the greatest term in (2.8) and (2.9). The coefficients $\alpha_{\rm reg}^{(n')}$ and $\widetilde{\alpha}_{\rm reg}^{(n')}$ depend on the regularization at short distances, whereas $c_{3,2m,n}$ and $\widetilde{c}_{3,2m+1,n}$ depend only on the dimension of space (here 3) and on the powers γ and n. For instance, the Fourier transform of $(1/r^3)_{\rm reg}$ is equal to $4\pi\ln|\mathbf{k}|$ plus a constant term that depends on the regularization at short distances. According to (2.8) and (2.9), if the dominant asymptotic behavior of $g(\mathbf{r})$ at large distances is $g_{\rm as}(\mathbf{r}) = (\ln r)^n/r^3$, while $[g - g_{\rm as,reg}](\mathbf{r})$ decays faster than $1/r^3$, then $g_{\rm as,reg}(\mathbf{k})$ is of order zero in $|\mathbf{k}|$ and $\widetilde{A}_{g,\rm reg} \equiv \int d\mathbf{r}[g(\mathbf{r}) - g_{\rm as,reg}(\mathbf{r})]$ is finite (if there is no short-distance nonintegrability). In this case,

$$g(\mathbf{k}) = \widetilde{A}_{g,\rm reg} + g_{\rm as,reg}(\mathbf{k}) + O(|\mathbf{k}|), \tag{2.10}$$

where $O(|\mathbf{k}|) = \int d\mathbf{r}[\exp(i\mathbf{k}\cdot\mathbf{r}) - 1][g - g_{\rm as,reg}](\mathbf{r})$ starts at the order $|\mathbf{k}|$. If $g_{\rm as}(\mathbf{r}) = (\ln r)^n/r^\gamma$ with $\gamma \geqslant 4$, the Fourier transform of $g(\mathbf{r})$ reads

$$g(\mathbf{k}) = \sum_{p=0}^{\gamma-4}\int d\mathbf{r}(i\mathbf{k}\cdot\mathbf{r})^p[g(\mathbf{r}) - g_{\rm as,reg}(\mathbf{r})] + \int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}}g_{\rm as,reg}(\mathbf{r}) + \int d\mathbf{r}\left[e^{i\mathbf{k}\cdot\mathbf{r}} - \sum_{p=0}^{\gamma-4}(i\mathbf{k}\cdot\mathbf{r})^p\right] \times [g(\mathbf{r}) - g_{\rm as,reg}(\mathbf{r})]. \tag{2.11}$$

When g and $g_{\rm as,reg}$ are sufficiently regular at short distances, the first term is a sum of analytic terms. Since $[g - g_{\rm as,reg}](\mathbf{r})$ decays at least as $(\ln r)^{n-1}/r^\gamma$, the second term contains $S^{(\gamma-3,n)}(\mathbf{k})$ plus terms of order $|\mathbf{k}|^{\gamma-3}$ times a possible $(\ln|\mathbf{k}|)^{n'}$ (with $n' \leqslant n-1$), while the third term contains only (nonanalytic and analytic) terms that are of order greater than that of $S^{(\gamma-3,n)}(\mathbf{k})$. If $n=0$, as will be the case in the following, (2.11) implies that

$$g(\mathbf{k})=[g-g_{\text{as,reg}}](\mathbf{k}=\mathbf{0})+\sum_{p=1}^{\gamma-4}k_{\mu_1}\cdots k_{\mu_p}\frac{\partial^p[g-g_{\text{as,reg}}]}{\partial k_{\mu_1}\cdots\partial k_{\mu_p}}(\mathbf{k}=\mathbf{0})+S^{(\gamma-3,0)}(\mathbf{k})+\widetilde{O}(S^{(\gamma-3,0)}(\mathbf{k}))\tag{2.12}$$

where $\widetilde{O}(S^{(\gamma-3,0)}(\mathbf{k}))$ denotes (analytic or nonanalytic) terms of order greater than that of $S^{(\gamma-3,0)}(\mathbf{k})$.

In the case of the algebraic tails of the $\widetilde{\Pi}$ diagrams (see Secs. III D and III E), there appears no $(\ln r)^n/r^\gamma$ with $n>0$, while there proves to be no need of short-distance regularization for $g_{\text{as}}(\mathbf{r})$. Indeed, if $g_{\text{as}}(\mathbf{r})=\partial_{\mu\nu}(1/r)$, then the singular term is $k_\mu k_\nu$ times the nonanalytic term in the Fourier transform of $1/r$, which reduces to $4\pi/\mathbf{k}^2$ (without any $\ln|\mathbf{k}|$); moreover, the Fourier transform of $1/r$ does not need any short-distance regularization, while its derivatives become integrable after integration over the orientation of $\mathbf{r}$ because only powers of the Laplacian of $1/r$ survive and the latter is short ranged. For instance, the Fourier transform of every purely algebraic term W_γ, defined in (5.33) of paper I and which involves only derivatives of $1/r$, reduces to one term $S^{(\gamma-3,0)}(\mathbf{k})$ of order $|\mathbf{k}|^{\gamma-3}$ and without any $\ln|\mathbf{k}|$, and $\int d\mathbf{r}\, W_\gamma(\mathbf{r},\chi_i,\chi_i')$ is conditionally convergent at short distances. According to the property (5.35) of paper I, the Fourier transforms of W_3 and W_4, respectively, read

$$W_3(\mathbf{k},\chi_i,\chi_i')=\beta e_{\alpha_i}e_{\alpha_i'}\int_0^{p_i}d\tau_i\int_0^{p_i'}d\tau_i'\times\{\delta([\tau_i-P(\tau_i)]-[\tau_i'-P(\tau_i')])-1\}4\pi\times\frac{[\mathbf{X}_i(\tau_i)\cdot\mathbf{k}][\mathbf{X}_i'(\tau_i')\cdot\mathbf{k}]}{\mathbf{k}^2}\tag{2.13}$$

and

$$W_4(\mathbf{k},\chi_i,\chi_i')=\beta e_{\alpha_i}e_{\alpha_i'}\int_0^{p_i}d\tau_i\int_0^{p_i'}d\tau_i'\{\delta([\tau_i-P(\tau_i)]-[\tau_i'-P(\tau_i')])-1\}\times4\pi\,\frac{[\mathbf{X}_i(\tau_i)\cdot\mathbf{k}]^2[\mathbf{X}_i'(\tau_i')\cdot\mathbf{k}]-[\mathbf{X}_i(\tau_i)\cdot\mathbf{k}][\mathbf{X}_i'(\tau_i')\cdot\mathbf{k}]^2}{2\mathbf{k}^2}.\tag{2.14}$$

Since F_R-W decays as $\frac{1}{2}[W_3(\mathbf{r},\chi_i,\chi_i')]^2$, plus faster tails, according to (2.11),

$$[F_R-W](\mathbf{k},\chi_i,\chi_i')=A(\chi_i,\chi_i')+A_\mu^{(1)}(\chi_i,\chi_i')k_\mu+A_{\mu\nu}^{(2)}(\chi_i,\chi_i')k_\mu k_\nu+A_{\mu\nu\sigma}^{(3)}(\chi_i,\chi_i')k_\mu k_\nu k_\sigma+\int d\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}\tfrac{1}{2}[W_3(\mathbf{r},\chi_i,\chi_i')]^2+O(|\mathbf{k}|^4),\tag{2.15}$$

where

$$A(\chi_i,\chi_i')\equiv\int d\mathbf{r}[F_R-W-\tfrac{1}{2}W_3^2](\mathbf{r},\chi_i,\chi_i'),\tag{2.16}$$

$A_\mu^{(1)}(\chi_i,\chi_i')=i\int d\mathbf{x}x_\mu[F_R-W-(1/2)W_3^2](\mathbf{x},\chi_i,\chi_j)$, and so on. k_μ (x_μ) is the component of the vector $\mathbf{k}$ ($\mathbf{x}$) with space index μ. $O(|\mathbf{k}|^\gamma)$ denotes a function starting at the order $|\mathbf{k}|^\gamma$.

Subsequently, the large-distance behavior of convolutions can be determined precisely. The asymptotic behavior of the convolution $[g_1*g_2](\mathbf{r})$ is dominated by the inverse Fourier transform of the lowest-order nonvanishing singular term in $g_1(\mathbf{k})g_2(\mathbf{k})$. We distinguish the cases where g_1 and g_2 decay with different inverse power laws from the cases where they decrease algebraically with the same exponent. Moreover, we assume the following properties. If $g(\mathbf{r})$ decays faster than $1/r^3$, then $\int d\mathbf{r}\,g(\mathbf{r})$ is supposed to be finite (there is no singularity at short distances), so that its Fourier transform is given by (2.12) with g_{as} in place of $g_{\text{as,reg}}$. If $g(\mathbf{r})$ decays as $1/r^3$, when $g_{\text{as}}(\mathbf{k})$ is a pure function of $\mathbf{k}/|\mathbf{k}|$ with no constant term, we assume that $A_g\equiv\int d\mathbf{r}[g(\mathbf{r})-g_{\text{as}}(\mathbf{r})]$ is finite [the integral is absolutely (conditionally) convergent at large (short) distances], and when $g_{\text{as,reg}}(\mathbf{k})=\widetilde{\alpha}_{\text{reg}}^{(0)}+A\times4\pi\ln|\mathbf{k}|$, we assume that $A_g\equiv\int d\mathbf{r}[g(\mathbf{r})-g_{\text{as,reg}}(\mathbf{r})]+\widetilde{\alpha}_{\text{reg}}^{(0)}$ is finite; then (2.10) becomes

$$g(\mathbf{k})=A_g+S_g^{(0,0)}(\mathbf{k})+O(|\mathbf{k}|).\tag{2.17}$$

If $g_1(\mathbf{r})$ decays as $1/r^{\gamma_1}$ and $g_2(\mathbf{r})$ as $1/r^{\gamma_2}$, with $3\leqslant\gamma_1<\gamma_2$, then, according to (2.12) or (2.17), the first singular term in $g_1(\mathbf{k})g_2(\mathbf{k})$ is $g_2(\mathbf{k}=\mathbf{0})S_{g_1}^{(\gamma_1-3)}(\mathbf{k})$ [if $g_2(\mathbf{k}=\mathbf{0})\neq0$] and g_1*g_2 decays as the slowest of the two functions g_1 and g_2:

$$[g_1*g_2](\mathbf{r})\underset{r\to\infty}{\sim}S_{g_1}^{(\gamma_1)}(\mathbf{r})\left(\int d\mathbf{x}\,g_2(\mathbf{x})\right)+o\left(\frac{1}{r^{\gamma_1}}\right),\tag{2.18}$$

where $o(1/r^{\gamma_1})$ denotes a term that decays faster than $1/r^{\gamma_1}$. Moreover, if $\gamma_1+1<\gamma_2$ and the first subleading term in the large-distance behavior of g_1 falls off as $1/r^{\gamma_1+1}$, then the first subleading asymptotic behavior of g_1*g_2 is the inverse Fourier transform of $g_2(\mathbf{k}=\mathbf{0})S_{g_1}^{(\gamma_1-2)}(\mathbf{k})+k_\mu(\partial g_2/\partial k_\mu)(\mathbf{k}=\mathbf{0})S_{g_1}^{(\gamma_1-3)}(\mathbf{k})$, namely,

$$[g_1*g_2](\mathbf{r})-S_{g_1}^{(\gamma_1)}(\mathbf{r})\left(\int d\mathbf{x}\, g_2(\mathbf{x})\right)$$

$$\underset{r\to\infty}{\sim} S_{g_1}^{(\gamma_1+1)}(\mathbf{r})\left(\int d\mathbf{x}\, g_2(\mathbf{x})\right)+\partial_\sigma[S_{g_1}^{(\gamma_1)}(\mathbf{r})]$$
$$\times\left(\int d\mathbf{x}\, x_\sigma g_2(\mathbf{x})\right)+o\left(\frac{1}{r^{\gamma_1+1}}\right). \tag{2.19}$$

If $g_1(\mathbf{r})$ and $g_2(\mathbf{r})$ decay with the same power law $1/r^\gamma$, with $\gamma>3$, $S_{g_1}^{(\gamma)}(\mathbf{k})$ and $S_{g_2}^{(\gamma)}(\mathbf{k})$ are of the same order in $|\mathbf{k}|$ and the asymptotic behavior of g_1*g_2 is given by the sum of the two terms $g_1(\mathbf{k}=\mathbf{0})S_{g_2}^{(\gamma-3)}(\mathbf{k})$ and $g_2(\mathbf{k}=\mathbf{0})S_{g_1}^{(\gamma-3)}(\mathbf{k})$,

$$[g_1*g_2](\mathbf{r})\underset{r\to\infty}{\sim}\left(\int d\mathbf{x}\, g_1(\mathbf{x})\right)S_{g_2}^{(\gamma)}(\mathbf{r})+\left(\int d\mathbf{x}\, g_2(\mathbf{x})\right)$$
$$\times S_{g_1}^{(\gamma)}(\mathbf{r})+o\left(\frac{1}{r^\gamma}\right). \tag{2.20}$$

If g_1 and g_2 are proportional to $1/r^3$ at large distances, then, with the notations of (2.17), $S_{g_i}^{(0,0)}(\mathbf{k})=c_i\times 4\pi\,\ln|\mathbf{k}|$ and g_1*g_2 behaves as the inverse Fourier transform of the zeroth order terms $A_{g_1}S_{g_2}^{(0,0)}(k)+A_{g_2}S_{g_1}^{(0,0)}(\mathbf{k})+S_{g_1}^{(0,0)}(\mathbf{k})S_{g_2}^{(0,0)}(\mathbf{k})$, namely,

$$[g_1*g_2](\mathbf{r})\underset{r\to\infty}{\sim}A_{g_2}S_{g_1}^{(3,0)}(\mathbf{r})+A_{g_1}S_{g_2}^{(3,0)}(\mathbf{r})$$
$$+[S_{g_1}^{(3,0)}*S_{g_2}^{(3,0)}](\mathbf{r}). \tag{2.21}$$

$[S_{g_1}^{(3,0)}*S_{g_2}^{(3,0)}](\mathbf{r})$ is the inverse Fourier transform of a term $(\ln|\mathbf{k}|)^2$ and it behaves as $(\ln r)/r^3$. However, in the following discussion, the singular terms of order zero in $|\mathbf{k}|$ are of the form $S_g^{(0,0)}(\mathbf{k})=k_{\mu_1}\cdots k_{\mu_p}/\mathbf{k}^2$, which contains no logarithm, so that $[S_{g_1}^{(3,0)}*S_{g_2}^{(3,0)}](\mathbf{r})$ decays as $1/r^3$ and not as $(\ln r)/r^3$.

III. PARTICLE-PARTICLE CORRELATIONS

According to Sec. V D of paper I, the exchange part of the particle-particle correlation decays faster than any inverse power law of the distance. The nonexchange part $\rho_{\alpha_a\alpha_b}^{(2)TQ}(r_{ab})|_{\text{nonexch}}$ of the particle-particle correlation, which is defined in (4.7) of paper I, is related to $h(\mathscr{L}_a,\mathscr{L}_b)$ by

$$\rho_{\alpha_a\alpha_b}^{(2)TQ}(r_{ab})|_{\text{nonexch}}=\sum_{p_a}\sum_{p_b} p_a p_b\int D(\mathbf{X}_a)\rho_{\alpha_a,p_a}(\mathbf{X}_a)$$
$$\times\int D(\mathbf{X}_b)\rho_{\alpha_b,p_b}(\mathbf{X}_b)h(\mathscr{L}_a,\mathscr{L}_b). \tag{3.1}$$

Its large-distance behavior is studied by using the Dyson equation (2.6).

A. Upper bound for the decay of K

The $\widetilde{\Pi}_{Wc}$ diagrams, and subsequently H, are shown to decay at least as $1/r_{ab}^6$, even before integration over the shapes of the root points, by considering two classes. In the first class, every diagram $\widetilde{\Pi}_{Wc}$ contains no bond W. Since an integrable diagram cannot decay more slowly than each of its bonds, namely, F^{cc}, F^{cm}, F^{mc}, and F_{R6}, it behaves at least as $1/r_{ab}^6$. In the second class, every diagram $\widetilde{\Pi}_{Wc}$ contains at least one W bond. If a bond W does contribute to the leading term in the asymptotic behavior of the $\widetilde{\Pi}_{Wc}$ diagram, its contribution is multiplied by that of other bonds and the product of these two contributions must decrease at least as $(1/r_{ab}^3)^2$, as explained in the following.

According to the topological definition of the $\widetilde{\Pi}_{Wc}$ diagrams, for each pair of points $\mathscr{P}_i$ and $\mathscr{P}_j$ that are linked by a W bond, there exists a path linking $\mathscr{L}_a$ to $\mathscr{L}_b$ that does not contain the bond $W(\mathscr{P}_i,\mathscr{P}_j)$. Thus the integral corresponding to a $\widetilde{\Pi}_{Wc}$ diagram of the second class may be written as

$$\int d\mathscr{P}_i\int d\mathscr{P}_j G(\mathbf{r}_a,\mathbf{r}_b;\chi_a,\chi_b;\mathscr{P}_i,\mathscr{P}_j)W(\mathscr{P}_i,\mathscr{P}_j), \tag{3.2}$$

where $G(\mathbf{r}_a,\mathbf{r}_b;\chi_a,\chi_b;\mathscr{P}_i,\mathscr{P}_j)$ is the value of a subdiagram of the $\widetilde{\Pi}_{Wc}$ diagram after integration over all its internal points except $\mathscr{P}_i$ and $\mathscr{P}_j$. This subdiagram is connected with respect to the root points $\mathscr{L}_a$ and $\mathscr{L}_b$, so that, at large distances, it decreases at least as $1/r_{ab}^3$, which is the decay of the slowest $\widetilde{F}$ bond. Then two cases might occur.

(i) If the topology of $\widetilde{\Pi}_{Wc}$ is such that none of the bonds W contributes to its dominant asymptotic behavior, then the latter is at least a $1/r^6$ falloff, since it is determined by the other bonds F^{cc}, F^{cm}, F^{mc}, and F_{R6}, as for the diagrams of the first class. For instance, in the diagram $\widetilde{\Pi}$ of Fig. 3(a), the bond $W(\mathscr{P}_1',\mathscr{P}_1)$ contributes only to a $1/r^9$ tail arising partly from the product of the bonds $W(\mathscr{P}_1',\mathscr{P}_1)$ and $F_{R6}(\mathscr{L}_a,\mathscr{P}_1)$ and the leading term in the asymptotic behavior of the diagram is a $1/r^6$ tail originating from the bond $F_{R6}(\mathscr{P}_1,\mathscr{L}_b)$.

(ii) If the bond $W(\mathscr{P}_i,\mathscr{P}_j)$ contributes to the leading term in the asymptotic behavior of the $\widetilde{\Pi}_{Wc}$ diagram, then the leading term is given by the Taylor expansion of $W(\mathbf{R}_{ij},\chi_i,\chi_j)$ around $\mathbf{R}_{ij}=\mathbf{r}_{ab}$ multiplied by the asymptotic behavior $[G]_{\text{as}}(\mathbf{r}_{ab};\chi_a,\chi_b;\mathscr{P}_i,\mathscr{P}_j)$ of $G(\mathbf{r}_a,\mathbf{r}_b;\chi_a,\chi_b,\mathscr{P}_i,\mathscr{P}_j)$ when r_{ab} goes to infinity and reads

$$\int d\mathscr{P}_i\int d\mathscr{P}_j[G]_{\text{as}}(\mathbf{r}_{ab};\chi_a,\chi_b;\mathscr{P}_i,\mathscr{P}_j)$$
$$\times\left\{W(\mathbf{r}_{ab},\chi_i,\chi_j)+O\left(\frac{1}{r_{ab}^4}\right)\right\}. \tag{3.3}$$

The product of these two contributions decays at least as the product $(1/r_{ab}^3)^2$. For instance, the two diagrams of Figs. 3(b) and 3(c) have a $1/r^6$ falloff arising from the product of the bond $W(\mathscr{L}_a,\mathscr{L}_b)$ with respectively, the bond $W(\mathscr{P}_1,\mathscr{L}_b)$ [Fig. 3(b)] or the convolution $W(\mathscr{L}_a,\mathscr{P}_1)*W(\mathscr{P}_1,\mathscr{L}_b)$ [Fig. 3(c)]. If $\int d\mathscr{P}_i\int d\mathscr{P}_j[G]_{\text{as}}(\mathbf{r}_{ab};\chi_a,\chi_b;\mathscr{P}_i,\mathscr{P}_j)W(\mathbf{r}_{ab},\chi_i,\chi_j)$ vanishes, the falloff is at least $1/r^7$. For instance, the slowest possible algebraic tail of the diagram of Fig. 3(d) behaves as $1/r^6$, but this algebraic contribution does not appear

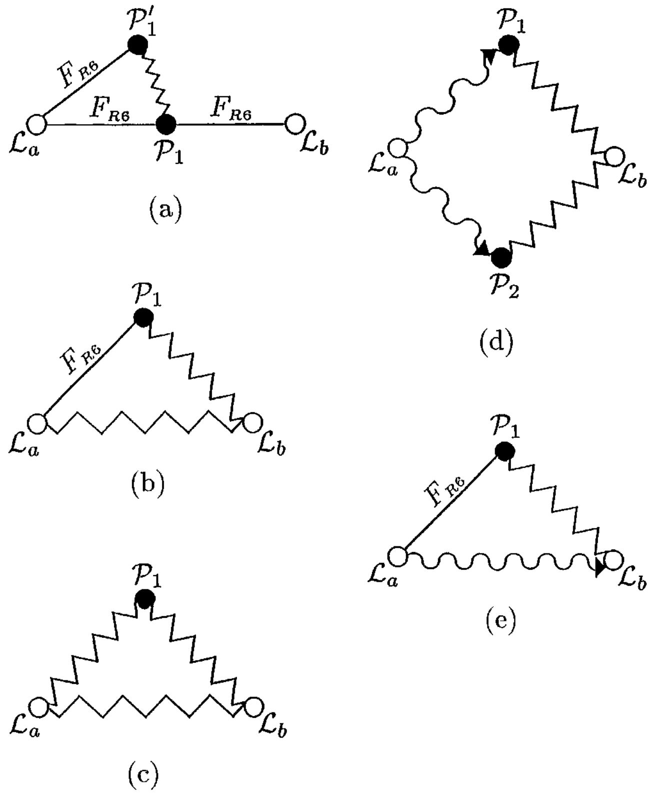

FIG. 3. Examples of diagrams $\tilde{\Pi}$ discussed in Sec. III A. (a)–(c) decay as $1/r^6$, (d) as $1/r^7$, and (e) has an exponential falloff.

because $\int d\mathbf{R}_1 \int d\chi_1 \rho(\chi_1) F^{cm}(\mathbf{R}_1,\chi_a,\chi_1) W_3(\mathbf{r}_{ab},\chi_1,\chi_b)=0$. (Indeed, W_3 involves one component of $\mathbf{X}_1$, so that, after integration over $\mathbf{X}_1$, the latter integral involves only terms $\int d\chi_1 \rho(\chi_1)[\mathbf{X}_1]_\mu (\mathbf{X}_1\cdot\nabla_{\mathbf{R}_1})^{2q+1}[\exp(-\kappa R_1)/R_1]$ that vanish after integration over the orientation of $\mathbf{R}_1$.) If $G(\mathbf{r}_{ab};\chi_a,\chi_b;\mathscr{P}_i,\mathscr{P}_j)$ has an exponential decay when r_{ab} goes to infinity, then (3.3) behaves as an exponential times $1/r^3$. For instance, the leading term in the asymptotic behavior of the $\tilde{\Pi}$ diagram of Fig. 3(e) is given by the product of the bonds $W(\mathscr{P}_1,\mathscr{L}_b)$ and $F^{cm}(\mathscr{L}_a,\mathscr{L}_b)$.

B. Upper bound for the decay of the convolution chains

The large-distance behavior of the chain made with $(I+1)$ graphs K linked by I bonds W in (2.6) can be analyzed rigorously in Fourier space. (The study is far more cumbersome in position space and, in Appendix A, we sketch the argument in position space only in the case of simple convolution chains.) Moreover, for the simplicity of the discussion, we use the decomposition of the Fourier transform of W as a series of purely nonanalytic terms $w^{[m_i,n_i]}$,

$$W(\mathbf{k},\chi_i,\chi_i') = -\beta e_{\alpha_i} e_{\alpha_i'} \int_0^{p_i} d\tau_i \int_0^{p_i'} d\tau_i' \{\delta([\tau_i - P(\tau_i)] - [\tau_i' - P(\tau_i')]) - 1\} \times \sum_{m_i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{m_i!\, n_i!} \times w^{[m_i,n_i]}(\mathbf{k},\mathbf{X}_i(\tau_i),\mathbf{X}_i'(\tau_i')), \quad (3.4)$$

where $w^{[m_i,n_i]}$ is a singular term of order $|\mathbf{k}|^{m_i+n_i-2}$ given by (5.33) of paper I,

$$w^{[m_i,n_i]}(\mathbf{k},\mathbf{X}_i(\tau_i),\mathbf{X}_i'(\tau_i')) \equiv [\mathbf{X}_i(\tau_i)\cdot\mathbf{k}]^{m_i}[-\mathbf{X}_i'(\tau_i')\cdot\mathbf{k}]^{n_i} \frac{4\pi}{\mathbf{k}^2}. \quad (3.5)$$

A chain in (2.6) with $(I+1)$, K graphs linked by I bonds W makes a contribution to $\rho^{(2)TQ}_{\alpha_a,\alpha_b}(\mathbf{k})$ [see (3.1)], which can be written as a series of chains involving purely algebraic terms $w^{[m_i,n_i]}$ instead of W,

$$(-\beta)^I \sum_{\{\alpha_i\}_{i=1,\ldots,I}} e^2_{\alpha_i} \sum_{\{\alpha_i'\}_{i=1,\ldots,I}} e^2_{\alpha_i'} \sum_{\{p_i\}_{i=1,\ldots,I}} \sum_{\{p_i'\}_{i=1,\ldots,I}} \left(\prod_{i=1}^{I} \frac{1}{m_i!\, n_i!} \int_0^{p_i} d\tau_i \int_0^{p_i'} d\tau_i' \{\delta([\tau_i - P(\tau_i)] - [\tau_i' - P(\tau_i')]) - 1\} \right)$$
$$\times \mathscr{C}_{\{\alpha_i\},\{\alpha_i'\},\{p_i\},\{p_i'\},\{\tau_i\},\{\tau_i'\};I}(\mathbf{k};\chi_a,\chi_b;\{m_i\},\{n_i\}), \quad (3.6)$$

where

$$\mathscr{C}_{\{\alpha_i\},\{\alpha_i'\},\{p_i\},\{p_i'\},\{\tau_i\},\{\tau_i'\};I}(\mathbf{k};\chi_a,\chi_b;\{m_i\},\{n_i\}) = \int D(\mathbf{X}_1)\cdots D(\mathbf{X}_I) D(\mathbf{X}_1')\cdots D(\mathbf{X}_I') K(\mathbf{k},\chi_a,\chi_1)$$
$$\times w^{[m_1,n_1]}(\mathbf{k},\mathbf{X}_1(\tau_1),\mathbf{X}_1'(\tau_1')) K(\mathbf{k},\chi_1',\chi_2) w^{[m_2,n_2]}(\mathbf{k},\mathbf{X}_2(\tau_2),\mathbf{X}_2'(\tau_2'))\cdots$$
$$\times w^{[m_I,n_I]}(\mathbf{k},\mathbf{X}_I(\tau_I),\mathbf{X}_I'(\tau_I')) K(\mathbf{k},\chi_I',\chi_b), \quad (3.7)$$

with $i=1,\ldots,I$ and m_i and n_i ranging from 1 to ∞. For conciseness, we omit the dependence upon the species α, the sizes p, and the times τ, in the notation $\mathscr{C}_I(\mathbf{k};\chi_a,\chi_b;\{m_i\},\{n_i\})$ in the following.

The integration over the shapes of the loops, including the root points, is performed with the result

$$\begin{aligned}\mathfrak{C}_I(\mathbf{k};\{m_i\},\{n_i\}) &\equiv \int D(\mathbf{X}_a)\int D(\mathbf{X}_b)\mathscr{C}_I(\mathbf{k};\chi_a,\chi_b;\{m_i\},\{n_i\})\\ &= \frac{1}{(\mathbf{k}^2)^I}\,\mathfrak{K}_{a,1}^{[m_1]}(\mathbf{k})\mathfrak{K}_{1,2}^{[n_1,m_2]}(\mathbf{k})\mathfrak{K}_{2,3}^{[n_2,m_3]}(\mathbf{k})\\ &\quad\cdots\mathfrak{K}_{I-1,I}^{[n_{I-1},m_I]}(\mathbf{k})\mathfrak{K}_{I',b}^{[n_I]}(\mathbf{k}).\end{aligned}\tag{3.8}$$

The quantity $\mathfrak{K}_{a,1}^{[m_1]}(\mathbf{k})[\mathfrak{K}_{I,b}^{[n_I]}(\mathbf{k})]$ at the end of the chain $\mathfrak{C}_I$ contains the integration over the shape of the root point and the shape $\mathbf{X}_1$ ($\mathbf{X}_I'$),

$$\mathfrak{K}_{a,1}^{[m_1]}(\mathbf{k})\equiv\int D(\mathbf{X}_a)\int D(\mathbf{X}_1)[\mathbf{k}\cdot\mathbf{X}_1(\tau_1)]^{m_1}K(\mathbf{k},\chi_a,\chi_1),\tag{3.9}$$

$$\mathfrak{K}_{I',b}^{[n_I]}(\mathbf{k})\equiv\int D(\mathbf{X}_I')\int D(\mathbf{X}_b)[\mathbf{k}\cdot\mathbf{X}_I'(\tau_I')]^{n_I}K(\mathbf{k},\chi_I',\chi_b).\tag{3.10}$$

$\mathfrak{K}_{a,1}^{[m_1]}(\mathbf{k})$ also depends on α_a, α_1, p_a, p_1, and τ_1. $\mathfrak{K}_{i',i+1}^{[n_i,m_{i+1}]}$ involves the integration over $\mathbf{X}_i'$ and $\mathbf{X}_{i+1}$ (and depends also on α_i, α_i', p_i, p_i', τ_i, and τ_i'),

$$\begin{aligned}\mathfrak{K}_{i,i+1}^{[n_i,m_{i+1}]}(\mathbf{k})\equiv\int D(\mathbf{X}_i')\int D(\mathbf{X}_{i+1})[\mathbf{k}\cdot\mathbf{X}_i'(\tau_i')]^{n_i}\\ \times[\mathbf{k}\cdot\mathbf{X}_{i+1}(\tau_{i+1})]^{m_{i+1}}K(\mathbf{k},\chi_i',\chi_{i+1}).\end{aligned}\tag{3.11}$$

Since the measure $D(\mathbf{X})$ is invariant under rotations of $\mathbf{X}$ while the bonds F are unchanged under a simultaneous rotation of $\mathbf{R}_{ij}$, $\mathbf{X}_i$, and $\mathbf{X}_j$, $K(\mathbf{k},\chi_i',\chi_{i+1})$ is also invariant under global rotations of $(\mathbf{k},\mathbf{X}_i',\mathbf{X}_{i+1})$ and the $\mathfrak{K}$'s are functions of $|\mathbf{k}|$. As a consequence, the small-$\mathbf{k}$ expansion of a $\mathfrak{K}$ contains only even powers of $|\mathbf{k}|$ up to the first nonanalytic term that characterizes the slowest algebraic tail in the large-distance behavior of $\mathfrak{K}(|\mathbf{r}|)$.

The structure of the first two terms in the small-$\mathbf{k}$ expansion of a $\mathfrak{K}$ can be readily determined from the rotational invariance of K and the measure $D(\mathbf{X})$ because the first nonanalytic term in the small-$\mathbf{k}$ expansion of K is of order $|\mathbf{k}|^3$. Indeed, according to Sec. III A, $H(\mathbf{r},\chi_i',\chi_{i+1})$ decays as $1/r^6$ at large distances and, according to (2.12), its Fourier transform at small $\mathbf{k}$ reads

$$\begin{aligned}H(\mathbf{k},\chi_i',\chi_{i+1})\underset{|\mathbf{k}|\to0}{\sim}&\int d\mathbf{x}\,H(\mathbf{x},\chi_i',\chi_{i+1})\\ &+ik_\mu\int d\mathbf{x}\,x_\mu H(\mathbf{x},\chi_i',\chi_{i+1})\\ &-\tfrac{1}{2}k_\mu k_\nu\int d\mathbf{x}\,x_\mu x_\nu H(\mathbf{x},\chi_i',\chi_{i+1})\\ &+S_H^{(3)}(|\mathbf{k}|,\chi_i',\chi_{i+1})+\widetilde{O}(|\mathbf{k}|^3).\end{aligned}\tag{3.12}$$

$\widetilde{O}(|\mathbf{k}|^3)$ contains analytic terms of order $|\mathbf{k}|^3$ and terms of higher orders. K has the same large-distance behavior as H and the small-$\mathbf{k}$ expansion of K is equal to that of H plus the constant $\rho(\chi_{i+1})\delta_{\chi_i',\chi_{i+1}}$. The first terms in the small-$\mathbf{k}$ expansion of $\mathfrak{K}_{a,1}^{[m_1]}(|\mathbf{k}|)$ at the end of a convolution chain is obtained by inserting the small-$\mathbf{k}$ expansion (3.12) of $K(\mathbf{k},\mathbf{X}_a,\mathbf{X}_1)$ into the definition (3.9). Since the measure $D(\mathbf{X}_a)$ and $K(\mathbf{r},\mathbf{X}_a,\mathbf{X}_1)$ are invariant under global rotation of their arguments, $\int D(\mathbf{X}_a)K(\mathbf{r},\mathbf{X}_a,\mathbf{X}_1)$ is invariant under the simultaneous rotations of $\mathbf{r}$ and $\mathbf{X}_1$ and the first two terms in the small-$|\mathbf{k}|$ expansion of $\mathfrak{K}_{a,1}^{[m_1]}(|\mathbf{k}|)$ are of order $|\mathbf{k}|^{m_1}$ and $|\mathbf{k}|^{m_1+2}$, respectively, if m_1 is even and of order $|\mathbf{k}|^{m_1+1}$ and $|\mathbf{k}|^{m_1+3}$, respectively, if m_1 is odd. This result can be summarized by introducing the function θ such that $\theta(n)=0$ if n is an even integer, whereas $\theta(n)=1$ if n is odd,

$$\mathfrak{K}_{a,1}^{[m_1]}(|\mathbf{k}|)=A_{a,1}^{[m_1]}|\mathbf{k}|^{m_1+\theta(m_1)}+O(|\mathbf{k}|^{m_1+\theta(m_1)+2}),\tag{3.13}$$

where $O(|\mathbf{k}|^p)$ denotes a term of order $|\mathbf{k}|^p$. Similarly,

$$\mathfrak{K}_{I',b}^{[n_I]}(\mathbf{k})=A_{I,b}^{[n_I]}|\mathbf{k}|^{n_I+\theta(n_I)}+O(|\mathbf{k}|^{n_I+\theta(n_I)+2}).\tag{3.14}$$

In the same way, the dimension of the first two terms in the small-$\mathbf{k}$ expansion of $\mathfrak{K}_{i',i+1}^{[n_i,m_{i+1}]}(|\mathbf{k}|)$ is obtained by inserting (3.12) in the definition (3.11),

$$\begin{aligned}\mathfrak{K}_{i',i+1}^{[n_i,m_{i+1}]}(\mathbf{k})=A_{i',i+1}^{[n_i,m_{i+1}]}|\mathbf{k}|^{n_i+m_{i+1}+\theta(n_i+m_{i+1})}\\ +O(|\mathbf{k}|^{n_i+m_{i+1}+\theta(n_i+m_{i+1})+2}).\end{aligned}\tag{3.15}$$

For instance, $\mathfrak{K}_{a,1}^{[1]}(\mathbf{k})$, $\mathfrak{K}_{I',b}^{[1]}(\mathbf{k})$, and $\mathfrak{K}_{i,i+1}^{[1,1]}(\mathbf{k})$ are of order $|\mathbf{k}|^2$ when $|\mathbf{k}|$ goes to zero, and the next term in their small-$\mathbf{k}$ expansion is of order $|\mathbf{k}|^4$. According to the structure (3.12) of the small-$\mathbf{k}$ expansion of $K(\mathbf{k},\chi_i,\chi_j)$, the first term in the small-$\mathbf{k}$ expansion of any $\mathfrak{K}$ is analytic in the components of $\mathbf{k}$ and the possible singularity due to the term $S_H^{(3)}(\mathbf{k},\chi_i',\chi_{i+1})$ may appear (if it is not canceled by integration over the shapes $\mathbf{X}$) only in the next nonzero higher-order term [because $\theta(n_I)+2\leqslant3$ and $\theta(n_i+m_{i+1})+2\leqslant3$].

According to the previous arguments of both dimensional analysis and rotational invariance, the first term in the small-$\mathbf{k}$ expansion of $\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ defined in (3.8) is *proportional* to $|\mathbf{k}|^{D_{\mathfrak{C}_I}}$, while the next term is *of order* $|\mathbf{k}|^{D_{\mathfrak{C}_I}+2}$, where, according to (3.13)–(3.15), the dimension $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ reads

$$D_{\mathfrak{C}_I}(\{m_i\},\{n_i\}) = -2I + [m_1 + \theta(m_1)] + [n_I + \theta(n_I)] + \sum_{i=1}^{I-1} [n_i + m_{i+1} + \theta(n_i + m_{i+1})]. \quad (3.16)$$

Since $n+\theta(n)$ is even, $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ can take only even values. Moreover, $m_i \geqslant 1$ and $n_i \geqslant 1$ for every $i=1,\ldots,I$, while $n+\theta(n)=2$ for $n=1,2$; $n+\theta(n)=4$ for $n=3,4$; and so on. Subsequently, the dimension $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ given by (3.16) can be equal to 2,4,6,... . $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ is positive and even and the first term in the small-**k** expansion of the chain $\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ is analytic, while the first possible nonanalytic term is the next term in the expansion, which is of order $|\mathbf{k}|^{D_{\mathfrak{C}_I}+2}$ with $D_{\mathfrak{C}_I}+2 \geqslant 4$. In other words, the convolution chains (3.8) decay at least as $1/r^7$ and, eventually, the part (3.1) of the particle-particle correlation behaves as $1/r^6$.

At this point, we make a comment on the origin of the upper bound $1/r^7$. First, before integration over the shapes of its end points, a convolution $\mathscr{C}_I(\mathbf{r},\chi_a,\chi_b;\{m_i\},\{n_i\})$ decays as $1/r^3$ because the Coulomb potential decays sufficiently fast for W to be at the borderline of integrability. Moreover, the fact that $W(\mathscr{P},\mathscr{P}')$ involves the components of the shapes of both $\mathscr{P}$ and $\mathscr{P}'$ together with the rotational invariance of the interactions and of the loop measures have two consequences, once the integration over the shapes of all loops has been performed. First, they ensure that the dimension $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ of the first term in the small-**k** expansion of $\mathfrak{C}_I(\mathbf{k};\{m_1\},\{n_1\})$ may take only values that differ by an even integer. Second, since they enforce $\mathfrak{K}_a^{[m_1]}$ and $\mathfrak{K}_b^{[n_I]}$ to start at least at the order $|\mathbf{k}|^2$, $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\}) \geqslant 2$. Then, the harmonicity of the Coulomb potential implies that the dimension of the first nonanalytic term is at least $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})+2$. Indeed, in the case of a potential v such that $v(\mathbf{k})=S^{(-2)}(\mathbf{k})$ is of order $|\mathbf{k}|^{-2}$ but not proportional to $1/\mathbf{k}^2$, the first term in the small-**k** expansion of $\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ would be proportional to

$$[S^{(-2)}(\mathbf{k})]^I \times |\mathbf{k}|^{\{m_1+\theta(m_1)+n_I+\theta(n_I)+\Sigma_{i=1}^{I-1}[n_i+m_{i+1}+\theta(n_i+m_{i+1})]\}}. \quad (3.17)$$

It would be of order $|\mathbf{k}|^{D_{\mathfrak{C}_I}}$ with $D_{\mathfrak{C}_I} \geqslant 2$ given by (3.16), but it would not be analytic and the chains (3.8) would decay at least as $1/r^5$. To sum up, because of the structure of W that arises from its definition as the difference between the loop potential and the electrostatic potential and once the basic rotational invariance has been taken into account, a mere dimensional analysis shows that the chains decay at least as $1/r^5$. Henceforth, the harmonicity of the Coulomb potential is crucial to ensure that the chains in the Dyson equation (2.6) decay faster than $1/r^6$. These arguments are exemplified in position space, in Appendix A.

C. Diagrammatic structure of the $1/r^6$ tail

The $1/r^6$ decay of the particle-particle correlations can only arise from the Π diagrams that give at least one $\widetilde{\Pi}_{Wc}$ diagram, when F is split into the two bonds W and $F_{R6}=F_R-W$. (We recall that a $\widetilde{\Pi}_{Wc}$ diagram remains connected when one bond W is removed.) At each order $\rho(\mathscr{L})^{2+N}$, with $N \geqslant 0$, diagrams that decrease as $1/r^6$ can be exhibited.

Let first consider convolution diagrams. At the order $\rho(\mathscr{L})^2$, the diagram with only one F_R bond is an example and its contribution, which is displayed in (2.3), comes from $\frac{1}{2}[W_3(\mathbf{r}_{ab},\chi_a,\chi_b)]^2$ because W leads to a fast decay after integration over the shapes of the end points. At the order $\rho(\mathscr{L})^3$, let consider the diagram of Fig. 4(a), which is the convolution of two F_R bonds. According to the general discussion of Sec. III B, after integration over the shapes of the end points, the convolutions (over the variables $\mathbf{R}$) $W*W$, $[F_R-W]*W$, and $W*[F_R-W]$ fall off faster than $1/r^6$. Thus the leading asymptotic behavior of the diagram $\Pi_{4(a)}$ is given by the convolution $[F_R-W]*[F_R-W]$. The $1/r^6$ tail of the convolution of two $1/r^6$-decaying functions is determined by the terms of order $|\mathbf{k}|^3$ in the small-**k** expansion of its Fourier transform. According to (2.15), the first singular term in $[F_R-W](\mathbf{k},\chi_i,\chi_j)$ is $\frac{1}{2}[W_3(\mathbf{k},\chi_i,\chi_j)]^2$. Subsequently, according to (2.20), the $1/r^6$ tail of the diagram in Fig. 4(a) originates from the integration with the measure $D(\mathbf{X}_a)p_a\rho(\chi_a)D(\mathbf{X}_b)p_b\rho(\chi_b)d\chi_1\rho(\chi_1)$ of

$$\tfrac{1}{2}[W_3(\mathbf{r}_{ab},\chi_a,\chi_1)]^2 A(\chi_1,\chi_b) + A(\chi_a,\chi_1)\tfrac{1}{2}[W_3(\mathbf{r}_{ab},\chi_1,\chi_b)]^2, \quad (3.18)$$

where $A(\chi_i,\chi_j)$ is given by (2.16). Only the $\frac{1}{2}[W_3]^2$ part of F_R does contribute to the $1/r^6$ tail of the diagram in Fig. 4(a).

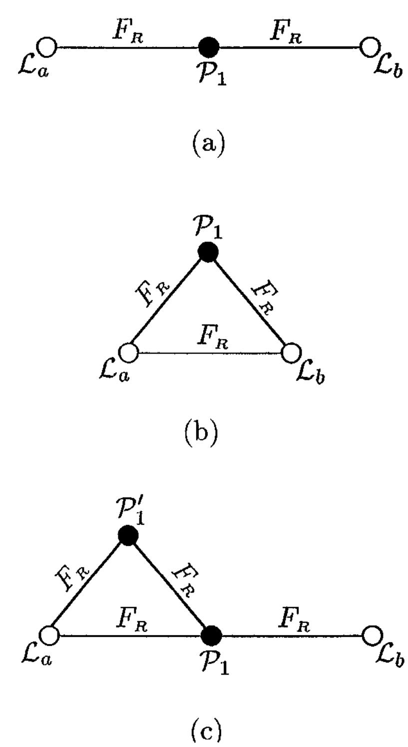

FIG. 4. Structure of the $1/r^6$ asymptotic behavior of the Π diagrams, given in Sec. III C.

At the order $N \geqslant 3$, the Π diagram made with a product of two convolution chains of bonds F_R, one chain with n bonds in it and the other one with $N-n$ bonds, decays as $1/r^6$ because a convolution chain of bonds F may decay as $1/r^3$ before integration over the shapes $\mathbf{X}_a$ and $\mathbf{X}_b$ of each end. Indeed, the asymptotic $1/r^3$ behavior of a convolution is determined by the singular terms of order zero in $|\mathbf{k}|$ in the expansion of the product of the small-$\mathbf{k}$ behaviors of the various functions involved in the convolution. The decomposition (2.17) allows one to disentangle the singular term $W_3(\mathbf{k},\chi_i,\chi_j)$ from the constant $A^*(\chi_i,\chi_j) = \int d\mathbf{r}[F_R(\mathbf{r},\chi_i,\chi_j) - W_3(\mathbf{r},\chi_i,\chi_j)]$ in the term of order zero in the small-$\mathbf{k}$ expansion of $F_R(\mathbf{k},\chi_i,\chi_j)$,

$$F_R(\mathbf{k},\chi_i,\chi_j) = W_3(\mathbf{k},\chi_i,\chi_j) + A^*(\chi_i,\chi_j) + O(|\mathbf{k}|). \tag{3.19}$$

For instance, let us consider the diagram of Fig. 4(b). This diagram leads in particular to the $\tilde{\Pi}_{Wc}$ diagrams of Figs. 3(b) and 3(c). According to (2.21), the asymptotic behavior of the diagram in Fig. 4(b) reads

$$W_3(\mathbf{r}_{ab},\chi_a,\chi_b)\int d\chi_1\rho(\chi_1)\Bigg\{A^*(\chi_1,\chi_b)W_3(\mathbf{r}_{ab},\chi_a,\chi_1) + A^*(\chi_a,\chi_1)W_3(\mathbf{r}_{ab},\chi_1,\chi_b) + \int d\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}_{ab}}W_3(\mathbf{k},\chi_a,\chi_1)W_3(\mathbf{k},\chi_1,\chi_b)\Bigg\}. \tag{3.20}$$

The last term in curly brackets in (3.20) is equal to

$$W_3(\mathbf{r}_{ab},\chi_a,\chi_b)\beta e_{\alpha_a}e_{\alpha_b}\int_0^{p_a}d\tau\int_0^{p_b}d\tau' \times[\mathbf{X}_a(\tau)\cdot\boldsymbol{\nabla}_{\mathbf{r}_{ab}}][\mathbf{X}_b(\tau')\cdot\boldsymbol{\nabla}_{\mathbf{r}_{ab}}]\left(\frac{1}{r_{ab}}\right) \times\Bigg\{\int_0^{p_1}d\tau_1\int_0^{p_1}d\tau_1'\{\delta([\tau-P(\tau)] - [\tau_1-P(\tau_1)])-1\}\{\delta([\tau_1'-P(\tau_1')] - [\tau'-P(\tau')])-1\}\int d\chi_1 4\pi\beta e_{\alpha_1}^2\rho(\chi_1) \times\tfrac{1}{3}[\mathbf{X}_1(\tau_1)\cdot\mathbf{X}_1(\tau_1')]\Bigg\}. \tag{3.21}$$

More generally, according to (2.21) and (3.19), the $1/r^3$ behavior of a convolution can only originate from one or several $W_3(\mathbf{k},\chi_i,\chi_j)$, i.e., from convolutions of W_3 with subdiagrams of Π. The asymptotic behavior of such convolutions is of the form $[\mathbf{X}_i(\tau_i)]_{\mu_i}[\mathbf{X}_j(\tau_j)]_{\nu_j}\partial_{\mu_i\nu_j}(v_C)\int d\mathbf{y}\; g_{\rm mid}(\mathbf{y})$, where the expression of $g_{\rm mid}$ is analogous to that of $G_{\rm mid}$ in Appendix A, with the $\mathfrak{K}^{[1,1]}$'s in $G_{\rm mid}$ replaced by moments of subdiagrams of Π calculated with respect to the shapes of their internal points.

As a final example, let us consider the Π diagram in Fig. 4(c), which leads in particular to the $\tilde{\Pi}$ diagram of Fig. 3(a) through the decomposition (2.4), $\Pi_{4(c)} = \int d\chi_1\rho(\chi_1)\Pi_{4(b)}(\mathscr{L}_a,\mathscr{L}_1)F_R(\mathscr{L}_1,\mathscr{L}_b)$. According to the general discussion of Sec. III B, after integration over $\mathbf{X}_a$ and $\mathbf{X}_b$, the contribution from $\Pi_{4(b)}*W$ decays faster than $1/r^6$ and the $1/r^6$ tail of $\int D(\mathbf{X}_a)D(\mathbf{X}_b)\rho(\chi_a)\rho(\chi_b)\Pi_{4(c)}(\mathscr{L}_a,\mathscr{L}_b)$ is given by the convolution of $\Pi_{4(b)}$ with F_R-W. Since both latter functions behaves as $1/r^6$, even before integration over their root points, the leading asymptotic behavior of $\Pi_{4(c)}$ is the sum of two contributions given by (2.20),

$$\int d\chi_1\rho(\chi_1)\Bigg\{[\Pi_{4(b)}]_{\rm as}(\mathbf{r}_{ab},\chi_a,\chi_1)A(\chi_1,\chi_b) + \left(\int d\mathbf{x}\;\Pi_{4(b)}(\mathbf{x},\chi_a,\chi_1)\right)\tfrac{1}{2}[W_3(\mathbf{r}_{ab},\chi_1,\chi_b)]^2\Bigg\}, \tag{3.22}$$

where $[\Pi_{4(b)}]_{\rm as}$ is the asymptotic behavior of the diagram $\Pi_{4(b)}$, which is written in (3.20).

As a conclusion, if the leading term in the asymptotic behavior of a $\tilde{\Pi}_{Wc}$ diagram (after integration over the root points) is $1/r^6$, then this tail comes either from the asymptotic behavior of a bond $F_{R6} = F_R - W$, which is equal to $(W_3)^2/2$, or from the product of two $1/r^3$ asymptotic behaviors, each of which arises from the asymptotic behavior W_3 of one or several W bonds involved in convolutions with subdiagrams of $\tilde{\Pi}_{Wc}$. If we return to Π diagrams [before the decomposition (2.4) into $\tilde{\Pi}$ diagrams], the result is the following: the $1/r^6$ tail of a Π diagram comes either from the term $(W_3)^2/2$ in the asymptotic behavior of one F_R bond or from the product of the asymptotic behavior of two convolutions, each of which is built with subdiagrams of Π linked by at least one F_R bond. Henceforth, after integration over the internal points of the chain, the general structure of the $A_{\alpha_a,\alpha_b}/r^6$ tail takes the form

$$\sum_{p_a}p_a\int D(\mathbf{X}_a)\rho_{\alpha_a,p_a}(\mathbf{X}_a)\sum_{p_b}p_b\int D(\mathbf{X}_b)\rho_{\alpha_b,p_b}(\mathbf{X}_b)\int d\chi_1\int d\chi_2\int d\chi_3\int d\chi_4 \times\int_0^{p_1}d\tau_1\int_0^{p_2}d\tau_2\int_0^{p_3}d\tau_3\int_0^{p_4}d\tau_4\, g(\mathbf{X}_a,\mathbf{X}_1,\mathbf{X}_2;\tau_1,\tau_2)g(\mathbf{X}_b,\mathbf{X}_3,\mathbf{X}_4;\tau_3,\tau_4) \times[\mathbf{X}_1(\tau_1)\cdot\boldsymbol{\nabla}_{\mathbf{r}_{ab}}][\mathbf{X}_3(\tau_3)\cdot\boldsymbol{\nabla}_{\mathbf{r}_{ab}}]\left(\frac{1}{r_{ab}}\right)[\mathbf{X}_2(\tau_2)\cdot\boldsymbol{\nabla}_{\mathbf{r}_{ab}}][\mathbf{X}_4(\tau_4)\cdot\boldsymbol{\nabla}_{\mathbf{r}_{ab}}]\left(\frac{1}{r_{ab}}\right). \tag{3.23}$$

D. Structure of algebraic tails in $\tilde{\Pi}$ diagrams

Now, in view of the discussion of Sec. IV, we turn to the $1/r^{\gamma}$ leading and subleading algebraic tails of $\tilde{\Pi}$ diagrams that have any exponent $\gamma \geqslant 6$. An algebraic tail in the asymptotic behavior of a $\tilde{\Pi}$ diagram arises either from a single elementary algebraic chain or from a product of the former elementary objects. An ''elementary algebraic chain'' refers either to an algebraic bond (W or F_{R6}) or to a convolution chain with such algebraic bonds at both ends and with subdiagrams of $\tilde{\Pi}$ and algebraic bonds in the middle of the convolution. The asymptotic behavior of a single elementary algebraic chain involves a series of purely ''elementary algebraic tails'' $S^{(\gamma)[q,q']}(\mathscr{P},\mathscr{P}')$, which decays as $1/r^{\gamma}$ and involves q components of the shape of $\mathscr{P}$ and q' components of the shape of $\mathscr{P}'$.

An algebraic tail $T(\mathbf{r}_{ab},\chi_a,\chi_b)$ of a $\tilde{\Pi}$ diagram that results from a product of L ($L \geqslant 1$) algebraic terms $S^{(\gamma_l)[q_l,q_l']}(\mathscr{P}_l,\mathscr{P}_l')$ is a term in the series of the algebraic decays of the function

$$\int \left[\prod_{l=1}^{L} d\mathscr{P}_l\right]\int\left[\prod_{l=1}^{L} d\mathscr{P}_l'\right] G(\mathscr{L}_a,\mathscr{P}_1,\ldots,\mathscr{P}_L) \times\left(\prod_{l=1}^{L} S^{(\gamma_l)[q_l,q_l']}(\mathscr{P}_l,\mathscr{P}_l')\right) G'(\mathscr{L}_b,\mathscr{P}_1',\ldots,\mathscr{P}_L'), \tag{3.24}$$

where $G(\mathscr{L}_a,\mathscr{P}_1,\ldots,\mathscr{P}_L)$ is equal to a subdiagram of $\tilde{\Pi}$ integrated over all its points except $(\mathscr{L}_a,\mathscr{P}_1,\ldots,\mathscr{P}_L)$, and so is $G'(\mathscr{L}_b,\mathscr{P}_1',\ldots,\mathscr{P}_L')$. (Examples have been given in Secs. III A and III C.) In the following, we use the notation $\zeta \equiv (\alpha,p,\mathbf{Z})$ for the internal degrees of freedom of a loop, in order to avoid confusion with the variables $\chi=(\alpha,p,\mathbf{X})$ used in the convolution chains of Eq. (2.6). The series of algebraic tails of (3.24) are obtained by expanding each $S^{(\gamma_l)[q_l,q_l']}(\mathscr{P}_l,\mathscr{P}_l') = S^{(\gamma_l)[q_l,q_l']}(\mathbf{R}_l-\mathbf{R}_l',\zeta_l,\zeta_l')$ around $(\mathbf{r}_{ab},\zeta_l,\zeta_l')$ in powers of the components of the $\mathbf{R}_l-\mathbf{R}_a$'s and $\mathbf{R}_l'-\mathbf{R}_b$'s. Subsequently, every algebraic tail $T(\mathbf{r}_{ab},\chi_a,\chi_b)$ can be written as

$$T(\mathbf{r}_{ab},\chi_a,\chi_b) = \int\left(\prod_{l=1}^{L} d\zeta_l d\zeta_l'\right) g^{[Q_a]}_{\sigma_1\cdots\sigma_{Q_a}}(\chi_a,\zeta_1,\ldots,\zeta_l) \times \partial_{\sigma_1\cdots\sigma_{Q_a}\sigma_1'\cdots\sigma_{Q_b}'} \times\left(\prod_{l=1}^{L} S^{(\gamma_l)[q_l,q_l']}(\mathbf{r}_{ab},\zeta_l,\zeta_l')\right) \times g'^{[Q_b]}_{\sigma_1'\cdots\sigma_{Q_b}'}(\chi_b,\zeta_1',\ldots,\zeta_L'), \tag{3.25}$$

where the partial derivative $\partial_{\sigma_1\cdots\sigma_{Q_a}\sigma_1'\cdots\sigma_{Q_b}'}$ operates on the variable $\mathbf{r}_{ab}$ and $g^{[Q_a]}_{\sigma_1\cdots\sigma_{Q_a}}(\chi_a,\zeta_1,\ldots,\zeta_l)$ is a moment of $G(\mathscr{L}_a,\mathscr{P}_1,\ldots,\mathscr{P}_L)$ with respect to the components with indices $\sigma_1,\ldots,\sigma_{Q_a}$ of Q_a vectors chosen among $\mathbf{R}_1-\mathbf{R}_a,\ldots,\mathbf{R}_L-\mathbf{R}_a$. (The same vector may appear several times.) The origin of $g'^{[Q_b]}_{\sigma_1'\cdots\sigma_{Q_b}'}(\chi_b,\zeta_1',\ldots,\zeta_L')$ is similar. Since both the measure $D(\mathbf{Z})$ and the bonds F are invariant under global rotations of their arguments, $g^{[Q_a]}_{\sigma_1\cdots\sigma_{Q_a}}(\chi_a,\zeta_1,\ldots,\zeta_l)$ is a tensor of rank Q_a with respect to global rotations of its arguments.

According to Appendix B, before integration over the shapes of its end points $\mathscr{P}$ and $\mathscr{P}'$, an elementary algebraic tail $S^{(\gamma)[q,q']}(\mathscr{P},\mathscr{P}')$ has the form of the tensorial product

$$S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta') = A^{[q]}_{\mu_1\cdots\mu_q}(\mathbf{Z})A^{[q']}_{\nu_1\cdots\nu_{q'}}(\mathbf{Z}') \times S^{(\gamma)}_{\mu_1\cdots\mu_q\nu_1\cdots\nu_{q'}}(\mathbf{r}), \tag{3.26}$$

where $A^{[q]}_{\mu_1\cdots\mu_q}(\mathbf{Z})$ and $A^{[q']}_{\nu_1\cdots\nu_{q'}}(\mathbf{Z}')$ are tensors of rank q and q', respectively: $A^{[q]}_{\mu_1\cdots\mu_q}(\mathbf{Z}) = [\mathbf{Z}]_{\mu_1}\cdots[\mathbf{Z}]_{\mu_q} f(|\mathbf{Z}|)$, where $f(|\mathbf{Z}|)$ is a scalar function that is invariant by rotation of $\mathbf{Z}$. In the definition (3.26)

$$\gamma = P + q + q', \tag{3.27}$$

where the allowed values for P depend on (q,q') and are given in Appendix B. The structure (3.26) arises from the decomposition of $W(\mathbf{r},\zeta,\zeta')$ as a series of purely algebraic terms $w^{[m,n]}$ given by (3.5)

$$w^{[m,n]}[\mathbf{r},\mathbf{Z}(\tau),\mathbf{Z}'(\tau')] \equiv (-1)^n [\mathbf{Z}(\tau)]_{\mu_1}\cdots[\mathbf{Z}(\tau)]_{\mu_m} \times [\mathbf{Z}'(\tau')]_{\nu_1}\cdots[\mathbf{Z}'(\tau')]_{\nu_n} \times \partial_{\mu_1\cdots\mu_m\nu_1\cdots\nu_n}\left(\frac{1}{r}\right). \tag{3.28}$$

$[\mathbf{Z}]_{\mu_j}$ is the component of the vector $\mathbf{Z}$ with space index μ_j. The space indices μ_j and $\nu_{j'}$ take the values 1,2,3, while j runs from 1 to m (j' from 1 to n).

The insertion of (3.26) into (3.25) shows that the tail T has a tensorial structure analogous to that of the elementary algebraic tails $S^{(\gamma)[q,q']}$,

$$T(\mathbf{r}_{ab},\chi_a,\chi_b) = \mathscr{A}^{[Q_a+\Sigma_l q_l]}_{\{\}}(\mathbf{X}_a)\mathscr{A}^{[Q_b+\Sigma_l q_l']}_{\{\}}(\mathbf{X}_b) S^{(\gamma_T)}_{\{\}\{\}}(\mathbf{r}_{ab}), \tag{3.29}$$

where

$$\gamma_T = Q_a + Q_b + \sum_{l=1}^{L}\gamma_l = \sum_{l=1}^{L} P_l + \sum_{l=1}^{L}(q_l+q_l') + Q_a + Q_b. \tag{3.30}$$

$\mathscr{A}^{[Q_a+\Sigma_l q_l]}_{\{\}}(\mathbf{X}_a)$ is a moment of the function $g^{[Q_a]}_{\{\}}(\chi_a,\zeta_1,\ldots,\zeta_l)$, which is of order q_l in the components of every $\mathbf{Z}_l$ ($l=1,\ldots,L$),

$$\mathscr{A}^{[Q_a+\Sigma_l q_l]}_{\{\}}(\mathbf{X}_a) \equiv \mathscr{A}_{\sigma_1\cdots\sigma_{Q_a}\{\mu_{l,1}\cdots\mu_{l,q_l}\}_{l=1,\ldots,L}}(\mathbf{X}_a) = \int\left(\prod_{l=1}^{L} d\zeta_l\right)\left(\prod_{l=1}^{L}\prod_{j_l=1}^{q_l}[\mathbf{Z}_l]_{\mu_{l,j_l}}\right) \times g^{[Q_a]}_{\sigma_1\cdots\sigma_{Q_a}}(\chi_a,\zeta_1,\ldots\zeta_L), \tag{3.31}$$

where $[\mathbf{Z}_l]_{\mu_{l,j_l}}$ is the component of the vector $\mathbf{Z}_l$ with space index μ_{l,j_l}, which takes the values 1,2,3, and j_l runs from 1 to q_l. The definition of $\mathscr{A}_{\{\}}^{[Q_b+\Sigma_l q_l']}(\mathbf{X}_b) \equiv \mathscr{A}_{\sigma_1'\cdots\sigma_{Q_b}'\{\nu_{l,1}\cdots\nu_{l,q_l'}\}_{l=1,\ldots,L}}(\mathbf{X}_b)$ is analogous to (3.31), with j_l' running from 1 to q_l'. $\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)$ depends only on $\chi_a=(\alpha_a,p_a,\mathbf{X}_a)$. [In the notation of (3.31), we write only the argument $\mathbf{X}_a$ because only the dependence upon $\mathbf{X}_a$ will be relevant in the following discussions.] The moments $\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)$ and $\mathscr{A}_{\{\}}^{[Q_b+\Sigma_l q_l']}(\mathbf{X}_b)$ are well defined because the weight $\rho(\mathbf{Z})$ is expected to decay faster than any inverse power law, according to paper I. Since $g_{\sigma_1\cdots\sigma_{Q_a}}^{[Q_a]}(\chi_a,\zeta_1,\ldots\zeta_L)$ is a tensor of rank Q_a with respect to global rotations of its arguments, $\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)$ is a tensor of rank $Q_a+\Sigma_{l=1}^{L}q_l$, while $\mathscr{A}_{\{\}}^{[Q_b+\Sigma_l q_l']}(\mathbf{X}_b)$ is a tensor of rank $Q_b+\Sigma_{l=1}^{L}q_l'$.

If $L=1$, the tail T originates from a convolution of algebraic bonds and subdiagrams of $\tilde{\Pi}$. If the convolution contains a W bond,

$$\gamma_T=P(q,q')+q+q'+Q_a+Q_b, \tag{3.32}$$

where $P(q,q')$ may take the values (B10). The exponent (3.30) of the decay of $T(\mathbf{r}_{ab},\zeta,\zeta')$ with $L\geqslant 2$ can be written as

$$\gamma_T=L+\sum_{l=1}^{L}(q_l+q_l')+Q_a+Q_b+\delta_T, \tag{3.33}$$

where $\delta_T\equiv(\Sigma_{l=1}^{L}P_l)-L$ does not depend on either L, Q_a, or Q_b. Since the allowed values for P_l depend on q_l and q_l', the allowed values of δ_T also depend on the values of $\{(q_l,q_l')\}_{l=1,\ldots,L}$ and we have to distinguish three cases (I)–(III), which are detailed in Appendix C.

E. Algebraic tails of *K*

The structure of one of the algebraic tails of a $\tilde{\Pi}_{Wc}$ diagram before integration over the shapes of the root points $\mathscr{L}_a$ and $\mathscr{L}_b$ is derived from the topological definition of a $\tilde{\Pi}_{Wc}$ diagram, as already done for the particular $1/r^6$ tail. Since a diagram $\tilde{\Pi}_{Wc}$ remains connected when one W bond is suppressed, an algebraic tail of the whole diagram may be of two kinds. First, it may come from the following single elementary algebraic chain convoluted with subdiagrams of $\tilde{\Pi}_{Wc}$ at both ends: either one F_{R6} bond or a convolution involving F_{R6} bonds (but no W bond) and subdiagrams of $\tilde{\Pi}_{Wc}$. In this case, the tail is an algebraic tail T (3.29) with $L=1$, $q\geqslant 2$, and $q'\geqslant 2$, and its exponent is given by (3.32) and (B11). Second, the algebraic tail may arise from the product of at least two elementary algebraic chains $S^{(\gamma)[q,q']}$ defined at the beginning of Sec. III D. Then, the exponent γ_T of the tail T (3.29) (with $L\geqslant 2$ and $P_l\geqslant 1$) is given by (3.33).

After integration over the shapes of the end points $\mathscr{L}_a$ and $\mathscr{L}_b$, the algebraic tail (3.29) in a $\tilde{\Pi}_{Wc}$ diagram is not canceled only if $\int D(\mathbf{X}_a)\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)\neq 0$ and $\int D(\mathbf{X}_b)\mathscr{A}_{\{\}}^{[Q_b+\Sigma_l q_l']}(\mathbf{X}_b)\neq 0$. Since $\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)$ is a tensor of rank $Q_a+\Sigma_{l=1}^{L}q_l$, $\int D(\mathbf{X}_a)\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)$ may be nonzero only if $Q_a+\Sigma_{l=1}^{L}q_l$ is even. The details are given in Appendix C. Eventually, in $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)\tilde{\Pi}_{Wc}(\mathbf{r}_{ab},\mathbf{X}_a,\mathbf{X}_b)$, the first three possible tails are $1/r^6, 1/r^8, 1/r^9$, the first two tails correspond only to $L=1,2$, while the $1/r^9$ tail comes from convolutions such as $F_{R6}*F_{R6}$. Any tail $1/r^\gamma$, with γ an integer and $\gamma\geqslant 9$, may appear.

F. Algebraic tails in the convolution chains

Next, we show that the interplay between the structure of W and both the rotational invariance and the harmonicity of the Coulomb potential ensures that the convolution chains may decay only as $1/r^8$, $1/r^{10}$, or $1/r^\gamma$, with $\gamma\geqslant 11$.

The order of the first nonanalytic term in the $\mathfrak{K}(|\mathbf{k}|)$'s is given by the study of the algebraic tails that survive after integration over the shapes of the root points in the diagrams $\tilde{\Pi}_{Wc}$ that contribute to H. This study is detailed in Appendix C and the result is the following. Because of the integration over the shapes $\mathbf{X}$ of all loops, the order of the first nonanalytic term in $\mathfrak{K}_{a,1}^{[m_1]}(|\mathbf{k}|)$ and $\mathfrak{K}_{i,i+1}^{[n_i,m_{i+1}]}(|\mathbf{k}|)$ proves to be *greater* than or equal to $|\mathbf{k}|^{m_1+3}$ and $|\mathbf{k}|^{n_i+m_{i+1}+3}$, respectively, though the first nonanalytic term in the small-$|\mathbf{k}|$ expansion of H is only of order 3. The reasons for this result are the following. First, W comes from the difference between the loop potential and the electrostatic potential, so that in every $w^{[m_i,n_i]}(\mathscr{P}_i,\mathscr{P}_i')$, the derivatives of the Coulomb potential are associated with the shapes $\mathbf{X}$ of *both* arguments $\mathscr{P}_i$ and $\mathscr{P}_i'$. Second, after integration over the shapes $\mathbf{X}$, only the terms that are not canceled by rotational invariance arguments remain. For instance, according to (C29),

$$\mathfrak{K}_{a,1}^{[m_1]}(\mathbf{k})=A_{a,1}^{[m_1]}|\mathbf{k}|^{m_1+\theta(m_1)}+B_{a,1}^{[m_1]}|\mathbf{k}|^{m_1+\theta(m_1)+2}+S_{a,1,m_1}^{m_1+\theta(m_1)+3}(|\mathbf{k}|)+\tilde{O}(|\mathbf{k}|^{m_1+\theta(m_1)+3}), \tag{3.34}$$

where $\tilde{O}(|\mathbf{k}|^{\gamma-3})$ denotes an analytic term of order $|\mathbf{k}|^{\gamma-3}$ plus terms of greater order. More precisely, as shown in Appendix C, the singular terms in (3.34) are of order $|\mathbf{k}|^{m_1+\theta(m_1)+3}$, $|\mathbf{k}|^{m_1+\theta(m_1)+5}$, $|\mathbf{k}|^{m_1+\theta(m_1)+6},\ldots$. Since $m_1\geqslant 1$, the expansion (3.34) starts at least at the order $|\mathbf{k}|^2$ and the first nonanalytic term is at least of order $|\mathbf{k}|^5$. In other words, $\int d\mathbf{r}\,\mathfrak{K}_{a,1}^{[m_1]}(|\mathbf{r}|)=0$, the algebraic tails of $\mathfrak{K}_{a,1}^{[m_1]}(|\mathbf{r}|)$ are $1/r^{6+m_1+\theta(m_1)}$, $1/r^{8+m_1+\theta(m_1)}$, and $1/r^{N+m_1+\theta(m_1)}$ with $N\geqslant 9$ and the $\mathfrak{K}_{a,1}^{[m_1]}(|\mathbf{r}|)$'s fall off at least as $1/r^8$. Since $n_I\geqslant 1$, the same mechanism holds for $\mathfrak{K}_{I',b}^{[n_I]}(\mathbf{k})$. Similarly, the first nonanalytic term in the small-$\mathbf{k}$ expansion of $\mathfrak{K}_{i',i+1}^{[n_i,m_{i+1}]}\times(\mathbf{k})$ is of order $|\mathbf{k}|^{\gamma-3}$ with $\gamma\geqslant n_i+m_{i+1}+6$ because, according to (C39),

$$\begin{aligned}\mathfrak{K}_{i',i+1}^{[n_i,m_{i+1}]}(\mathbf{k})=&A_{i',i+1}^{[n_i,m_{i+1}]}|\mathbf{k}|^{n_i+m_{i+1}+\theta(n_i+m_{i+1})}\\&+B_{i',i+1}^{[n_i,m_{i+1}]}|\mathbf{k}|^{n_i+m_{i+1}+\theta(n_i+m_{i+1})+2}\\&+\cdots+S_{i',i+1,n_i,m_{i+1}}^{n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+3}(|\mathbf{k}|)\\&+\tilde{O}(|\mathbf{k}|^{n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+3}).\end{aligned} \tag{3.35}$$

$\theta(n_i)+\theta(m_{i+1})\geqslant\theta(n_i+m_{i+1})$, so that the first two terms in the small-$\mathbf{k}$ expansion of $\mathfrak{K}^{[n_i,m_{i+1}]}_{i',i+1}(\mathbf{k})$ are indeed analytic. In (3.35), the ellipsis represents the possible analytic term of order $n_i+m_{i+1}+\theta(n_i+m_{i+1})+4$ in the case where $\theta(n_i+m_{i+1})+4<\theta(n_i)+\theta(n_{i+1})+3$. Since $m_i\geqslant 1$ and $n_i\geqslant 1$, the expansion (3.35) starts at least at the order $|\mathbf{k}|^2$ and the first nonanalytic term is at least of order $|\mathbf{k}|^7$. The possible algebraic tails of $\mathfrak{K}^{[n_i,m_{i+1}]}_{i',i+1}(\mathbf{r})$ are given in (C38) and since the slowest tail is $1/r^{6+n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})}$, it falls off at least as $1/r^{10}$.

The first terms in the small-$\mathbf{k}$ expansion of a chain $\mathfrak{C}_I(\mathbf{k};\{m_i\},\{n_i\})$ given by (3.8) are derived from (3.34) and (3.35),

$$\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})=A^{[m_1]}_{a,1}\left(\prod_{i=1}^{I-1}A^{[n_i,m_{i+1}]}_{i',i+1}\right)A^{[n_I]}_{I',b}|\mathbf{k}|^{D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})}+\left(\sum A\cdots ABA\cdots A\right)|\mathbf{k}|^{D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})+2}+o(|\mathbf{k}|^{D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})+2}) \tag{3.36}$$

where $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ is given by (3.16) and $\Sigma A\cdots ABA\cdots A$ is a notation for a sum of products involving I coefficients $A_{i',i+1}$ and one coefficient $B_{i',i+1}$ (with $i'=a,1,\ldots,I'$ and $i+1=1,\ldots,I,b$). Since $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ is positive and even, the first two terms in the small-$\mathbf{k}$ expansion of $\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ are analytic and at least of order $|\mathbf{k}|^{D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})}$ and $|\mathbf{k}|^{D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})+2}$, respectively. The first singular term can only appear at the order $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})+3$. Since $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})\geqslant 2$, the first nonanalytic term is at least of order $|\mathbf{k}|^5$, namely, the convolutions decay at least as $1/r^8$.

More precisely, the dimension $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})$ takes its minimal value $D_{I,\min}=2$ [namely, the small-$\mathbf{k}$ expansion of $\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ starts at the order $|\mathbf{k}|^2$] for the convolutions with $n_i=m_{i+1}=1$ for all $i=1,\ldots,I-1$ and (m_1,n_I) being equal to one of the four values

$$(m_1=1,n_I=1),\ (m_1=2,n_I=1),\ (m_1=1,n_I=2),\ (m_1=2,n_I=2). \tag{3.37}$$

For these convolutions, according to (3.34) and (3.35)

$$\begin{aligned}\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})=&\left(\frac{1}{\mathbf{k}^2}\right)^I[A^{[1]\text{or}[2]}_{a,1}|\mathbf{k}|^2+B^{[1]\text{or}[2]}_{a,1}|\mathbf{k}|^4\\&+S^{(5)}_{a,1}(|\mathbf{k}|)+\cdots+S^{(7)}_{a,1}(|\mathbf{k}|)+S^{(8)}_{a,1}(|\mathbf{k}|)+\widetilde{O}(|\mathbf{k}|^8)]\\&\times[A^{[1]\text{or}[2]}_{I',b}|\mathbf{k}|^2+B^{[1]\text{or}[2]}_{I',b}|\mathbf{k}|^4+S^{(5)}_{I',b}(|\mathbf{k}|)+\cdots+S^{(7)}_{I',b}(|\mathbf{k}|)+S^{(8)}_{I',b}(|\mathbf{k}|)+\widetilde{O}(|\mathbf{k}|^8)]\\&\times\prod_{i=1}^{I-1}[A^{[1,1]}_{i',i+1}|\mathbf{k}|^2+B^{[1,1]}_{i',i+1}|\mathbf{k}|^4+\cdots+S^{(7)}_{i',i+1}(|\mathbf{k}|)+S^{(8)}_{i',i+1}(|\mathbf{k}|)+\widetilde{O}(|\mathbf{k}|^8)],\end{aligned} \tag{3.38}$$

with the same notations as in (3.34) and (3.35). In (3.38), we have omitted the analytic terms in order to point out the singular ones and we have simplified the notation of the singular terms. The first three nonanalytic terms in the expansion of (3.38) are of order $|\mathbf{k}|^5$, $|\mathbf{k}|^7$, and $|\mathbf{k}|^8$ and the corresponding chains have algebraic tails decaying as $1/r^8$, $1/r^{10}$, $1/r^{11}$,... .

The case $D_{\mathfrak{C}_I}(\{m_i\},\{n_i\})=4$ corresponds to two kinds of chains, which prove to decay at least as $1/r^{10}$. In the first case, $n_i=m_{i+1}=1$ for all $i=1,\ldots,I-1$ and (m_1,n_I) is equal to one of the eight values

$$(3,1),(4,1),(3,2),(4,2),(1,3),(1,4),(2,3),(2,4) \tag{3.39}$$

According to (3.34),

$$\mathfrak{K}^{[3]\text{or}[4]}_{a,1}(\mathbf{k})=A_{a,1}|\mathbf{k}|^4+B_{a,1}|\mathbf{k}|^6+\cdots+S^{(7)}_{a,1}(|\mathbf{k}|)+\cdots+S^{(9)}_{a,1}(|\mathbf{k}|)+S^{(10)}_{a,1}(|\mathbf{k}|)+\widetilde{O}(|\mathbf{k}|^{10}), \tag{3.40}$$

where we have omitted the superscript [3] or [4] and the analytic terms. Thus, according to (3.38) with $\mathfrak{K}^{[3]\text{or}[4]}_{a,1}(\mathbf{k})$ in place of $\mathfrak{K}^{[1]\text{or}[2]}_{a,1}(\mathbf{k})$, the first three singular terms in $\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ are of order $|\mathbf{k}|^7$, $|\mathbf{k}|^9$, and $|\mathbf{k}|^{10}$, respectively.

In the second case, $n_i=m_{i+1}=1$ for all $i=1,\ldots,I-1$, except i_0, for which (n_{i_0},m_{i_0+1}) is equal to (1,2), (1,3), (2,1), (3,1), or (2,2), while (m_1,n_L) is equal to one of the four values in (3.37). According to (3.35)

$$\mathfrak{K}^{[1,2]\text{or}[2,2]}_{i',i+1}(\mathbf{k})=A_{i',i+1}|\mathbf{k}|^4+B_{i',i+1}|\mathbf{k}|^6+\cdots+S^{(7)}_{i',i+1}(|\mathbf{k}|)+\cdots+S^{(9)}_{i',i+1}(|\mathbf{k}|)+\widetilde{O}(|\mathbf{k}|^9), \tag{3.41a}$$

while

$$\mathfrak{K}^{[1,3]}_{i',i+1}(\mathbf{k})=A_{i',i+1}|\mathbf{k}|^4+B_{i',i+1}|\mathbf{k}|^6+\cdots+C_{i',i+1}|\mathbf{k}|^8+\cdots+S^{(9)}_{i',i+1}(|\mathbf{k}|)+\widetilde{O}(|\mathbf{k}|^9). \tag{3.41b}$$

According to (3.38) with one $\mathfrak{K}^{[1,1]}_{i',i+1}(\mathbf{k})$ replaced by a $\mathfrak{K}^{[1,2],[2,2],\text{or}[1,3]}_{i',i+1}(\mathbf{k})$ and (3.34) with $m_1=1,2$, the first singular terms in $\mathfrak{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ are of order $|\mathbf{k}|^7$, $|\mathbf{k}|^9$, $|\mathbf{k}|^{10}$,... . In both cases, in position space, the chain may decay as

$1/r^{10}$,$1/r^{12}$,$1/r^{\gamma}$ with $\gamma \geqslant 13$. As for the chains corresponding to $D_{\mathfrak{C}_I}=6$, since the first two terms of the small-$\mathbf{k}$ expansion of a chain are analytic [see (3.36)], these chains decay at least as $1/r^{12}$.

As a conclusion, the convolution chains of bonds W and graphs K have algebraic tails decaying as $1/r^8$, $1/r^{10}$, $1/r^{11}$, and so on. Since K involves algebraic tails $1/r^6$, $1/r^8$, $1/r^9$,..., according to (2.6), the algebraic tails of $\rho^{(2)TQ}(r)$ are only $1/r^6$, $1/r^8$, $1/r^9$, and $1/r^{\gamma}$, with $\gamma \geqslant 10$.

As an introduction to next section, we notice that, since $\int_0^{p'} d\tau' e^{-i\mathbf{k}\cdot\mathbf{X}'(\tau')}\rho^{(2)TQ}_{\alpha,\alpha'}(\mathbf{k},\chi,\chi')$ has the same global rotational invariance property as $\rho^{(2)TQ}_{\alpha,\alpha'}(\mathbf{k},\chi,\chi')$ and since they coincide, up to a p factor, at $\mathbf{k}=\mathbf{0}$, the same discussion as that developed for the nonexchange part of the particle-particle correlation can be applied to

$$\sum_{p=1}^{\infty} p \sum_{p'=1}^{\infty} \int D(\mathbf{X}) \int D(\mathbf{X}') \rho^{(2)T}_{\alpha,p;\alpha,p'}(\mathbf{k},\mathbf{X},\mathbf{X}') \int_0^{p'} d\tau' \times e^{-i\mathbf{k}\cdot\mathbf{X}'(\tau')}$$

and the latter term proves to have its first singular term at least at the order $|\mathbf{k}|^3$. Subsequently, according to the linear response relation (4.27) of paper I, the induced charge density in the presence of an infinitesimal external charge decays at least as $1/r^4$. However, as shown in Sec. IV E, the induced charge density has a faster falloff because of Coulomb screening.

IV. CHARGE-CHARGE CORRELATION AND INDUCED CHARGE DENSITY

A. Classical case

First, we investigate the classical screening mechanism for the induced charge density in the presence of an external infinitesimal charge. This mechanism is exhibited by a suitable reorganization of the resummed Mayer-Meeron diagrammatics for the classical system.

As recalled at the end of Sec. V C of paper I, the classical prototype diagrams $\Pi^{\rm cl}$ are built with two kinds of bonds $F^{cc}_{\rm DH}(r_{ij},\alpha_i,\alpha_j)=-\beta_{ij}\phi_{\rm DH}(r_{ij})$ and $F_{R,{\rm DH}}(r_{ij},\alpha_i,\alpha_j)=\exp[-\beta_{ij}\phi_{\rm DH}(r_{ij})]-1+\beta_{ij}\phi_{\rm DH}(r_{ij})$, where $\beta_{ij}\equiv\beta e_{\alpha_i}e_{\alpha_j}$. (We recall that $\phi_{\rm DH}$ has the same form as ϕ with the inverse Debye screening length $\kappa_{\rm DH}$ in place of κ, $\kappa^2_{\rm DH}=4\pi\beta\Sigma_\alpha e^2_\alpha\rho_\alpha$.) Moreover, the $\Pi^{\rm cl}$ diagrams must satisfy the following excluded-convolution rule: there cannot exist convolution chains $F^{cc}_{\rm DH}*F^{cc}_{\rm DH}$. Let us introduce two kinds of root points. A root point is called a ''Coulomb-root'' point if it is involved in one and only one $F^{cc}_{\rm DH}$ bond and it is called a ''non-Coulomb-root'' point in the other cases, namely, if it is involved either in one $F_{R,{\rm DH}}$ bond or in at least two bonds, whatever they are. The sum of the $\Pi^{\rm cl}$ diagrams whose both root points are non-Coulomb-root points is denoted by $h^{nn}_{\alpha_a\alpha_b}(r)$. By definition of the non-Coulomb root points, the classical Ursell function $h^{\rm cl}_{\alpha_a\alpha_b}(r)$ can be decomposed as the sum

$$\begin{aligned} h^{\rm cl}_{\alpha_a\alpha_b}(\mathbf{r}_a-\mathbf{r}_b) &= F^{cc}_{\rm DH}(\mathbf{r}_a-\mathbf{r}_b,\alpha_a,\alpha_b)+h^{nn}_{\alpha_a\alpha_b}(\mathbf{r}_a-\mathbf{r}_b) \\ &+\sum_{\alpha'_a}\rho_{\alpha'_a}\int d\mathbf{r}'_a F^{cc}_{\rm DH}(\mathbf{r}_a-\mathbf{r}'_a,\alpha_a,\alpha'_a)h^{nn}_{\alpha'_a\alpha_b}(\mathbf{r}'_a-\mathbf{r}_b) \\ &+\sum_{\alpha'_b}\rho_{\alpha'_b}\int d\mathbf{r}'_b h^{nn}_{\alpha_a\alpha'_b}(\mathbf{r}_a-\mathbf{r}'_b)F^{cc}_{\rm DH}(\mathbf{r}'_b-\mathbf{r}_b,\alpha'_b,\alpha_b) \\ &+\sum_{\alpha'_a}\rho_{\alpha'_a}\int d\mathbf{r}'_a\sum_{\alpha'_b}\rho_{\alpha'_b}\int d\mathbf{r}'_b F^{cc}_{\rm DH}(\mathbf{r}_a-\mathbf{r}'_a,\alpha_a,\alpha'_a)h^{nn}_{\alpha'_a\alpha'_b}(\mathbf{r}'_a-\mathbf{r}'_b)F^{cc}_{\rm DH}(\mathbf{r}'_b-\mathbf{r}_b,\alpha'_b,\alpha_b). \end{aligned} \tag{4.1}$$

Equation (4.1) can be written in a more compact form that involves only one convolution (where the convolution operates on the position variable) as

$$\begin{aligned} &\rho_{\alpha_a}\rho_{\alpha_b}[h^{\rm cl}_{\alpha_a\alpha_b}(r)-F^{cc}_{\rm DH}(r,\alpha_a,\alpha_b)] \\ &=\rho_{\alpha_a}\rho_{\alpha_b}\sum_{\alpha'_a}\sum_{\alpha'_b}[\delta_{\alpha_a,\alpha'_a}\delta(\mathbf{r}) +\rho_{\alpha'_a}F^{cc}_{\rm DH}(\mathbf{r},\alpha_a,\alpha'_a)]*h^{nn}_{\alpha'_a\alpha'_b}(\mathbf{r})*[\delta_{\alpha'_b,\alpha_b}\delta(\mathbf{r}) +\rho_{\alpha'_b}F^{cc}_{\rm DH}(\mathbf{r},\alpha'_b,\alpha_b)] \\ &=\sum_{\alpha'_a,\alpha'_b}\rho_{\alpha'_a}\rho_{\alpha'_b}\Sigma^{\rm cl}_{\rm DH}(\mathbf{r},\alpha_a|\alpha'_a)*h^{nn}_{\alpha'_a\alpha'_b}(\mathbf{r})*\Sigma^{\rm cl}_{\rm DH}(\mathbf{r},\alpha_b|\alpha'_b), \end{aligned} \tag{4.2}$$

where

$$\Sigma^{\rm cl}_{\rm DH}(\mathbf{r}_a-\mathbf{r}'_a,\alpha_a|\alpha'_a)\equiv\delta_{\alpha_a,\alpha'_a}\delta(\mathbf{r}_a-\mathbf{r}'_a)+\rho_{\alpha_a}\times F^{cc}_{\rm DH}(\mathbf{r}_a-\mathbf{r}'_a,\alpha_a,\alpha'_a) \tag{4.3}$$

is the Debye approximation of the density $\Sigma^{\rm cl}(\mathbf{r}_a-\mathbf{r}'_a,\alpha_a|\alpha'_a)$ of particles of species α_a in the polarization cloud around a particle of species α'_a, $\Sigma^{\rm cl}(\mathbf{r}_a-\mathbf{r}'_a,\alpha_a|\alpha'_a)\equiv\delta_{\alpha_a,\alpha'_a}\delta(\mathbf{r}_a-\mathbf{r}'_a)+\rho_{\alpha_a}h^{\rm cl}_{\alpha_a\alpha'_a}(\mathbf{r}_a-\mathbf{r}'_a)$.

The charge density of the system made by a particle of species α'_a and its polarization cloud is $\Sigma_{\alpha_a}e_{\alpha_a}\Sigma^{\rm cl}(\mathbf{r}_a-\mathbf{r}'_a,\alpha_a|\alpha'_a)$ and the Fourier transform of its Debye approximation reads

$$\sum_{\alpha_a} e_{\alpha_a}\Sigma^{\rm cl}_{\rm DH}(\mathbf{k},\alpha_a|\alpha'_a)=e_{\alpha'_a}\left[1-4\pi\beta\,\frac{(\Sigma_{\alpha_a}e^2_{\alpha_a}\rho_{\alpha_a})}{\kappa^2_{\rm DH}+\mathbf{k}^2}\right]$$
$$=e_{\alpha'_a}\frac{\mathbf{k}^2}{\kappa^2_{\rm DH}+\mathbf{k}^2}. \qquad (4.4)$$

The Debye approximation satisfies the screening rule (1.2) of paper I: the net charge of the polarization cloud around a charge of the medium exactly compensates this charge.

Moreover, the charge-charge correlation defined in (4.30) of paper I can be written as $C^{\rm cl}(\mathbf{r})=\Sigma_{\alpha'_a}e_{\alpha'_a}\rho_{\alpha'_a}\Sigma_{\alpha_a}e_{\alpha_a}\Sigma^{\rm cl}(\mathbf{r},\alpha_a|\alpha'_a)$ and the Fourier transform of its Debye approximation is equal to

$$C^{\rm cl}_{\rm DH}(\mathbf{k})=\frac{\kappa^2_{\rm DH}}{4\pi\beta}\frac{\mathbf{k}^2}{\kappa^2_{\rm DH}+\mathbf{k}^2}. \qquad (4.5)$$

The small-$\mathbf{k}$ expansion of $C^{\rm cl}_{\rm DH}(\mathbf{k})$ starts as $(1/4\pi\beta)\mathbf{k}^2$, so that, according to the classical response relation (4.36) of paper I, the total Debye induced charge $\int d\mathbf{r}\Sigma_\alpha e_\alpha\rho^{\rm ind}_{\alpha,\rm DH}(\mathbf{r})$ in the presence of an infinitesimal external charge δq is exactly opposite to δq. In other words, $C^{\rm cl}_{\rm DH}(r)$ also satisfies the basic screening rule (1.3) of paper I.

The Fourier transform of the classical charge-charge correlation $C^{\rm cl}(\mathbf{r})$ can be decomposed as the sum of the contributions from the Debye correlation $\rho_{\alpha_a}\rho_{\alpha_b}F^{cc}_{\rm DH}(r,\alpha_a,\alpha_b)$ and from $\rho_{\alpha_a}\rho_{\alpha_b}[h^{\rm cl}_{\alpha_a\alpha_b}(r)-F^{cc}_{\rm DH}(r,\alpha_a\alpha_b)]$ given by (4.2). After exchange of the order of summations over α_a and α'_a, on one hand, and α_b and α'_b, on the other hand, the contributions (4.4) factor out and

$$C^{\rm cl}(\mathbf{k})=C^{\rm cl}_{\rm DH}(\mathbf{k})+\left(\frac{\mathbf{k}^2}{\kappa^2_{\rm DH}+\mathbf{k}^2}\right)^2$$
$$\times\sum_{\alpha'_a,\alpha'_b}e_{\alpha'_a}e_{\alpha'_b}\rho_{\alpha'_a}\rho_{\alpha'_b}h^{nn}_{\alpha'_a\alpha'_b}(\mathbf{k}). \qquad (4.6)$$

The small-$\mathbf{k}$ expansion of $h^{nn}_{\alpha'_a\alpha'_b}(|\mathbf{k}|)$ is an analytic function of $\mathbf{k}^2$ because all the $\Pi^{\rm cl}$ diagrams decay exponentially. $h^{nn}(\mathbf{k}=\mathbf{0})$ is finite and the small-$\mathbf{k}$ expansion of $C^{\rm cl}(\mathbf{k})-C^{\rm cl}_{\rm DH}(\mathbf{k})$ starts only at the order $|\mathbf{k}|^4$. Eventually, the first term in the small-$\mathbf{k}$ expansion of $C^{\rm cl}(\mathbf{k})$ is given by the corresponding term in $C^{\rm cl}_{\rm DH}(\mathbf{k})$, so that, according to (4.5), both the Debye approximated correlation and the exact classical correlation obey the Stillinger-Lovett sum rule (4.42) of paper I, which is the classical version of the general sum rule (1.3) of paper I. We note that a consequence of the previous discussion is that any approximated particle-particle correlation that is devised by replacing h^{nn} by an approximated function in the relation (4.1) satisfies the Stillinger-Lovett sum rule by construction.

B. Basic mechanisms in the quantum case

In the quantum case, the charge-charge correlation $C^Q(\mathbf{k})$ can be decomposed into the sum of two terms: the Fourier transform of a short-ranged contribution given by (4.29) and (5.14) of paper I,

$$\sum_\alpha e^2_\alpha\rho_\alpha+\sum_{\alpha_a,\alpha_b}e_{\alpha_a}e_{\alpha_b}\rho^{(2)TQ}_{\alpha_a\alpha_b}|_{\rm exch}(\mathbf{k})$$
$$=\frac{\kappa^2}{4\pi\beta}+\int d\chi_a e^2_{\alpha_a}p_a\rho(\chi_a)\int_0^{p_a}d\tau[e^{-i\mathbf{k}\cdot\mathbf{X}_a[P(\tau)]}-1], \qquad (4.7)$$

where $\kappa^2=4\pi\beta\int d\chi_a e^2_{\alpha_a}p^2_a\rho(\chi_a)$ and the contribution from the correlations between particles that are not exchanged under the same cyclic permutation,

$$C^Q|_{\rm nonexch}(\mathbf{k})\equiv\sum_{\alpha_a,\alpha_b}e_{\alpha_a}e_{\alpha_b}\rho^{(2)TQ}_{\alpha_a\alpha_b}|_{\rm nonexch}(\mathbf{k}). \qquad (4.8)$$

In Sec. IV C we will show that the part $C^Q|_{\rm nonexch}(\mathbf{r})$ of the charge-charge correlation decays only as $1/r^{10}$, whereas the particle-particle correlation decays as $1/r^6$. In Fourier space, this means that the order of the first nonanalytic term in the small-$\mathbf{k}$ expansion of $C^Q(\mathbf{k})$ is increased by 4 with respect to the corresponding order for $\rho^{(2)TQ}_{\alpha_a\alpha_b}(\mathbf{k})$. The basic mechanisms are of two kinds.

First, in the loop system as in the classical particle system, the sum $\Pi+F^{cc}\rho*\Pi$ (where the convolution operates on the loop-position variable and ρ is the density of the intermediate point of the convolution) gives a contribution to $C^Q|_{\rm nonexch}(\mathbf{r})$, the Fourier transform of which is proportional to $\mathbf{k}^2\times\int d\chi'_a\int d\chi_b e_{\alpha'_a}p'_a\rho(\chi'_a)e_{\alpha_b}p_b\rho(\chi_b)\Pi(\mathbf{k},\chi'_a,\chi_b)$. Indeed, in position space, the contribution reads

$$\sum_{\alpha_a}\sum_{\alpha_b}e_{\alpha_a}e_{\alpha_b}p_ap_b\int D(\mathbf{X}_a)\rho(\mathbf{X}_a)\int D(\mathbf{X}_b)\rho(\mathbf{X}_b)$$
$$\times\Big[\Pi(\mathcal{L}_a,\mathcal{L}_b)+\int d\mathcal{L}'_a\rho(\mathcal{L}'_a)$$
$$\times F^{cc}(\mathcal{L}_a,\mathcal{L}'_a)\Pi(\mathcal{L}'_a,\mathcal{L}_b)\Big]$$
$$=\int d\chi_a e_{\alpha_a}p_a\int d\chi_b e_{\alpha_b}p_b\rho(\chi_b)\int d\mathbf{R}'_a\int d\chi'_a$$
$$\times\rho(\chi'_a)\Sigma_D(\mathbf{R}_a-\mathbf{R}'_a,\chi_a|\chi'_a)\Pi(\mathcal{L}'_a,\mathcal{L}_b), \qquad (4.9)$$

where $\Sigma_D(\mathbf{R}_a-\mathbf{R}'_a,\chi_a|\chi'_a)$ is the loop-cloud density around a loop $\mathcal{L}'_a$ in some kind of ''Debye'' approximation where $h(\mathcal{L}_a,\mathcal{L}_b)$ is replaced by $F^{cc}(\mathcal{L}_a,\mathcal{L}_b)$. The loop-cloud density around a loop $\mathcal{L}'_a$ is defined by analogy with (4.3), $\Sigma(\mathbf{R}_a-\mathbf{R}'_a,\chi_a|\chi'_a)\equiv\delta_{\chi_a,\chi'_a}\delta(\mathbf{R}_a-\mathbf{R}'_a)+\rho(\chi_a)h(\mathcal{L}_a,\mathcal{L}'_a)$, and

$$\Sigma_D(\mathbf{R}_a-\mathbf{R}'_a,\chi_a|\chi'_a)=\delta_{\chi_a,\chi'_a}\delta(\mathbf{R}_a-\mathbf{R}'_a)$$
$$+\rho(\chi_a)F^{cc}(\mathcal{L}_a,\mathcal{L}'_a). \qquad (4.10)$$

As in the classical case [see Eq. (4.4)], the total charge of the loop and its Debye polarization cloud is zero and, after averaging over the shapes of the loops, the charge density of this system satisfies

$$\int d\chi_a e_{\alpha_a} p_a \Sigma_D(\mathbf{k},\chi_a|\chi'_a) = e_{\alpha'_a} p'_a \int d\chi_a \left[\delta_{\chi'_a,\chi_a} - \frac{4\pi\beta e^2_{\alpha_a} p^2_a \rho(\chi_a)}{\kappa^2+\mathbf{k}^2} \right] = e_{\alpha'_a} p'_a \frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}^2}. \quad (4.11)$$

The structure of the Fourier transform of the convolution (4.9) is very similar to that of the convolution $\Sigma_{\alpha_a} e_{\alpha_a} \Sigma^{\mathrm{cl}}(\mathbf{r},\alpha_a|\alpha'_a) * \Pi^{\mathrm{cl}}(\mathbf{r},\alpha'_a|\alpha_b)$. After exchange of the integrations over χ_a and χ'_a, according to (4.11),

$$\int d\chi_a e_{\alpha_a} p_a \rho(\chi_a) \int d\chi_b e_{\alpha_b} p_b \rho(\chi_b) \int d\chi'_a \Sigma_D(\mathbf{k},\chi'_a|\chi_a) \Pi(\mathbf{k},\chi'_a,\chi_b)$$
$$= \frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}^2} \times \int d\chi'_a e_{\alpha'_a} p'_a \rho(\chi'_a) \int d\chi_b e_{\alpha_b} p_b \rho(\chi_b) \Pi(\mathbf{k},\chi'_a,\chi_b), \quad (4.12)$$

because $\rho(\chi_a)\Sigma_D(\mathbf{k},\chi'_a|\chi_a) = \rho(\chi'_a)\Sigma_D(\mathbf{k},\chi_a|\chi'_a)$. An analogous mechanism takes place in the case of the induced charge density and will be detailed in Sec. IV F.

The other partial-screening mechanism involved in the falloff from the exponent 6 to the exponent 10 lies in the fact that the diagrams that have the structure $F^{cm}\rho*\Pi$ and $F^{cm}\rho*\Pi*\rho F^{mc}$ decay as $1/r^8$ and $1/r^{10}$, respectively. Indeed, the Fourier transforms of these diagrams are proportional to an analytic function of $\mathbf{k}$ [namely, $\phi(\mathbf{k})$] times

$$\int D(\mathbf{X}_a)\rho(\chi_a) \int D(\mathbf{X}_b)\rho(\chi_b)[e^{i\mathbf{k}\cdot\mathbf{X}_a}-1]\Pi(\mathbf{k},\chi_a,\chi_b) \quad (4.13)$$

and

$$\int D(\mathbf{X}_a)\rho(\chi_a) \int D(\mathbf{X}_b)\rho(\chi_b)[e^{i\mathbf{k}\cdot\mathbf{X}_a}-1] \times [e^{-i\mathbf{k}\cdot\mathbf{X}_b}-1]\Pi(\mathbf{k},\chi_a,\chi_b), \quad (4.14)$$

respectively. As in the case of the particle-particle correlation, the Dyson equation (2.6) can be used. According to Appendix C, rotational invariance arguments and the structure (3.29) of the algebraic tails T of the $\tilde{\Pi}$ diagrams imply that, if $\tilde{\Pi}$ is a $\tilde{\Pi}_{Wc}$ diagram, which decays at least as $1/r^6$, the ''dressed'' diagrams (4.13) and (4.14) fall off at least as $1/r^8$ and $1/r^{10}$, respectively, and the same is true for H. If $\tilde{\Pi}$ is a chain made with $I+1$ diagrams contributing to K linked by I bonds W, the discussion about the large-distance behavior of the chains can be adapted from that given in Sec. III B for the particle-particle correlation and the dressed diagrams (4.13) and (4.14) also prove to decay at least as $1/r^8$ and $1/r^{10}$, respectively, in the case of the above convolution chains.

Eventually, a suitable reorganization of the diagrams analogous to that performed in the classical case [see (4.2)] is introduced in next subsection. It leads to integral relations that allow one to show that $C^Q(r)$ behaves as $1/r^{10}$.

C. Integral relations

By analogy with the classical case described in Sec. IV A, we introduce two kinds of root points with the same terminology as above. In the quantum resummed Mayer-like diagrams, the excluded-convolution rules lead to the following definitions. A so-called Coulomb-root point $\mathscr{L}_a$ in a Π diagram is a root point that is involved either in one and only one F^{cc} bond or in one and only one bond $F^{cm}(\mathscr{L}_a,\mathscr{P}_i)$. A non-Coulomb-root point $\mathscr{L}_a$ is involved either in one bond F_R, or $F^{mc}(\mathscr{L}_a,\mathscr{P}_i)$, or in at least two bonds, whatever they are.

The basic integral relations that are useful for our purpose involve sums of graphs as follows. The loop Ursell function $h(\mathscr{L}_a,\mathscr{L}_b)$, which is the sum of the Π diagrams [as defined in (5.10) of paper I] may be decomposed into $h^{n-}(\mathscr{L}_a,\mathscr{L}_b)+h^{c-}(\mathscr{L}_a,\mathscr{L}_b)$, where $h^{n-}(\mathscr{L}_a,\mathscr{L}_b)$ is the sum of the Π diagrams where $\mathscr{L}_a$ is a non-Coulomb root point, whereas $\mathscr{L}_b$ is of any kind (Coulomb-root or non-Coulomb root point) and h^{c-} is the sum of the Π diagrams, where $\mathscr{L}_a$ is a Coulomb root point and $\mathscr{L}_b$ is a root point of any kind. With these definitions, h^{c-} may be written as the sum of four contributions: the two diagrams with a single bond $F^{cc}(\mathscr{L}_a,\mathscr{L}_b)$ and $F^{cm}(\mathscr{L}_a,\mathscr{L}_b)$, and the sums of the diagrams $[F^{cc}\rho*\Pi](\mathscr{L}_a,\mathscr{L}_b)$ and $[F^{cm}\rho*\Pi](\mathscr{L}_a,\mathscr{L}_b)$ in which $\mathscr{L}_a$ is involved in only one bond $F^{cc}(\mathscr{L}_a,\mathscr{L}'_a)$ or $F^{mc}(\mathscr{L}_a,\mathscr{L}'_a)$ and where $\mathscr{L}'_a$ is linked to $\mathscr{L}_b$ by a subdiagram that is also a Π diagram with respect to the root points $\mathscr{L}'_a$ and $\mathscr{L}_b$. According to the excluded-convolution rules, if $\mathscr{L}'_a$ is linked to $\mathscr{L}_a$ by a bond $F^{cc}(\mathscr{L}_a,\mathscr{L}'_a)$, $\mathscr{L}'_a$ is a non-Coulomb root point for the subdiagram $\Pi(\mathscr{L}'_a,\mathscr{L}_b)$ and the corresponding sum of diagrams linking $\mathscr{L}_a$ to $\mathscr{L}_b$ is equal to the convolution $\rho(\chi'_a)F^{cc}(\mathscr{L}_a,\mathscr{L}'_a)*h^{n-}(\mathscr{L}'_a,\mathscr{L}_b)$. If $\mathscr{L}'_a$ is linked to $\mathscr{L}_a$ by a bond $F^{cm}(\mathscr{L}_a,\mathscr{L}'_a)$, $\mathscr{L}'_a$ is a root point of any kind for the subdiagram $\Pi(\mathscr{L}'_a,\mathscr{L}_b)$, and the corresponding sum of diagrams linking $\mathscr{L}_a$ to $\mathscr{L}_b$ is equal to the convolution $\rho(\chi'_a)F^{cm}(\mathscr{L}_a,\mathscr{L}'_a)*h(\mathscr{L}'_a,\mathscr{L}_b)$. Eventually, we get an integral relation involving h and h^{n-},

$$h(\mathscr{L}_a,\mathscr{L}_b) = h^{n-}(\mathscr{L}_a,\mathscr{L}_b) + F^{cc}(\mathscr{L}_a,\mathscr{L}_b) + \int d\mathscr{L}'_a \rho(\mathscr{L}'_a) F^{cc}(\mathscr{L}_a,\mathscr{L}'_a) h^{n-}(\mathscr{L}'_a,\mathscr{L}_b) + F^{cm}(\mathscr{L}_a,\mathscr{L}_b) + \int d\mathscr{L}'_a \rho(\mathscr{L}'_a) \times F^{cm}(\mathscr{L}_a,\mathscr{L}'_a) h(\mathscr{L}'_a,\mathscr{L}_b). \quad (4.15)$$

This ''left-dressing'' relation can be written in a more compact form by using the short notation $g\rho*$, where ρ is the density of the intermediate point of the convolution,

$$h=F^{cc}+F^{cm}+h^{n-}+F^{cc}\rho*h^{n-}+F^{cm}\rho*h$$
$$=F^{cc}+F^{cm}+\Sigma_D*h^{n-}+F^{cm}\rho*h. \tag{4.16}$$

Σ_D is defined in (4.10) and we use the convention that $[\Sigma_D*F^{mc}](\mathscr{L}_a,\mathscr{L}_b) = \int d\mathscr{L}'_a\Sigma_D(\mathscr{L}'_a|\mathscr{L}_a)F^{mc}(\mathscr{L}'_a,\mathscr{L}_b)$, while $[F^{cm}*\Sigma_D](\mathscr{L}_a,\mathscr{L}_b)=\int d\mathscr{L}'_aF^{cm}(\mathscr{L}_a,\mathscr{L}'_a)\times\Sigma_D(\mathscr{L}'_a|\mathscr{L}_b)$. Of course, there also exists the symmetric relation of (4.16). With the short notation $*\rho g$, where ρ is the density of the intermediate point of the convolution, the ''right-dressing'' relation between h and h^{-n} reads

$$h=F^{cc}+F^{mc}+h^{-n}*\Sigma_D+h*\rho F^{mc}. \tag{4.17}$$

By iterating the argument that leads to (4.17), we get an integral relation between h^{n-} and h^{nn}, where $h^{nn}(\mathscr{L}_a,\mathscr{L}_b)$ is the sum of the Π diagrams where both $\mathscr{L}_a$ and $\mathscr{L}_b$ are non-Coulomb root points. The right-dressing relation between h^{n-} and h^{nn} is

$$h^{n-}(\mathscr{L}_a,\mathscr{L}_b)=h^{nn}(\mathscr{L}_a,\mathscr{L}_b)+F^{mc}(\mathscr{L}_a,\mathscr{L}_b)+\int d\mathscr{L}'_b\rho(\mathscr{L}'_b)h^{nn}(\mathscr{L}_a,\mathscr{L}'_b)F^{cc}(\mathscr{L}'_b,\mathscr{L}_b)+\int d\mathscr{L}'_b\rho(\mathscr{L}'_b)h^{n-}(\mathscr{L}_a,\mathscr{L}'_b)F^{mc}(\mathscr{L}'_b,\mathscr{L}_b). \tag{4.18}$$

With short notations, (4.18) reads

$$h^{n-}=F^{mc}+h^{nn}*\Sigma_D+h^{n-}*\rho F^{mc} \tag{4.19}$$

and the symmetric (left-dressing) relation is

$$h^{-n}=F^{cm}+\Sigma_D*h^{nn}+F^{cm}\rho*h^{-n}. \tag{4.20}$$

D. Algebraic tails of the charge-charge correlation

The diagrams contributing to $h(\mathscr{L}_a,\mathscr{L}_b)$ can be reorganized by applying the right-dressing relation (4.17) to h in the term $F^{cm}\rho*h$ of the left-dressing formula (4.16) and by applying the right-dressing relation (4.19) to h^{n-} in Σ_D*h^{n-}. We get

$$h=h_{(A)}+h_{(B)}+h_{(C)}+h_{(D)}+h_{(E)}, \tag{4.21}$$

with the definitions

$$h_{(A)}=F^{cc}+F^{mc}+F^{cm}+F^{cc}\rho*F^{mc}+F^{cm}*\rho F^{cc}+F^{cm}*\rho F^{mc}$$
$$=F^{cc}+\Sigma_D*F^{mc}+F^{cm}*\Sigma_D+F^{cm}*\rho F^{mc}, \tag{4.22}$$

$$h_{(B)}=\Sigma_D*h^{nn}*\Sigma_D, \tag{4.23}$$

$$h_{(C)}=\Sigma_D*h^{n-}*\rho F^{mc}, \tag{4.24}$$

$$h_{(D)}=F^{cm}\rho*h^{-n}*\Sigma_D, \tag{4.25}$$

$$h_{(E)}=F^{cm}\rho*h*\rho F^{mc}. \tag{4.26}$$

In the following $C^Q_{(L)}(r)$, with $L=A,\ldots,E$, denotes the contribution from $h_{(L)}(\mathscr{L}_a,\mathscr{L}_b)$ to the nonexchange part $C^Q|_{\mathrm{nonexch}}(r)$ of the quantum charge-charge correlation function $C^Q(r)$ given by (4.8).

The diagrams in $h_{(A)}$ involve only exponentially decaying bonds and $h_{(A)}(\mathscr{L}_a,\mathscr{L}_b)$ decays faster than any inverse power law of the distance $|\mathbf{R}_a-\mathbf{R}_b|$. This result remains valid after integration over the shapes of the root points and $C^Q_{(A)}(r)$ has a fast falloff.

According to the screening property (4.12) of the Debye loop cloud Σ_D, the Fourier transform of $C^Q_{(B)}(r)$ reads

$$C^Q_{(B)}(\mathbf{k})=\left(\frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}^2}\right)^2\int d\chi'_a e_{\alpha'_a}p'_a\rho(\chi'_a)\times\int d\chi'_b e_{\alpha'_b}p'_b\rho(\chi'_b)h^{nn}(\mathbf{k},\chi'_a,\chi'_b). \tag{4.27}$$

According to Sec. III, $\int D(\mathbf{X}_1)\int D(\mathbf{X}_2)\Pi(\mathscr{P}_1,\mathscr{P}_2)$ decays at least as $1/r^6$, $1/r^8$, $1/r^9$,..., so that the first nonanalytic terms in $\int D(\mathbf{X}_a)\rho(\chi_a)\int D(\mathbf{X}_b)\rho(\chi_b)h^{nn}(\mathbf{k},\chi'_a,\chi'_b)$ are of order $|\mathbf{k}|^3$, $|\mathbf{k}|^5$, $|\mathbf{k}|^6$,... . Henceforth, the first nonanalytic terms in $C^Q_{(B)}(\mathbf{k})$ are of order $|\mathbf{k}|^7$, $|\mathbf{k}|^9$, $|\mathbf{k}|^{10}$,..., and $C^Q_{(B)}(r)$ decays as $1/r^{10}$, $1/r^{12}$, $1/r^{13}$,... .

By the same mechanism of Debye screening (4.12), the Fourier transform of $C^Q_{(C)}(r)$ reads

$$C^Q_{(C)}(\mathbf{k})=\frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}_2}\int d\chi'_a e_{\alpha'_a}p'_a\rho(\chi'_a)\int d\chi'_b e_{\alpha'_b}p'_b\times\rho(\chi'_b)h^{n-}(\mathbf{k},\chi'_a,\chi'_b)\int_0^{p'_b}d\tau\times[e^{i\mathbf{k}\cdot\mathbf{X}'_b(\tau)}-1]\,\frac{\kappa^2}{\kappa^2+\mathbf{k}^2}. \tag{4.28}$$

The decomposition (2.6) of h into H and convolution chains of graphs K linked by bonds W can be performed for h^{n-}, which is a partial sum of the graphs contributing to h,

$$h^{n-}(\mathbf{k},\chi_a,\chi_b)=H^{n-}(\mathbf{k},\chi_a,\chi_b)+\sum_{I=1}^{\infty}\int d\chi_1\cdots d\chi_I d\chi'_1\cdots d\chi'_I\times K^{n-}(\mathbf{k},\chi_a,\chi_1)W(\mathbf{k},\chi_1,\chi'_1)\times K(\mathbf{k},\chi'_1,\chi_2)\cdots W(\mathbf{k},\chi_I,\chi'_I)K(\mathbf{k},\chi'_I,\chi_b). \tag{4.29}$$

The functions relative to h^{n-} have the same notations as those corresponding to h, except that they carry an extra supercript $n-$. According to Appendix D, $C^Q_{(C)}$ and $C^Q_{(D)}$ decay as $1/r^{10}$, $1/r^{12}$, $1/r^{13}$,... .

The Fourier transform of the contribution $C^Q_{(E)}(r)$ from $h_{(E)}$ given by (4.26) reads

$$C^Q_{(E)}(\mathbf{k})=\int d\chi'_a e_{\alpha'_a}p'_a\rho(\chi'_a)\int d\chi'_b e_{\alpha'_b}p'_b\rho(\chi'_b)\int_0^{p'_a}d\tau\times\int_0^{p'_b}d\tau'\,\frac{\kappa^2}{\kappa^2+\mathbf{k}^2}\,[e^{-i\mathbf{k}\cdot\mathbf{X}'_a(\tau)}-1]h(\mathbf{k},\chi'_a,\chi'_b)\times[e^{i\mathbf{k}\cdot\mathbf{X}'_b(\tau')}-1]\,\frac{\kappa^2}{\kappa^2+\mathbf{k}^2}. \tag{4.30}$$

According to Appendix D, $C^Q_{(E)}(r)$ decays as $1/r^{10}$, $1/r^{11}$,... .

E. Algebraic tails of the induced charge

First, we notice that, according to (4.27) of paper I, the Fourier transform of the induced charge density in the presence of an infinitesimal external charge distribution $\delta q(\mathbf{r})$ can be decomposed as

$$\frac{1}{\delta q(\mathbf{k})}\sum_{\alpha} e_{\alpha}\rho_{\alpha}^{\text{ind}}(\mathbf{k})=-\frac{\kappa^2(\mathbf{k})}{\mathbf{k}^2}+\sum_{\alpha} e_{\alpha}\rho_{\alpha}^{\text{ind}}|_{\text{nonexch}}(\mathbf{k}), \tag{4.31}$$

with the function $\kappa^2(\mathbf{k})$ defined as

$$\frac{\kappa^2(\mathbf{k})}{4\pi\beta}\equiv\int d\chi\, e_{\alpha}^2 p\rho(\chi)\int_0^p d\tau\, e^{i\mathbf{k}\cdot\mathbf{X}(\tau)} \tag{4.32}$$

and $\kappa^2(\mathbf{k}=\mathbf{0})=\kappa^2$. The nonexchange part is

$$\sum_{\alpha} e_{\alpha}\rho_{\alpha}^{\text{ind}}|_{\text{nonexch}}(\mathbf{k})\equiv-\beta v_C(\mathbf{k})\int d\chi_a e_{\alpha_a}p_a\rho(\chi_a)\times\int d\chi_b e_{\alpha_b}\rho(\chi_b)\times\int_0^{p_b}d\tau\, e^{i\mathbf{k}\cdot\mathbf{X}_b(\tau)}h(\mathbf{k},\chi_a,\chi_b). \tag{4.33}$$

The value obtained by replacing h by a single bond F^{cc} in (4.33) is $\sum_{\alpha}e_{\alpha}\rho_{\alpha}^{\text{ind}}|_{\text{nonexch},D}(\mathbf{k})=\kappa^2\kappa^2(\mathbf{k})/[\mathbf{k}^2(\kappa^2+\mathbf{k}^2)]$, so that

$$-\frac{\kappa^2(\mathbf{k})}{\mathbf{k}^2}+\sum_{\alpha} e_{\alpha}\rho_{\alpha}^{\text{ind}}|_{\text{nonexch},D}(\mathbf{k})=-\frac{\kappa^2(\mathbf{k})}{\kappa^2+\mathbf{k}^2}=-1+O(|\mathbf{k}|^2). \tag{4.34}$$

The diagram F^{cc} ensures that the induced charge density exactly screens the infinitesimal external charge, though it does not contribute alone to the term of order $\mathbf{k}^2$ in $C^Q(\mathbf{k})$, as discussed in Sec. IV F.

The contribution from the convolution $\Pi*\Sigma_D$ to the Fourier transform of the charge-charge correlation $C^Q(\mathbf{k})$ starts at the order $|\mathbf{k}|^2$ and its contribution to the nonexchange part of the induced charge density $\sum_{\alpha}e_{\alpha}\rho_{\alpha}^{\text{ind}}|_{\text{nonexch}}(\mathbf{k})$ starts at the order zero. Indeed, this contribution reads

$$\frac{-4\pi\beta}{\mathbf{k}^2}\int d\chi_a e_{\alpha_a}p_a\rho(\chi_a)\int d\chi_b' e_{\alpha_b'}p_b'\rho(\chi_b')\Pi(\mathbf{k},\chi_a,\chi_b')\times\left\{\frac{1}{p_b'}\int_0^{p_b'}d\tau\, e^{i\mathbf{k}\cdot\mathbf{X}_b'(\tau)}-\frac{\kappa^2(\mathbf{k})}{\kappa^2+\mathbf{k}^2}\right\}. \tag{4.35}$$

The term in curly brackets in (4.35) can be written as the sum of $(1/p_b')\int_0^{p_b'}d\tau\exp[i\mathbf{k}\cdot\mathbf{X}_b'(\tau)-1]$ and $1-[\kappa^2(\mathbf{k})/(\kappa^2+\mathbf{k}^2)]$. Thus the contribution (4.35) is finite when $|\mathbf{k}|$ goes to zero, whereas the screening rule (1.3) of paper I is already ensured by the F^{cc} part of h. In fact, this constant term is compensated by the contribution from the diagram $\Pi*\rho F^{mc}$ and we rather consider the convolution $\Pi+\Pi*\rho F^{cc}+\Pi*\rho F^{mc}=\Pi*\Sigma_D^{\star}$, with

$$\Sigma_D^{\star}(\mathbf{R}_b-\mathbf{R}_b',\chi_b|\chi_b')\equiv\delta(\mathbf{R}_b-\mathbf{R}_b')\delta_{\chi_b',\chi_b}+\rho(\chi_b)F^{cc}(\mathscr{L}_b',\mathscr{L}_b)+\rho(\chi_b)F^{mc}(\mathscr{L}_b',\mathscr{L}_b). \tag{4.36}$$

$\Sigma_D^{\star}$ has a property similar to (4.11)

$$\int d\chi_b e_{\alpha_b}\int_0^{p_b}d\tau\, e^{i\mathbf{k}\cdot\mathbf{X}_b(\tau)}\Sigma_D^{\star}(\mathbf{k},\chi_b|\chi_b')=e_{\alpha_b'}\int_0^{p_b'}d\tau\, e^{i\mathbf{k}\cdot\mathbf{X}_b'(\tau)}\left[1-\frac{\kappa^2(\mathbf{k})}{\kappa^2+\mathbf{k}^2}\right]. \tag{4.37}$$

According to (4.32), the function (4.37) starts at the order $\mathbf{k}^2$ when $|\mathbf{k}|$ goes to zero. After exchange of the order of the summations over χ_b and χ_b', the contribution from $\Pi*\Sigma^{\star}$ to $\sum_{\alpha}e_{\alpha}\rho_{\alpha}^{\text{ind}}|_{\text{nonexch}}(\mathbf{k})$ reads

$$\frac{-4\pi\beta}{\mathbf{k}^2}\int d\chi_a e_{\alpha_a}p_a\rho(\chi_a)\int d\chi_b e_{\alpha_b}\rho(\chi_b)\int_0^{p_b}d\tau\, e^{-i\mathbf{k}\cdot\mathbf{X}_b(\tau)}\Pi(\mathbf{k},\chi_a,\chi_b')\Sigma_D^{*}(\mathbf{k},\chi_b'|\chi_b)=-\frac{4\pi\beta}{\mathbf{k}^2}\left[1-\frac{\kappa^2(\mathbf{k})}{\kappa^2+\mathbf{k}^2}\right]\int d\chi_a e_{\alpha_a}p_a\rho(\chi_a)\int d\chi_b' e_{\alpha_b'}\rho(\chi_b')\int_0^{p_b'}d\tau\, e^{-i\mathbf{k}\cdot\mathbf{X}_b'(\tau)}\Pi(\mathbf{k},\chi_a,\chi_b'), \tag{4.38}$$

because $\rho(\chi_b)\Sigma_D^{*}(\mathbf{k},\chi_b'|\chi_b)=\rho(\chi_b')\Sigma_D^{*}(\mathbf{k},\chi_b|\chi_b')$.

In order to study both the first term and the first nonanalytic term in $\sum_{\alpha}e_{\alpha}\rho_{\alpha}^{\text{ind}}|_{\text{nonexch}}(\mathbf{k})$ by using the property (4.38), we introduce a decomposition of h that is different from that exhibited in (4.21) for the contribution to $C^Q(\mathbf{k})$. In a recurrence scheme, we write the right-dressing relation (4.17) for h, then we use it again for $h*\rho F^{mc}$ in (4.17), and we get

$$h=F^{cc}+\Sigma_D*F^{mc}+F^{mc}*\rho F^{mc}+h^{-n}*\Sigma_D^{\star}+h^{-n}*\rho F^{cc}*\rho F^{mc}+h*\rho F^{mc}*\rho F^{mc}. \tag{4.39}$$

Then we apply the left-dressing relation (4.20) to $h^{-n}*\Sigma_D^\star$, $h^{-n}*\rho F^{cc}*\rho F^{mc}$, and $h*\rho F^{mc}*\rho F^{mc}$ in (4.39) with the result

$$h=F^{cc}+h_{(A^\star)}+h_{(B^\star)}+h_{(C^\star)}, \tag{4.40}$$

with

$$h_{(A^\star)}=\Sigma_D*\{F^{mc}+F^{mc}*\rho F^{mc}\}+F^{cm}*\{\Sigma_D^\star+\rho F^{mc}*\rho F^{mc}\}+F^{cm}*\rho F^{cc}*\rho F^{mc}, \tag{4.41}$$

$$h_{(B^\star)}=\Sigma_D*\{h^{nn}*[\Sigma_D^\star+\rho F^{cc}*\rho F^{mc}]+h^{-n}*\rho F^{mc}*\rho F^{mc}\}, \tag{4.42}$$

$$h_{(C^\star)}=F^{cm}*\{h^{-n}*[\Sigma_D^\star+\rho F^{cc}*\rho F^{mc}]+h*\rho F^{mc}*\rho F^{mc}\}. \tag{4.43}$$

Σ_D and $\Sigma_D^\star$ have the properties (4.12) and (4.38), respectively, while, if $g(\mathbf{k},\mathbf{X})$ and $g(\mathbf{k},\mathbf{X}_a',\mathbf{X}_b')$ are invariant under global rotations of their arguments, $\int D(\mathbf{X})F^{cm}(\mathbf{k},\mathbf{X})g(\mathbf{k},\mathbf{X})$ and $\int D(\mathbf{X}_a')\int D(\mathbf{X}_b')F^{cm}(\mathbf{k},\mathbf{X}_a')g(\mathbf{k},\mathbf{X}_a',\mathbf{X}_b')F^{mc}(\mathbf{k},\mathbf{X}_b)$ start at least as $|\mathbf{k}|^2$, when $|\mathbf{k}|$ goes to zero. Henceforth, the contribution from $h_{(A^\star)}+h_{(B^\star)}+h_{(C^\star)}$ to $\Sigma_\alpha e_\alpha \rho_\alpha^{\rm ind}(\mathbf{k})/4\pi\beta v_C(\mathbf{k})\delta q(\mathbf{k})$ is of order $|\mathbf{k}|^4$ and the constant term in $\Sigma_\alpha e_\alpha \rho_\alpha^{\rm ind}(\mathbf{k})/\delta q(\mathbf{k})$ is determined by F^{cc}.

The diagrams in $h_{(A^\star)}$ decay faster than any inverse power law and so do their contributions to $\Sigma_\alpha e_\alpha \rho_\alpha^{\rm ind}(\mathbf{k})$. The discussion of the algebraic tails of the contributions from $h_{(B^\star)}$ and $h_{(C^\star)}$ is very similar to that of $h_{(B)}$, on one hand, and $h_{(C)}$ and $h_{(D)}$, on the other hand.

According to (4.12), the Fourier transform of the contribution to $\Sigma_\alpha e_\alpha \rho_\alpha^{\rm ind}(\mathbf{k})/\delta q(\mathbf{k})$ from $h_{(B^\star)}$ reads

$$-\frac{4\pi\beta}{\mathbf{k}^2}\frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}^2}\int d\chi_a' e_{\alpha_a'}p_a'\rho(\chi_a')\int d\chi_b' e_{\alpha_b'}\rho(\chi_b')\times\{\mathscr{G}_1^{nn}(\mathbf{k},\chi_a',\chi_b')+\mathscr{G}_2^{-n}(\mathbf{k},\chi_a',\chi_b')\}, \tag{4.44}$$

where, according to (4.38) and to the property that $\int d\chi' e_{\alpha'}p'\rho(\chi')F^{cm}(\mathbf{k},\chi,\chi')=-pe_\alpha[\kappa^2(\mathbf{k})-\kappa^2]/(\kappa^2+\mathbf{k}^2)$,

$$\mathscr{G}_1^{nn}(\mathbf{k},\chi_a',\chi_b')=\left\{\left[1-\frac{\kappa^2(\mathbf{k})}{\kappa^2+\mathbf{k}^2}\right]\int_0^{p_b'}d\tau\, e^{-\mathbf{k}\cdot\mathbf{X}_b'(\tau)}+\frac{\kappa^2(\mathbf{k})[\kappa^2(\mathbf{k})-\kappa^2]}{(\kappa^2+\mathbf{k}^2)^2}\right\}h^{nn}(\mathbf{k},\chi_a',\chi_b') \tag{4.45}$$

and

$$\mathscr{G}_2^{-n}(\mathbf{k},\chi_a',\chi_b')=\frac{\kappa^2(\mathbf{k})[\kappa^2(\mathbf{k})-\kappa^2]}{(\kappa^2+\mathbf{k}^2)^2}\int_0^{p_b'}d\tau\times[e^{-i\mathbf{k}\cdot\mathbf{X}_b'(\tau)}-1]h^{-n}(\mathbf{k},\chi_a',\chi_b'). \tag{4.46}$$

When $|\mathbf{k}|$ goes to zero, $\mathscr{G}_1^{nn}(\mathbf{k},\chi_a',\chi_b')$ and $\mathscr{G}_2^{-n}(\mathbf{k},\chi_a',\chi_b')$ are proportional to $\mathbf{k}^2$ times $h^{nn}(\mathbf{k},\chi_a',\chi_b')$ and $[\exp(i\mathbf{k}\cdot\mathbf{X}_b')-1]h^{-n}(\mathbf{k},\chi_a',\chi_b')$, respectively, because of the dressing by $\Sigma_D^\star$ and $\rho F^{cc}*\rho F^{mc}$. Moreover, $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)h^{nn}(\mathbf{k},\chi_a',\chi_b')$ falls off at least as $1/r^6$, while $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)[\exp(-i\mathbf{k}\cdot\mathbf{X}_b')-1]h^{-n}(\mathbf{k},\chi_a',\chi_b')$ decreases at least as $1/r^8$, according to (4.13). Thus $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)\mathscr{G}_1^{nn}(\mathbf{k},\chi_a',\chi_b')$ and $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)\mathscr{G}_2^{-n}(\mathbf{k},\chi_a',\chi_b')$ decay at least as $1/r^8$ and $1/r^{10}$, respectively, and the same is true for their contributions to (4.44).

The Fourier transform of the contribution to $\Sigma_\alpha e_\alpha \rho_\alpha^{\rm ind}(\mathbf{k})/\delta q(\mathbf{k})$ from $h_{(C^\star)}$ reads

$$-\frac{4\pi\beta}{\mathbf{k}^2}\int d\chi_a' e_{\alpha_a'}p_a'\rho(\chi_a')\int d\chi_b' e_{\alpha_b'}\rho(\chi_b')\int_0^{p_a'}d\tau\times[e^{i\mathbf{k}\cdot\mathbf{X}_a'(\tau)}-1]\{\mathscr{G}_1^{-n}(\mathbf{k},\chi_a',\chi_b')+\mathscr{G}_2(\mathbf{k},\chi_a',\chi_b')\}, \tag{4.47}$$

where the definitions of $\mathscr{G}_1^{-n}$ and $\mathscr{G}_2$ are similar to those of $\mathscr{G}_1^{nn}$ and $\mathscr{G}_2^{-n}$, respectively. $[\exp(i\mathbf{k}\cdot\mathbf{X}_a')-1]\mathscr{G}_1^{n}(\mathbf{k},\chi_a',\chi_b')$ and $[\exp(i\mathbf{k}\cdot\mathbf{X}_a')-1]\mathscr{G}_2(\mathbf{k},\chi_a',\chi_b')$ are proportional to $\mathbf{k}^2$ times $[\exp(i\mathbf{k}\cdot\mathbf{X}_a')-1]h^{-n}$ and $[\exp(i\mathbf{k}\cdot\mathbf{X}_a')-1][\exp(-i\mathbf{k}\cdot\mathbf{X}_b')-1]h^{nn}$, respectively. According to (4.13) and (4.14), after integration over $\mathbf{X}_a$ and $\mathbf{X}_b$, the inverse Fourier transforms of the latter functions decay as $1/r^8$ and $1/r^{10}$, respectively, and (4.47) has similar algebraic tails.

F. Second moments

We make a comment about the first term in the small-$\mathbf{k}$ expansion of $C^Q(\mathbf{k})$. Let us call the ''Debye'' contribution to $C^Q|_{\rm nonexch}(\mathbf{k})$ the sum

$$C_D^Q|_{\rm nonexch}(\mathbf{k})\equiv\int d\chi_a e_{\alpha_a}p_a\rho(\chi_a)\int d\chi_b e_{\alpha_b}p_b\rho(\chi_b)\times F^{cc}(\mathbf{k},\chi_a,\chi_b)=-\frac{\kappa^2}{4\pi\beta}\frac{\kappa^2}{\kappa^2+\mathbf{k}^2}. \tag{4.48}$$

According to (4.7) and (4.8),

$$C^Q(\mathbf{k})=\frac{\kappa^2}{4\pi\beta}\frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}^2}+\int d\chi_a e_{\alpha_a}^2 p_a\rho(\chi_a)\int_0^{p_a}d\tau[e^{-i\mathbf{k}\cdot\mathbf{X}_a[P(\tau)]}-1]+\int d\chi_a e_{\alpha_a}p_a\rho(\chi_a)\int d\chi_b e_{\alpha_b}p_b\rho(\chi_b)\times[h(\mathbf{k},\chi_a,\chi_b)-F^{cc}(\mathbf{k},\chi_a,\chi_b)], \tag{4.49}$$

where the first term on the right-hand side is equal to $\Sigma_\alpha e_\alpha^2[\rho_\alpha+\rho_{\alpha\alpha}^{(2)TQ}|_{\rm exch}(\mathbf{k}=\mathbf{0})]+C_D^Q|_{\rm nonexch}(\mathbf{k})$, the second term is merely $\Sigma_\alpha e_\alpha^2[\rho_{\alpha\alpha}^{(2)TQ}|_{\rm exch}(\mathbf{k})-\rho_{\alpha\alpha}^{(2)TQ}|_{\rm exch}(\mathbf{k}=\mathbf{0})]$, and the third term comes from the interactions between particles that are not exchanged under the same cyclic permutation.

According to the previous discussion, only the part $F^{cc}+F^{cm}*\rho F^{mc}+F^{cm}\rho*h*\rho F^{mc}$ in the integral relation (4.21) contributes at the order $\mathbf{k}^2$ in the third term of (4.49). Indeed, after integration over the shapes of the root points, $[\Sigma_D*F^{mc}](\mathbf{k})$ and $[F^{cm}*\Sigma_D](\mathbf{k})$ are proportional to $\mathbf{k}^2$ times $\int D(\mathbf{X}_a')\rho(\chi_a')[\exp(i\mathbf{k}\cdot\mathbf{X}_a')-1]\phi(\mathbf{k})$, while the structures of $C^Q_{(B)}$, $C^Q_{(C)}$ or $C^Q_{(D)}$, and $C^Q_{(E)}$ are given by (4.27), (4.28), and (4.30), respectively. Rotational invariance arguments imply that all these contributions, except $C^Q_{(E)}$, start at the order $|\mathbf{k}|^4$, when $|\mathbf{k}|$ goes to zero, while $C^Q_{(E)}$ starts only at the order $|\mathbf{k}|^2$. In fact, after applying the left-dressing relation (4.16) to h in the term $F^{cm}\rho*h*\rho F^{mc}$ of (4.49) and then the right-dressing relation (4.19) to h^{n-} in $F^{cm}\rho*h^{n-}*\rho F^{mc}$, after taking into account rotational invariances, only $F^{cm}\rho*h^{nn}*\rho F^{mc}$ contributes at the order $|\mathbf{k}|^2$ to $C^Q_{(E)}$. Thus the only terms that survive at the order $|\mathbf{k}|^2$ in the integral relation (4.49) are

$$
\begin{aligned}
C^Q(\mathbf{k}) \underset{|\mathbf{k}|\to 0}{\sim} & \frac{\mathbf{k}^2}{4\pi\beta}+\mathbf{k}^2\Bigg\{-\tfrac{1}{6}\int d\chi\, e_\alpha^2 p\rho(\chi)\sum_{l=2}^{p_a}(\mathbf{x}_l-\mathbf{x}_1)^2 \\
&+\tfrac{1}{3}\int d\chi\, e_\alpha^2\rho(\chi)\int_0^p d\tau\int_0^p d\tau'[\mathbf{X}(\tau)\cdot\mathbf{X}(\tau')]\Bigg\} \\
&+\tfrac{1}{3}\int d\chi_a e_{\alpha_a}p_a\rho(\chi_a)\int d\chi_b e_{\alpha_b}p_b\rho(\chi_b)\int_0^{p_a}d\tau \\
&\times\int_0^{p_b}d\tau'[\mathbf{X}_a(\tau)\cdot\mathbf{X}_b(\tau')]h^{nn}(\mathbf{k}=\mathbf{0},\chi_a,\chi_b).
\end{aligned}
\tag{4.50}
$$

Equation (4.50), which originates from the Mayer-like diagrammatics, and Eq. (4.40) of paper I, which is derived from the external screening equation (1.3) and the linear response (4.32), are different expressions of the second moment of the charge-charge correlation. The second contribution in curly brackets on the right-hand side of (4.50) contains a term of order $\hbar^2$ ($p_a=p_b=1$), a term of order $\hbar\exp(-G_{12}/\hbar)$ ($p_a=1$ and $p_b=2$), a term of order $\exp(-G_{22}/\hbar)$ ($p_a=p_b=2$), and so on. Eventually, at the order $|\mathbf{k}|^2$, $C^Q(\mathbf{k})$ involves contributions from both $\rho^{(2)TQ}_{\alpha_a\alpha_a}|_{\text{exch}}$ and $\rho^{(2)TQ}_{\alpha_a\alpha_b}|_{\text{nonexch}}$ and the only terms in h that contribute to the term of order $|\mathbf{k}|^2$ in $C^Q(\mathbf{k})$ are F^{cc}, $F^{cm}*\rho F^{mc}$, and $F^{cm}*\rho h^{nn}\rho*F^{mc}$.

However, as shown in Sec. IV E, F^{cc} contributes alone to the constant term in $\Sigma_\alpha e_\alpha\rho_\alpha^{\text{ind}}(\mathbf{k})$, while $F^{cm}*\Sigma_D^\star$ and $F^{cm}*\rho h^{nn}*\Sigma_D^\star$ contribute at the greater order $|\mathbf{k}|^2$. In other words, the contributions to $\Sigma_\alpha e_\alpha\rho_\alpha^{\text{ind}}(\mathbf{k}=\mathbf{0})$ from $F^{cm}*\rho F^{mc}$ and $F^{cm}*\rho h^{nn}\rho*F^{mc}$, both of which contribute to $C^Q(\mathbf{k})$ at the order $|\mathbf{k}|^2$, are compensated by the contributions to $\Sigma_\alpha e_\alpha\rho_\alpha^{\text{ind}}(\mathbf{k}=\mathbf{0})$ from $F^{cm}*\Sigma_D$ and $F^{cm}*\rho h^{nn}*\Sigma_D$, respectively, both of which contribute to $C^Q(\mathbf{k})$ only at the order $|\mathbf{k}|^4$. $F^{cm}*\rho F^{mc}$ and $F^{cm}*\Sigma_D$ are both contained in $h_{(A)}$, while $F^{cm}*\rho h^{nn}*F^{mc}$ appears in $h_{(E)}$ when the left- and right-dressing relations (4.16) and (4.19) are used and $F^{cm}*\rho h^{nn}*\Sigma_D$ arises in $h_{(D)}$ when the left-dressing relation (4.20) is inserted into it.

Subsequently, in order to devise an approximation for h that satisfies the screening rule (1.3) of paper I, we must include the diagram F^{cc}, which ensures this property by itself, and if we take a diagram Π into account, we must add other "dressing" diagrams, so that any spurious contribution to $\Sigma_\alpha e_\alpha\rho_\alpha^{\text{ind}}(\mathbf{k}=\mathbf{0})$ should not appear. If we include a diagram Π^{nn}, the root points of which are non-Coulomb root points, then we must take the whole set of graphs $\Sigma_D*\Pi^{nn}*\Sigma_D^\star$ into account. If we consider a diagram $F^{cm}\rho*\Pi^{-n}$, $F^{cm}\rho*\Pi^{-n}*F^{cc}$, or $F^{cm}\rho*\Pi^{-n}*F^{mc}$, we must in fact consider the whole set $F^{cm}\rho*\Pi^{-n}*\Sigma_D^\star$. If we add a diagram $\Pi^{n-}*\rho F^{mc}$, we must add the whole set $\Sigma_D*\Pi^{n-}*\rho F^{mc}$. We notice that, in all cases, every diagram of the dressed set has the same order in $\rho(\mathscr{L})$ because $\int d\chi\,\rho(\chi)F^{cc}(\mathbf{k},\chi,\chi')$ and $\int d\chi\,\rho(\chi)F^{cm}(\mathbf{k},\chi,\chi')$ are of order zero in $\rho(\mathscr{L})$.

V. COMPARISON WITH SCREENING IN OTHER FORMALISMS

A. Correlation in the chain approximation

A more microscopic approach than that of the mean-field models can be investigated by means of formalisms in which the linear response theory gives the relation between the induced charge density and some kind of charge-charge correlation function. As recalled in paper I, in classical statistics, it is the charge-charge correlation (4.30) itself that is involved [see (4.36)]. In quantum mechanics, the linear response theory (4.34) relates the static induced charge density to the zero-frequency component of the time-ordered charge-charge correlation function in imaginary time $C_T(\mathbf{r},s)$. In the classical case, when the static structure factor is approximated by

$$S_{\text{DH},\alpha\alpha'}=\delta_{\alpha,\alpha'}\rho_\alpha\delta(\mathbf{r})+\rho_\alpha\rho_{\alpha'}(-\beta e_\alpha e_{\alpha'})\phi_{\text{DH}}(\mathbf{r}),\tag{5.1}$$

where $\phi_{\text{DH}}(\mathbf{r})=\exp(-\kappa_{\text{DH}}r)/r$ and κ_{DH} is the Debye-Hückel screening length, the corresponding approximated induced charge density given by the classical linear response (4.36) is the same as in the linearized mean-field Debye-Hückel model. From the diagrammatic point of view, this means that the Debye-Hückel theory can be retrieved by approximating the Ursell function by one graph, namely, the graph that contains only one resummed bond obtained by chain summation (without any exponentiation). More generally, after a systematic resummation of the Coulomb divergencies has been performed and has introduced a chain potential (as explained in Sec. VI of paper I), we call a "chain approximation" an approximation in which the basic object of the diagrammatics (Ursell function or effective potential associated with the proper polarization) is calculated by retaining only the diagram with one "linearized" bond equal to the chain potential.

In the loop formalism, if we do not make any hypothesis about $\rho_{\alpha,p}(\mathbf{X})$ (which contributes to the part of the correlation function due to the exchange), a chain approximation for the loop Ursell function [see (3.1)] is to retain only the sum of the four properly dressed diagrams that are built from the four linearized bonds F^{cc}, F^{mc}, F^{cm}, or F^{mm} [see the comment before (6.16) in paper I]. According to Sec. IV F, the dressed diagrams are, respectively, F^{cc}, Σ_D*F^{mc}, $F^{cm}*\Sigma_D^\star$, and $\Sigma_D*F^{mm}*\Sigma_D^\star$. The corresponding chain structure factor $S_{\text{linearized chain}}(\mathscr{L},\mathscr{L}')$ is given by (6.10) of paper I, where, according to (5.28) and (6.16) of paper I, $\rho(\mathscr{L})\rho(\mathscr{L}')h_{\text{linearized chain}}(\mathscr{L},\mathscr{L}')$ may be written as fast-decaying terms plus the convolution

$$\int d\chi_1\rho(\chi_1)\int d\chi_2\rho(\chi_2)\Sigma_D(\mathscr{L},\mathscr{L}_1)* \times[-\beta e_{\alpha_1}e_{\alpha_2}\phi_{\text{linearized chain}}](\mathscr{L}_1,\mathscr{L}_2)*\Sigma_D^{\star}(\mathscr{L}_2,\mathscr{L}'), \tag{5.2}$$

with the linearized chain potential

$$-\beta e_{\alpha_1}e_{\alpha_2}\phi_{\text{linearized chain}}=F^{cc}+F^{mc}+F^{cm}+F^{mm} =-\beta e_{\alpha_1}e_{\alpha_2}\phi_{\text{elect}}+W.$$

In the RPA, the proper polarization Π^* of the standard perturbation formalism is approximated by its simplest value, namely, its value for an ideal gas Π^0. The effective potential $U^{\text{eff}}_{\text{RPA},\alpha\alpha'}(\mathbf{r},n)$ proves to be equal to $\beta e_\alpha e_{\alpha'}\phi_{\text{RPA}}(\mathbf{r},n)$ [see (6.3) and (6.12) of paper I] and the correlation function is derived from the latter objects through the relation (6.9) of paper I,

$$S_{\text{RPA},\alpha\alpha'}(\mathbf{k})=\delta_{\alpha,\alpha'}\sum_{n=-\infty}^{+\infty}\Pi^0_\alpha(\mathbf{k},n)+\sum_{n=-\infty}^{+\infty}\Pi^0_\alpha(\mathbf{k},n) \times(-\beta e_\alpha e_{\alpha'})\phi_{\text{RPA}}(\mathbf{k},n)\Pi^0_{\alpha'}(\mathbf{k},n). \tag{5.3}$$

Though algebraic tails appear in the chain potentials, these tails give a short-ranged contribution to the induced charge density in the chain approximation. In the loop formalism, the fast decay of the induced charge density corresponding to $h_{\text{linearized chain}}(\mathscr{L},\mathscr{L}')$ (5.2) through the linear response (4.27) of paper I is enforced by the same mechanism as that shown in (2.2). In the RPA, the induced charge density [see (6.6) of paper I) is

$$\frac{\Sigma_\alpha e_\alpha\rho^{\text{ind}}_{\text{RPA},\alpha}(\mathbf{k})}{\delta q(\mathbf{k})}=\frac{4\pi\beta\Sigma_\alpha e_\alpha^2\Pi^0_\alpha(\mathbf{k},n=0)}{\mathbf{k}^2-4\pi\beta\Sigma_\alpha e_\alpha^2\Pi^0_\alpha(\mathbf{k},n=0)} \tag{5.4}$$

and the corresponding total potential created by a point charge δq, $V^{\text{tot}}_{\text{RPA}}(\mathbf{r})=(\delta q/\beta e_\alpha e_{\alpha'})U^{\text{eff}}_{\text{RPA}}(\mathbf{r},n=0)$, is equal to the RPA chain potential

$$V^{\text{tot}}_{\text{RPA}}(\mathbf{r})=\delta q\,\phi_{\text{RPA}}(\mathbf{r},n=0). \tag{5.5}$$

At zero temperature the first derivative of $\Pi^0(\mathbf{k},n=0)$ is infinite at $|\mathbf{k}|=2k_F$, because of the sharpness of the Fermi surface for noninteracting particles, and the so-called Friedel algebraic oscillations appear: at large distances the induced charge density is proportional to $\cos(2k_F)/r^3$. At finite temperature, $\Pi^0(\mathbf{k},n)$ is an analytic function of $\mathbf{k}$, $\Pi^0(\mathbf{k}=\mathbf{0},n=0)\neq 0$, and these algebraic oscillations are exponentially damped. Thus, according to Eqs. (6.6) and (6.8) of paper I, $e\rho^{\text{ind}}_{\text{RPA}}(\mathbf{r})$ and $V^{\text{tot}}_{\text{RPA}}(\mathbf{r})$ decrease faster than any inverse power of the distance. The leading term in the asymptotic behavior of the RPA induced charge density is

$$\sum_\alpha e_\alpha\rho^{\text{ind}}_{\text{RPA},\alpha}(\mathbf{r})\underset{r\to\infty}{\sim}-\delta q\,\kappa^2_{\text{RPA}}\frac{e^{-\kappa_{\text{RPA}}r}}{r} \tag{5.6}$$

and that of $V^{\text{tot}}_{\text{RPA}}(\mathbf{r})$ (5.5), which is given by (6.13) in paper I, is a Yukawa potential with a screening length equal to $[Z_{\text{RPA}}\kappa_{\text{RPA}}]^{-1}$.

The structure factors in the chain approximations are also short ranged. In the loop formalism, the mechanism for $\int D(\mathbf{X})\int D(\mathbf{X}')\rho_{\alpha,p}(\mathbf{X})\rho_{\alpha',p'}(\mathbf{X}')h_{\text{linearized chain}}(\mathscr{L},\mathscr{L}')$ has been displayed in (2.2). A similar phenomenon takes place in the RPA theory, where, according to (6.2) and (6.5) of paper I, $S_{\text{RPA}}(\mathbf{r})$ decays faster than any inverse power law, though the nonzero frequency components of the effective potential $U^{\text{eff}}_{\text{RPA}}$ are purely Coulombic. The mechanism can be viewed as follows. $\Pi^0_\alpha(\mathbf{k},s)$ is an analytic function of $\mathbf{k}$ and the small-$|\mathbf{k}|$ behavior of $\Pi^0_\alpha(\mathbf{k},n\neq 0)$ starts as $\mathbf{k}^2$. Subsequently, the $1/\mathbf{k}^2$ singularity of the nonzero-frequency components of $\phi_{\text{RPA}}(\mathbf{k},s)$ is canceled in (5.3) by the same mechanism as in (2.2). In position space, the argument is the following. $\Pi^0_\alpha(\mathbf{r},s)$ decays faster than any inverse power law, so that all the moments of $\Pi^0_\alpha(\mathbf{r},s)$ in the components of $\mathbf{r}$ are well defined. Moreover, $\int d\mathbf{r}\,\Pi^0_\alpha(\mathbf{r},s)$ is independent from s and $\Pi^0_\alpha(\mathbf{r},s)$ is invariant under rotations of $\mathbf{r}$ for every time s. Subsequently, the possibly long-ranged contributions from $h(s_1-s_2)/r$ to $S_{\text{RPA},\alpha\alpha'}(\mathbf{r})$ [see (6.9) and (6.15) of paper I] are terms proportional to

$$\begin{aligned}&\int_0^1 ds_1\int_0^1 ds_2 h(s_1-s_2)\int d\mathbf{r}_1\Pi^0_{\alpha_1}(\mathbf{r}_1,s_1)\int d\mathbf{r}_2\Pi^0_{\alpha_2}(\mathbf{r}_2,s_2)\\&\quad\times(\mathbf{r}_1\cdot\boldsymbol{\nabla}_{\mathbf{R}})^{2m_1}(\mathbf{r}_2\cdot\boldsymbol{\nabla}_{\text{R}})^{2m_2}\left(\frac{1}{r}\right)\\&\propto\Delta^{m_1+m_2}\left(\frac{1}{r}\right)\int_0^1 ds_1\int_0^1 ds_2 h(s_1-s_2)\\&\quad\times\left(\int d\mathbf{r}_1[\mathbf{r}_1^2]^{m_1}\Pi^0_{\alpha_1}(\mathbf{r}_1,s_1)\right)\\&\quad\times\left(\int d\mathbf{r}_2[\mathbf{r}_2^2]^{m_2}\Pi^0_{\alpha_2}(\mathbf{r}_2,s_2)\right),\end{aligned} \tag{5.7}$$

with $m_1\geqslant 1$ and $m_2\geqslant 1$. Only powers of the Laplacian of $1/r$ appear and the corresponding approximated correlations prove to be short ranged.

B. Macroscopic screening in the chain approximation

The RPA theory is not so bad for the description of macroscopic screening in one-component and multicomponent plasmas, as is the case for the mean-field theories, because, in all those theories, the small-$\mathbf{k}$ expansion of the induced charge density has the same structure as the exact behavior. Indeed, since the mean-field theories can be valid only on large-distance scales, they can only mimic the first terms in the small-$\mathbf{k}$ expansion of the exact induced charge density around an external point charge δq: only the small-$\mathbf{k}$ expansion of the mean-field value derived from (1.1) and (1.2),

$$\frac{\Sigma_\alpha e_\alpha \rho^{\text{ind}}_{\text{MF},\alpha}(\mathbf{k})}{\delta q(\mathbf{k})} = -\frac{\kappa^2_{\text{MF}}}{\mathbf{k}^2+\kappa^2_{\text{MF}}} = -1 + \frac{\mathbf{k}^2}{\kappa^2_{\text{MF}}} + o(|\mathbf{k}|^2), \tag{5.8}$$

is to be considered. In the same way, in the RPA theory, since $\Pi^*_{\text{RPA}}(\mathbf{k}=\mathbf{0},n=0)\neq 0$, Eq. (5.4) implies that

$$\frac{\Sigma_\alpha e_\alpha \rho^{\text{ind}}_{\text{RPA},\alpha}(\mathbf{k})}{\delta q(\mathbf{k})} = -1 + \frac{\mathbf{k}^2}{\kappa^2_{\text{RPA}}} + o(|\mathbf{k}|^2). \tag{5.9}$$

The first term -1 ensures that the total charge of the polarization cloud exactly compensates the external charge δq. In the classical case, this property is equivalent to the fact that the exact charge-charge correlation, as well as its Debye-Hückel approximation, obeys the classical Stillinger sum rule [see (4.42) of paper I], as recalled in Sec. IV A. In the quantum OCP, there exists the second-moment sum rule (4.43) of paper I, which is not equivalent to the above screening property, but comes from the fact that there is only one species of moving charges in a uniform background. It happens that, in the RPA theory, the second moment of the charge-charge correlation (which is proportional to the particle-particle correlation) satisfies the above quantum sum rule, but only with the ideal-gas density $\rho^0(\beta,\mu)$ in place of the plasma density ρ_B determined by the density of the background.

The second terms on the right-hand side of (5.8) and (5.9) are not exact because κ depends on the model, but the structure is correct in a one-component as well as in a multicomponent plasma. Indeed, in multicomponent plasmas, $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{r})$ decays only as $1/r^8$, as shown in Sec. IV E, and the first singular term in the small-$\mathbf{k}$ expansion of the induced charge density is of order $|\mathbf{k}|^5$. (A mere dimensional analysis would have implied the existence of a term proportional to $|\mathbf{k}|$, but the latter in fact vanishes, as explained in Sec. IV E.) On the other hand, in the very special case of the OCP, where the density of charge is proportional to the density of particles, the structure of the first two terms in the exact small-$\mathbf{k}$ expansion of $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha(\mathbf{k})$ originates from the exact "compressibility" sum rule (6.19) of paper I and allows one to define an inverse screening length as the square root of the coefficient of $\mathbf{k}^2$. The various screening lengths that are derived from the approximated small-$\mathbf{k}$ expansions (5.8) and (5.9) are comparable to the exact inverse screening length κ_{OCP} given by (6.19)

$$\kappa^2_{\text{OCP}} = 4\pi e^2 \rho_B^2 \chi_T = 4\pi e^2 \frac{\partial \rho}{\partial \mu^*}, \tag{5.10}$$

where μ^* is defined in (2.11) of paper I. The RPA inverse length κ_{RPA} has a similar expression given in (6.13) of paper I,

$$\kappa^2_{\text{RPA}} = 4\pi\beta \sum_\alpha e^2_\alpha \left.\frac{\partial \rho^0_\alpha(\beta,\mu_\alpha)}{\partial(\beta\mu_\alpha)}\right|_\beta . \tag{5.11}$$

In the case of the OCP,

$$\kappa^2_{\text{OCP,RPA}} = 4\pi e^2 \frac{\partial \rho^0(\beta,\mu)}{\partial \mu}\beta = 4\pi e^2 [\rho^0(\beta,\mu)]^2 \chi^0_T(\beta,\mu). \tag{5.12}$$

The similarity between (5.10) and (5.12) might be linked to the fact that the density fluctuations of the OCP in the thermodynamic limit obey the same rule $\int d\mathbf{r}\, \rho^{(2)T}(\mathbf{r}) = -\rho$ (see Sec. II of paper I) as those of an ideal gas with a *finite* number of particles in the canonical ensemble, before the thermodynamic limit is taken.

Moreover, in the weak coupling limit (high-density regime in the semiclassical fermionic case and low-density or high-temperature regimes in the classical case), the RPA value of the screening length tends to the corresponding mean-field values (Thomas-Fermi and Debye-Hückel, respectively). Indeed, according to Eq. (1.1) and the assumption that the quasiparticles do not interact together, the Thomas-Fermi inverse length κ_{TF} is given by

$$\kappa^2_{\text{TF}} = 4\pi\beta \sum_\alpha e^2_\alpha \left.\frac{\partial \rho^0_\alpha(\beta,\tilde{\mu}_\alpha)}{\partial(\beta\tilde{\mu}_\alpha)}\right|_\beta = 4\pi \sum_\alpha e^2_\alpha [\rho^0_\alpha(\beta,\tilde{\mu}_\alpha)]^2 \chi^0_{\alpha,T}(\beta,\tilde{\mu}_\alpha) \tag{5.13}$$

at zero as well as at finite temperature, with $\rho^0_\alpha(\beta,\tilde{\mu}_\alpha) = \rho_\alpha(\beta,\{\rho_{\alpha'}\}_{\alpha'=1,\ldots,n_s-1})$. (The mean interparticle distance a is far smaller than the Thomas-Fermi length $a\kappa_{\text{TF}} \ll 1$ and the Thomas-Fermi model is coherent). In the case of the OCP, $\rho^0_\alpha(\beta,\tilde{\mu}_\alpha) = \rho_B$ and

$$\kappa^2_{\text{OCP,TF}} = 4\pi e^2 [\rho_B]^2 \chi^0_T .$$

κ_{RPA} has nearly the same expression as the Thomas-Fermi inverse length κ_{TF}, with $\rho^0_\alpha(\beta,\mu_\alpha)$ in place of $\rho^0_\alpha(\beta,\tilde{\mu}_\alpha)$. In the strict high-density limit, $\rho^0_\alpha(\beta,\mu_\alpha)$ coincides with $\rho_\alpha(\beta,\{\rho_{\alpha'}\}_{\alpha'=1,\ldots,n_s-1})$, while $\partial^2\rho^0_\alpha/(\partial\mu_\alpha)^2$ tends to zero: κ_{RPA} becomes equal to κ_{TF}, while Z_{RPA} [see (6.14) of paper I] tends to 1 and the asymptotic behavior of $V^{\text{tot}}_{\text{RPA}}(\mathbf{r}) = \phi_{\text{RPA}}(\mathbf{r})$ coincides with $V^{\text{tot}}_{\text{TF}}(\mathbf{r})$. In fact, the structure of the Thomas-Fermi model is retrieved by replacing $\Pi^0(\mathbf{k},n=0)$ by $\Pi^0(\mathbf{k}=\mathbf{0},n=0)$ in Eqs. (6.6) and (6.8) of paper I. In the classical limit, $\partial \rho^{0\,\text{cl}}_\alpha / \partial(\beta\mu_\alpha)|_{\rho^{0\,\text{cl}}_\alpha = \rho_\alpha}$ becomes equal to ρ_α, and κ^2_{RPA} leads to the Debye-Hückel value

$$\kappa^2_{\text{DH}} = 4\pi\beta \sum_\alpha e^2_\alpha \rho_\alpha . \tag{5.14}$$

In the loop formalism, the screening length associated with the induced charge density (by means of the small-$\mathbf{k}$ expansion of $\Sigma_\alpha e_\alpha \rho^{\text{ind}}_\alpha$ in the chain approximation) does not coincide with the screening length of the charge-charge interaction because the structure of the linear response (4.27) of paper I is not similar to (6.6). The total induced charge is equal to $-\delta q$ in the chain approximation, but the mean-field equation (5.8) is not valid: the small-$\mathbf{k}$ expansion of the corresponding induced charge starts as $-1+\mathbf{k}^2/\kappa'^2$, where κ' is not equal to κ. Even if one chooses for the Ursell function the graph with one bond F^{cc}, $\kappa'^2 = \kappa^2/(1-A)$ with $A = \frac{1}{6}\int d\chi\, \rho(\chi) e^2 p \int_0^p d\tau [\mathbf{X}(\tau)]^2$. However, we recall that, for fermions in a regime of high density (in which the interactions become negligible with respect to the quantum kinetic energy), the value κ given by (5.14) and (4.9) of paper I tends to κ_{RPA},

$$\kappa^2=4\pi\beta\sum_\alpha e_\alpha^2\left[\frac{\partial\rho(\beta,\mu_\alpha)}{\partial(\beta\mu_\alpha)}\bigg|_\beta-\int d\mathbf{r}\ \rho^{(2)TQ}_{\alpha\alpha}|_{\text{nonexch}}(r)\right].$$

C. Beyond the chain approximations

In the quantum case, the algebraic tails of the chain potential induce algebraic tails in the particle-particle correlation, as soon as the next corrections to the chain approximation are considered. For instance, in the linearized loop formalism, the next correction to the chain approximation (5.2) of $\rho(\mathscr{L})\rho(\mathscr{L}')h(\mathscr{L},\mathscr{L}')$ is to consider the diagrams where the two root points are linked by two linearized bonds. According to the dressing rules displayed at the end of Sec. IV F, this correction involves a term $\Sigma_D*\frac{1}{2}[F^{mm}]^2*\Sigma_D^\star$. When there is no summation over (α,p) and (α',p'), the leading asymptotic behavior of this correction is given by $\frac{1}{2}[W_3(\mathscr{L},\mathscr{L}')]^2$ and its contribution to the particle-particle correlation decays as the $1/r^6$ tail of the diagram $\Sigma_D*F_R*\Sigma_D^\star$ with one nonlinearized F_R bond [which is similar to (2.3)]. This particular correction is an example of the more general structure of the diagrams that are responsible for the $1/r^6$ tail of the correlation, as shown in Sec. III D.

In the standard perturbation many-body theory, the next natural correction to the RPA for the proper polarization is to include the graph Π_1^* made with two free-propagator loops linked by two interaction lines $-\beta e_\alpha e_{\alpha'}\phi_{\text{RPA}}$. In the case of the OCP, all the frequency components of Π_1^* have an algebraic falloff [5]: $\Pi_1^*(\mathbf{r},n=0)$ decays as $1/r^6$ and $\Pi_1^*(\mathbf{r},n\neq0)$ as $1/r^{10}$. Moreover, the exact sum rules (4.43) and (6.19) of paper I, which are specific to the OCP, allow one to derive two sum rules about the small-$\mathbf{k}$ behavior of $\Pi(\mathbf{k},n)$. Thus, under the assumption that every frequency component $\Pi^*(\mathbf{k},n)$ is invariant under rotations of $\mathbf{k}$, it can be shown (see Sec. VI of paper I) that, if the exact proper polarization Π^* decays as Π_1^*, then $S(\mathbf{r})$ decays as $1/r^{10}$, $\rho^{\text{ind}}(\mathbf{r})$ as $1/r^8$, and $V^{\text{tot}}(\mathbf{r})$ as $1/r^6$. (We notice that the so-called ladder diagrams with more than two RPA interaction lines do not contribute to the leading asymptotic behavior, as is the case for the tails T with $L\geqslant2$ in the loop formalism.)

The mechanism can be exemplified in the Maxwell-Boltzmann approximation of the RPA theory that is given in Ref. [5]. As recalled in Sec. VI of paper I, the effective potential (6.17) between two filaments is a dipole-dipole-like interaction (plus a Debye-Hückel term) before the shapes of the filaments are averaged over. After integration over those internal degrees of freedom, the potential decays faster than any inverse power law. However, the graph where two filaments interact through two effective potential lines decays as $1/r^6$. The mechanism is the same as in the more general loop formalism. In the case of the OCP, it can be shown exactly that, at the order $\hbar^4$, the correlation function decays in fact not as $1/r^6$ but as $1/r^{10}$, because of the classical Stillinger-Lovett sum rule [4,5,35]. According to Ref. [5], the quantum second-moment sum rule (4.43) implies that this result is expected to be true at any order in $\hbar$. The underlying reason is that the particle-particle correlation coincides with the charge-charge correlation in the one-component plasma.

As a conclusion, we mention a very simple model [4] that exhibits the basic mechanisms involved in the present series of papers and are responsible for the $1/r^6$ decay of the particle-particle correlation. First, we stress that the absence of exponential screening for the $1/r^3$ dipole-dipole-like interaction between the charges surrounded by their polarization cloud, before the average is taken over the quantum fluctuations, is due to the fact that the corresponding classical loops do not interact via the electrostatic potential: only curve elements with the same parameter $\tau-P(\tau)$ interact together and the resummed bonds involve an algebraic part $W=-\beta e_\alpha e_{\alpha'}[v(\mathscr{L},\mathscr{L}')-v_{\text{elect}}(\mathscr{L},\mathscr{L}')]$. In a regime of Maxwell-Boltzmann statistics with quantum dynamics, only loops with size $p=1$, i.e., closed filaments $\mathscr{L}^{\alpha,1}$ with shapes $\boldsymbol{\xi}$, contribute to the grand partition function and W [see (5.25) of paper I] becomes

$$W(\mathscr{L}^{\alpha_i,1},\mathscr{L}^{\alpha_j,1})=\beta_{ij}\int_0^1 ds\int_0^1 ds'[\delta(s-s')-1]v_C[\mathbf{r}_{ij}+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s)-\lambda_{\alpha_j}\boldsymbol{\xi}_j(s')]. \tag{5.15}$$

This potential already appeared in the model of two quantum charges embedded in a classical plasma, which is solved in Ref. [4]. In this case the energy of the two corresponding filaments interacting with a given configuration of particles of a classical bath can be split into the pure electrostatic energy plus a quantum ''correction'' $W(\mathscr{L}^{\alpha 1},\mathscr{L}^{\alpha' 1})$, which does not involve the bath. Second, the correlation between the two quantum charges of the previous model decays algebraically and its inverse-power asymptotic expansion exactly starts with a term B/r^6 $(B>0)$. This term involves W^2 because of the rotational invariance of the quantum fluctuations and the short-range of the Laplacian of the Coulomb potential. In the present paper, the mechanism is generalized to the exchange loops: there is no exponential screening at nonequal (imaginary) times for an interaction between open filaments that takes place at equal times and the leading large-distance behavior of the quantum correlation comes from the product of two convolutions, each of which involves at least one W bond.

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APPENDIX A

In this appendix we sketch the direct study of the large-distance behavior of the convolutions (2.6) in position space in the two simplest cases, in order to exhibit the mechanisms in position space. The roles played, on one hand, by the rotational invariance of the quantum fluctuations and the interactions and, on the other hand, by the harmonicity of the Coulomb potential are clearly disentangled (though this makes the discussion a little longer).

The chain made with $I+1$ graphs K linked by I bonds W in (2.6) can be decomposed as a series of chains involving purely algebraic terms W_γ [defined in (5.33) of paper I] instead of W. Such a chain reads

$$\int d\chi_a p_a \rho(\chi_a) \int d\chi_b p_b \rho(\chi_b) \int \prod_{i=1}^{I} [d\chi_i \rho(\chi_i) d\chi_i' \rho(\chi_i')] K(\mathbf{r},\chi_a,\chi_1) * W_{\gamma_1}(\mathbf{r},\chi_1,\chi_1') * K(\mathbf{r},\chi_1',\chi_2)$$
$$* \cdots * K(\mathbf{r},\chi_{i-1}',\chi_i) * W_{\gamma_i}(\mathbf{r},\chi_i,\chi_i') * K(\mathbf{r},\chi_i',\chi_{i+1}) * \cdots * K(\mathbf{r},\chi_{I-1}',\chi_I) * W_{\gamma_I}(\mathbf{r},\chi_I,\chi_I') * K(\mathbf{r},\chi_I',\chi_b), \quad \text{(A1)}$$

where the convolution operates on the loop-position variable $\mathbf{r}$. After integration over the shapes of the loops, (A1) becomes a sum of terms, each of which is a convolution of derivatives $\partial_{\mu_{i,1}\cdots\mu_{i,m_i}} \partial_{\nu_{i,1}\cdots\nu_{i,n_i}}(1/r)$ of the Coulomb potential and functions $\mathcal{K}^{[n_i,m_{i+1}]}_{\nu_{i,1}\cdots\nu_{i,n_i}\mu_{i+1,1}\cdots\mu_{i+1,m_{i+1}}}(\mathbf{r})$, which are the moments of the functions $K(\mathbf{r},\chi_i',\chi_{i+1})$ that are of order n_i $[m_{i+1}]$ in the components of $\mathbf{X}_i'(\tau_i')$ $[\mathbf{X}_{i+1}(\tau_{i+1})]$. These moments depend only on $\mathbf{r}$, (α_i',p_i',τ_i'), and $(\alpha_{i+1},p_{i+1},\tau_{i+1})$; they are well defined because the weight $\rho(\mathbf{X})$ is expected to decay faster than any inverse power law, according to paper I. Because of the invariance of both the measure $D(\mathbf{X})$ and the bonds F under global rotations of their arguments, the $\mathcal{K}^{[n_i,m_{i+1}]}_{\nu_{i,1}\cdots\nu_{i,n_i}\mu_{i+1,1}\cdots\mu_{i+1,m_{i+1}}}(\mathbf{r})$ are tensors of rank n_i+m_{i+1}. For instance, the chain with $\gamma_1=\cdots=\gamma_I=3$ is proportional to

$$\mathcal{K}^{[1]}_{\mu_1} * \partial_{\mu_1\nu_1}(v_C) * \mathcal{K}^{[1,1]}_{\nu_1\mu_2} * \cdots * \mathcal{K}^{[1,1]}_{\nu_{i-1}\mu_i} * \partial_{\mu_i\nu_i}(v_C) * \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}} * \cdots * \mathcal{K}^{[1,1]}_{\nu_{I-1}\mu_I} * \partial_{\mu_I\nu_I}(v_C) * \mathcal{K}^{[1]}_{\nu_I} \quad \text{(A2)}$$

with

$$\mathcal{K}^{[1]}_{\mu_1}(\mathbf{r};\alpha_a,p_a;\alpha_1,p_1,\tau_1) \equiv \int D(\mathbf{X}_a) \int D(\mathbf{X}_1)[\mathbf{X}_1(\tau_1)]_{\mu_1} K(\mathbf{r},\chi_a,\chi_1) \quad \text{(A3)}$$

and

$$\mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}}(\mathbf{r};\alpha_i',p_i',\tau_i';\alpha_{i+1},p_{i+1},\tau_{i+1}) \equiv \int D(\mathbf{X}_i') \int D(\mathbf{X}_{i+1})[\mathbf{X}_i'(\tau_i')]_{\nu_i}[\mathbf{X}_{i+1}(\tau_{i+1})]_{\mu_{i+1}} K(\mathbf{r},\chi_i',\chi_{i+1}). \quad \text{(A4)}$$

The summation over the space indices is implicit in (A2). In the same way, the chain with $\gamma_1=4$ and $\gamma_2=\cdots=\gamma_I=3$ is proportional to the sum of two terms

$$\mathcal{K}^{[2]}_{\mu_{1,1}\mu_{1,2}} * \partial_{\mu_{1,1}\mu_{1,2}\nu_1}(v_C) * \mathcal{K}^{[1,1]}_{\nu_1\mu_2} * \cdots * \mathcal{K}^{[1,1]}_{\nu_{i-1}\mu_i} * \partial_{\mu_i\nu_i}(v_C) * \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}} * \cdots * \mathcal{K}^{[1,1]}_{\nu_{I-1}\mu_I} * \partial_{\mu_I\nu_I}(v_C) * \mathcal{K}^{[1]}_{\nu_I}, \quad \text{(A5)}$$

where

$$\mathcal{K}^{[2]}_{\mu_{1,1}\mu_{1,2}}(\mathbf{r};\alpha_a,p_a;\alpha_1,p_1,\tau_1) = \int D(\mathbf{X}_a) \int D(\mathbf{X}_1)[\mathbf{X}_1(\tau_1)]_{\mu_{1,1}}[\mathbf{X}_1(\tau_1)]_{\mu_{1,2}} K(\mathbf{r},\chi_a,\chi_1), \quad \text{(A6)}$$

plus the term

$$\mathcal{K}^{[1]}_{\mu_1} * \partial_{\mu_1\nu_{1,1}\nu_{1,2}}(v_C) * \mathcal{K}^{[2,1]}_{\nu_{1,1},\nu_{1,2}\mu_2} * \cdots * \mathcal{K}^{[1,1]}_{\nu_{i-1}\mu_i} * \partial_{\mu_i\nu_i}(v_C) * \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}} * \cdots * \mathcal{K}^{[1,1]}_{\nu_{I-1}\mu_I} * \partial_{\mu_I\nu_I}(v_C) * \mathcal{K}^{[1]}_{\nu_I}, \quad \text{(A7)}$$

where the definition of $\mathcal{K}^{[2,1]}_{\nu_{1,1}\nu_{1,2}\mu_2}$ is similar to (A4). In Fourier space, the convolutions such as (A2), (A5), and (A7) are products of singular terms $k_{\mu_{i,1}}\cdots k_{\mu_{i,m_i}} k_{\nu_{i,1}}\cdots k_{\nu_{i,n_i}}/k^2$ and tensors $\mathcal{K}^{[n_i,m_{i+1}]}_{\nu_{i,1}\cdots\nu_{i,n_i}\mu_{i+1,1}\cdots\mu_{i+1,m_{i+1}}}(\mathbf{k})$.

At this point, we put the results of Sec. III A in terms of the decomposition of W into pure algebraic terms W_γ, each of which behaves as $1/r^\gamma$. The W_γ, defined in (5.33) of paper I, are related to the $w^{[m,n]}$ defined in (3.5) by

$$W_\gamma(\mathbf{k},\chi_i,\chi_i') = -\beta e_{\alpha_i} e_{\alpha_i'} \int_0^{p_i} d\tau_i \int_0^{p_i'} d\tau_i'$$
$$\times \{\delta([\tau_i - P(\tau_i)] - [\tau_i' - P(\tau_i')]) - 1\}$$
$$\times \sum_{(m_i,n_i)/m_i+n_i=\gamma-1} \frac{1}{m_i!\,n_i!}$$
$$\times w^{[m_i n_i]}(\mathbf{k},\mathbf{X}_i(\tau_i),\mathbf{X}_i'(\tau_i')). \quad \text{(A8)}$$

W_3 is proportional to $w^{[1,1]}$, W_4 to $w^{[1,2]}+w^{[2,1]}$, and $w^{[1,1]}$, $w^{[1,2]}$, and $w^{[2,1]}$ appear only in W_3 and W_4. The discussion of Sec. III B in Fourier space shows that rotational invariance together with dimensional analysis ensure that any convolution $K*W_{\gamma_1}*K*W_{\gamma_2}\cdots*W_{\gamma_{I-1}}*K*W_{\gamma_I}*K$ decays at least as $1/r^7$, either if $\gamma_1\geqslant 5$ or $\gamma_I\geqslant 5$ or if any of the γ_i with $i=2,\ldots,I-1$ is greater than 3 (because the corresponding small-$\mathbf{k}$ expansion starts at the order $|\mathbf{k}|^4$). Only the convolutions with $W_{\gamma_2}=\cdots=W_{\gamma_{I-1}}=W_3$ ($n_i=m_{i+1}=1$ for all $i=1,\ldots,I-1$) and γ_1 and γ_I equal to 3 or 4 may decay as $1/r^5$ (because their small-$\mathbf{k}$ expansions start at the order $|\mathbf{k}|^2$). ($W_{\gamma_1}=W_3$ corresponds to $m_1=1$, and in the case $W_{\gamma_1}=W_4$, according to the dimensional analysis, only the term in (2.14) corresponding to $m_1=2$ and $n_1=1$, $[\mathbf{X}_1(\tau_1)\cdot\mathbf{k}]^2[\mathbf{X}_1'(\tau_1')\cdot\mathbf{k}]$, may give a tail decreasing more slowly than $1/r^7$.) In fact, rotational invariance arguments and the harmonicity of the Coulomb potential enforce the latter convolutions to decrease at least as $1/r^7$ [namely, the first two terms in the small-$\mathbf{k}$ expansion of the convolutions (3.8) with $m_{i+1}=n_i=1$, $i=1,\ldots,I-1$ and (m_1,n_I) equal to one of the four cases in (3.37) are analytic and of order $|\mathbf{k}|^2$ and $|\mathbf{k}|^4$, respectively].

The general discussion in position space is more cumbersome than in Fourier space, but it can be summed up as follows in the special case of the convolutions with the slowest possible decay. The derivatives and the convolutions commute according to the property $\partial_\mu f * \partial_\nu g = -\partial_{\mu\nu}(f*g)$ and, after integration over the shapes of the loops, the convolution chain (A2) with all the γ_i's equal to 3 appears as the convolution

$$\mathcal{K}^{[1]}_{\mu_1} * \partial_{\mu_1\nu_I}(v_C) * G_{\text{mid}} * \mathcal{K}^{[1]}_{\nu_1}, \tag{A9}$$

where G_{mid} is a scalar function that depends only on $|\mathbf{r}|$,

$$G_{\text{mid}}(|\mathbf{r}|) = \partial_{\nu_1\mu_2}[v_C * \mathcal{K}^{[1,1]}_{\nu_1\mu_2}] * \cdots * \partial_{\nu_i\mu_{i+1}}[v_C * \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}}] * \cdots * \partial_{\nu_{I-1}\mu_I}[v_C * \mathcal{K}^{[1,1]}_{\nu_{I-1}\mu_I}]. \tag{A10}$$

The convolution (A.5) reads

$$\mathcal{K}^{[2]}_{\mu_{1,1}\mu_{1,2}} * \partial_{\mu_{1,1}\mu_{1,2}\nu_I}(v_C) * G_{\text{mid}} * \mathcal{K}^{[1]}_{\nu_I}. \tag{A11}$$

A mere dimensional analysis implies that each term $\partial_{\nu_i\mu_{i+1}}[v_C * \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}}]$ decays at least as $1/r^3$. Henceforth, G_{mid} and $\partial_{\mu_1\nu_I v_C}(r)$ fall off as $1/r^3$, whereas the $\mathcal{K}$'s decay as $1/r^6$. Moreover, because of rotational invariance, only the odd-order moments of $\mathcal{K}^{[1]}_{\mu_1}(\mathbf{r})$ and $\mathcal{K}^{[1]}_{\nu_I}(\mathbf{r})$ at the ends of the chain do not vanish. Consequently, according to the general analysis recalled in Sec. II C, the slowest nonzero algebraic tail of the convolution (A9) reads

$$\partial_{\sigma\sigma'}[\partial_{\mu_1\nu_I} v_C * G_{\text{mid}}](\mathbf{r}) \left(\int d\mathbf{x}\, x_\sigma \mathcal{K}^{[1]}_{\mu_1}(\mathbf{x}) \right) \times \left(\int d\mathbf{x}' x'_{\sigma'} \mathcal{K}^{[1]}_{\nu_I}(\mathbf{x}') \right), \tag{A12}$$

with $\sigma=\mu_1$ and $\sigma'=\nu_I$. Because of rotational invariance, the even-order moments of $\mathcal{K}^{[2]}_{\mu_{1,1}\mu_{1,2}}(\mathbf{x})$ do not vanish and the slowest possible tail of (A11) is

$$\partial_{\sigma'}[\partial_{\mu_{1,1}\mu_{1,2}\nu_I} v_C * G_{\text{mid}}](\mathbf{r}) \left(\int d\mathbf{x}\, \mathcal{K}^{[2]}_{\mu_{1,1}\mu_{1,2}}(\mathbf{x}) \right) \times \left(\int dx' x'_{\sigma'} \mathcal{K}^{[1]}_{\nu_I}(\mathbf{x}') \right), \tag{A13}$$

with $\sigma'=\nu_I$. The tails (A12) and (A13) behave as $1/r^5$.

In Fourier space, the singular terms corresponding to the $1/r^5$ tails are canceled because, on one hand, $v_C(\mathbf{k}) \propto 1/\mathbf{k}^2$ and, on the other hand, $\mathcal{K}^{[1,1]}_{\nu_i,\mu_{i+1}}(\mathbf{k}=\mathbf{0}) = A^{[1,1]}_{i,i+1}\delta_{\nu_i,\mu_{i+1}}$ according to (3.15), while $\mathcal{K}^{[1]}_{\mu_1}(\mathbf{k})$ starts as $A^{[1]}_{a,1}k_{\mu_1}$ and $\mathcal{K}^{[2]}_{\mu_{1,1}\mu_{1,2}}(\mathbf{k}=\mathbf{0}) = A^{[2]}_{a,1}\delta_{\mu_{1,1},\mu_{1,2}}$ according to (3.13). The mechanism is more intricate to write in position space than in Fourier space, but the argument that has been developed for the previous simple convolutions can be pursued as follows.

The harmonicity of the Coulomb potential implies that $\partial_{\nu_i\mu_{i+1}}[v_C * \mathcal{K}_{\nu_i\mu_{i+1}}](\mathbf{r})$ does not decay as the $1/r^3$ law given by dimensional analysis, but, in fact, falls off at least as $1/r^5$. Indeed, the invariance of K under global rotations implies that $\int d\mathbf{x}\, \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}}(\mathbf{x})$ is proportional to $\delta_{\nu_i,\mu_{i+1}}$ times a term independent from ν_i, and $\int d\mathbf{x}\, x_\sigma \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}}(\mathbf{x}) = 0$. Therefore, the asymptotic behavior of the term $\partial_{\nu_i\mu_{i+1}}(v_C * \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}})$ starts as

$$\partial_{\nu_i\mu_{i+1}}[v_C(\mathbf{r})] \int d\mathbf{x}\, \mathcal{K}^{[1,1]}_{\nu_i\mu_{i+1}}(\mathbf{x}) + O\left(\frac{1}{r^5}\right) = \Delta[v_C(\mathbf{r})] \frac{1}{3} \int d\mathbf{x} \sum_\nu \mathcal{K}^{[1,1]}_{\nu\nu}(\mathbf{x}) + O\left(\frac{1}{r^5}\right). \tag{A14}$$

Hence $G_{\text{mid}}(|\mathbf{r}|)$ (A10) decays in fact at least as $1/r^5$ and the leading terms in the asymptotic behavior of $[\partial_{\mu_1\nu_I} v_C * G_{\text{mid}}](\mathbf{r})$ is $\partial_{\mu_1\nu_I} v_C(\mathbf{r}) \times \int d\mathbf{y}\, G_{\text{mid}}(|\mathbf{y}|) + O(1/r^5)$. An algebraic tail of the convolution (A9) that would be slower than $1/r^7$, according to dimensional analysis, can only arise from the asymptotic behavior

$$\begin{aligned}
&\partial_{\sigma\sigma'\mu_1\nu_I}[v_C(r)] \left(\int d\mathbf{x}\, x_\sigma \mathcal{K}^{[1]}_{\mu_1}(\mathbf{x}) \right) \left(\int d\mathbf{x}' x'_{\sigma'} \mathcal{K}^{[1]}_{\nu_I}(\mathbf{x}') \right) \\
&\quad \times \left(\int d\mathbf{y}\, G_{\text{mid}}(|\mathbf{y}|) \right) + O\left(\frac{1}{r^7}\right) \\
&= \Delta\Delta\left(\frac{1}{r}\right) \left(\int d\mathbf{x}\, \frac{1}{3} \sum_{\mu_1} x_{\mu_1} \mathcal{K}^{[1]}_{\mu_1}(\mathbf{x}) \right) \\
&\quad \times \left(\int d\mathbf{x}' \frac{1}{3} \sum_{\nu_I} x'_{\nu_I} \mathcal{K}^{[1]}_{\nu_I}(\mathbf{x}') \right) \\
&\quad \times \left(\int d\mathbf{y}\, G_{\text{mid}}(|\mathbf{y}|) \right) + O\left(\frac{1}{r^7}\right).
\end{aligned} \tag{A15}$$

An analogous mechanism arises for (A11). As a conclusion, in the diagrammatic decomposition (2.6), all the convolutions decay faster than $1/r^6$ and only K falls off as $1/r^6$.

APPENDIX B

In this appendix, we study the structure of the ‘‘elementary’’ algebraic tails defined at the beginning of Sec. III D, before integration over the shapes of their end points.

It is readily shown that the structure (3.26) holds for the elementary algebraic tails that come from a single algebraic bond (W or F_{R6}). Indeed, a bond $W(\mathbf{r},\zeta,\zeta')$ is a series of terms $w^{[m,n]}[\mathbf{r},\mathbf{Z}(\tau),\mathbf{Z}'(\tau')]$ [given by (3.28)], which have the structure (3.26) with $q=m$, $q'=n$, and $P=1$. According to the expression of F_R given in (5.30) of paper I, a bond F_{R6}, defined in (2.4), is a sum of two terms: $[\exp(-\beta e_\alpha e_{\alpha'}\phi_{\text{elect}})-1]\exp(W) - F^{cc} - F^{cm} - F^{mc}$, which decays faster than any inverse power law, and $\exp(W)-1-W$, which is a series of algebraic terms, each of which is proportional to the product of at least two $w^{[m,n]}(\mathbf{r},\mathbf{Z},\mathbf{Z}')$ with various values of (m,n) but with the same argument $(\mathbf{r},\zeta,\zeta')$. Such an algebraic term reads $\Pi_{p=1}^{P_w} w^{[m_p,n_p]}(\mathbf{r},\zeta,\zeta')$ (up to counting factors), with $P_w \geqslant 2$, and, according to (3.28), has the structure (3.26) with P equal to the number P_w of w, $q=\Sigma_{p=1}^{P_w} m_p$, and $q' =\Sigma_{p=1}^{P_w} n_p$. Since the m's and n's are greater than or equal to 1, $q \geqslant P \geqslant 2$ and $q' \geqslant P \geqslant 2$. Subsequently, an elementary algebraic tail that originates from one algebraic W or F_{R6} bond has the structure (3.26), where P takes the following values. If (q,q') is equal either to (1,1), (1,2), or (2,1), the algebraic tail originates from one bond W and cannot arise from an F_{R6} bond and $P=1$. On the contrary, if $q \geqslant 2$ and $q' \geqslant 2$, the structure (3.26) may originate from the product of two $w^{[m,n]}(\mathbf{r},\zeta,\zeta')$ in an F_{R6} bond and P may take the values 1 and 2.

Now we turn to the structure of the algebraic tails of the convolutions of algebraic bonds (W or F_{R6}) with subdiagrams of a given $\tilde{\Pi}$ diagram and that have algebraic bonds at both ends. The structure is given before integration over the root points. The discussion is decomposed into two steps.

In the first step, every purely algebraic tail $S_{\{1\}I}^{(\gamma)}(\mathbf{r},\zeta,\zeta')$ is the inverse Fourier transform of one nonanalytic term arising from one algebraic bond (W or F_{R6}) times a product of I analytic terms originating from the small-$\mathbf{k}$ expansions of the other terms in the Fourier transform of the convolution. We proceed by recurrence. If $I=1$, the analytic term is of the form $\int d\mathbf{x}(i\mathbf{k}\cdot\mathbf{x})^n G_1(\mathbf{x},\chi_1,\zeta')$ and, according to the structure (3.26) of the nonanalytic term $S^{(\gamma_1-3)[q_1,q_1']}(\mathbf{k},\zeta,\chi_1)$ arising from an algebraic bond, with $\gamma_1=P_1+q_1+q_1'$, $P_1 \leqslant q_1$, and $P_1 \leqslant q_1'$,

$$S_{(1)1}^{(\gamma_1+n-3)}(\mathbf{k},\zeta,\zeta')=\int d\chi_1\rho(\chi_1)A^{[q_1]}_{\mu_1\cdots\mu_{q_1}}(\mathbf{Z}) \times S^{(\gamma_1-3)[q_1,q_1']}_{\{\}\{\}}(\mathbf{k}) \times A^{[q_1']}_{\nu_1\cdots\nu_{q_1'}}(\mathbf{X}_1)(i)^n k_{\sigma_1}\cdots k_{\sigma_n} \times \int d\mathbf{x}\, x_{\sigma_1}\cdots x_{\sigma_n} G_1(\mathbf{x},\chi_1,\zeta'), \tag{B1}$$

where k_σ (x_σ) is the component with index σ of the vector $\mathbf{k}$ ($\mathbf{x}$) and $S^{(\gamma_1-3)[q_1,q_1']}_{\{\}\{\}}(\mathbf{k})$ is a short notation for $S^{(\gamma_1-3)[q_1,q_1']}_{\mu_1\cdots\mu_{q_1}\nu_1\cdots\nu_{q_1'}}(\mathbf{k})$. Because of the rotational invariance of both the bonds $F(\mathscr{P},\mathscr{P}')$ and the weights $\rho(\mathscr{P})$, $G_1(\mathbf{x},\chi_1,\zeta')$is a function invariant by global rotation of its arguments. As a consequence, after integration over $\mathbf{x}$ and $\mathbf{X}_1$, we get a tensor $A^{[q_1'+n]}_{\{\}}(\mathbf{Z}')$ of rank $q_1'+n$,

$$\int d\chi_1\rho(\chi_1)A^{[q_1']}_{\nu_1\cdots\nu_{q_1'}}(\mathbf{X}_1)\int d\mathbf{x}\, x_{\sigma_1}\cdots x_{\sigma_n}G_1(\mathbf{x},\chi_1,\zeta') =[\mathbf{Z}']_{\nu_1}\cdots[\mathbf{Z}']_{\nu_{q_1'}}[\mathbf{Z}']_{\sigma_1}\cdots[\mathbf{Z}']_{\sigma_n}f(|\mathbf{Z}'|) \equiv A^{[q_1'+n]}_{\{\}}(\mathbf{Z}'), \tag{B2}$$

where $f(|\mathbf{Z}'|)$ is a function of $|\mathbf{Z}'|$. The tensor $A^{[q_1'+n]}_{\{\}}(\mathbf{Z}')$ of rank $q_1'+n$ is nonzero only if $q_1'+n$ is even. The inverse Fourier transform of $S_{(1)1}^{(\gamma_1+n-3)}(\mathbf{k},\zeta,\zeta')$ is of the form $A^{[q_1]}_{\{\}}(\mathbf{Z})A^{[q_1'+n]}_{\{\}}(\mathbf{Z}')$ times an algebraic term decaying as $1/r^\gamma$ with $\gamma=\gamma_1+n=P_1+q_1+(q_1'+n)$. Thus we retrieve the structure (3.26)

$$S_{(1)1}^{(\gamma_1+n)}(\mathbf{r},\zeta,\zeta')=A^{[q_1]}_{\{\}}(\mathbf{Z})A^{[q_1'+n]}_{\{\}}(\mathbf{Z}') \times S^{(\gamma_1+n)[q_1,q_1'+n]}_{\{\}\{\}}(\mathbf{r},\zeta,\zeta'), \tag{B3}$$

with $q=q_1 \geqslant P_1$ and $q'=q_1'+n \geqslant P_1$. When the number I of analytic terms increases, a discussion by recurrence shows that this structure is still preserved after integration over the shapes of the I internal points of the convolution. For instance, if $I=2$, there appear terms such as

$$S_{(1)2}^{(\gamma_1+n_1+n_2-3)}(\mathbf{k},\zeta,\zeta')=\int d\chi_1\rho(\chi_1)\int d\chi_2\rho(\chi_2) \times A^{[q_1]}_{\{\}}(\mathbf{Z})S^{(\gamma_1-3)[q_1,q_1']}_{\{\}\{\}}(\mathbf{k}) \times A^{[q_1']}_{\{\}}(\mathbf{X}_1)\int d\mathbf{x}(i\mathbf{k}\cdot\mathbf{x})^{n_1} \times G_1(\mathbf{x},\chi_1,\chi_2) \times\int d\mathbf{y}(i\mathbf{k}\cdot\mathbf{y})^{n_2}G_2(\mathbf{y},\chi_2,\zeta'). \tag{B4}$$

By using (B2) twice, we get

$$S_{(1)2}^{(\gamma_1+n_1+n_2-3)}(\mathbf{k},\zeta,\zeta') =\int d\chi_2\rho(\chi_2)A^{[q_1]}_{\{\}}(\mathbf{Z}) \times S^{(\gamma_1+n_1-3)[q_1,q_1'+n_1]}_{\{\}\{\}}(\mathbf{k}) \times A^{[q_1'+n_1]}_{\{\}}(\mathbf{X}_2)\int d\mathbf{y}(i\mathbf{k}\cdot\mathbf{y})^{n_2}G_2(\mathbf{y},\chi_2,\zeta') =A^{[q_1]}_{\{\}}(\mathbf{Z})S^{(\gamma_1+n_1+n_2-3)[q_1,q_1'+n_1+n_2]}_{\{\}\{\}}(\mathbf{k}) \times A^{[q_1'+n_1+n_2]}_{\{\}}(\mathbf{Z}'). \tag{B5}$$

$S_{(1)2}^{(\gamma_1+n_1+n_2)}(\mathbf{r},\zeta,\zeta')$ still has the structure (3.26), with $\gamma=\gamma_1+n_1+n_2=P_1+q+q'$ with $q=q_1 \geqslant P_1$ and $q'=q_1'+n_1+n_2 \geqslant P_1$.

In the second step, the algebraic tail $S_{\{J\}I}^{(\gamma)}(\mathbf{r},\zeta,\zeta')$ is the inverse Fourier transform of a product of J ($J \geqslant 2$) nonanalytic terms $S^{(\gamma_j)[q_j,q_j']}(\mathbf{k},\zeta_{j-1},\zeta_j)$, which arise from J bonds W or F_{R6}, times I analytic terms. According to the previous discussion, the product by analytic terms does not change the structure (3.26). Thus, after integration over the intermediate points of every product made of a $S^{(\gamma)}$ (coming from an algebraic bond) with a product of analytic terms, $S^{(\gamma)}_{\{J\}I}(\mathbf{r},\zeta,\zeta')$ comes from a product of $JS^{(\gamma_j-3)[q_j,q'_j]}_{(1)I_j}(\mathbf{k},\chi_{j-1},\chi_j)$,

$$\int\left[\prod_{j=1}^{J-1}d\chi_j\rho(\chi_j)\right]S^{(\gamma_1-3)[q,q'_1]}_{(1)I_1}(\mathbf{k},\zeta,\chi_1)$$
$$\times S^{(\gamma_2-3)[q_2,q'_2]}_{(1)I_2}(\mathbf{k},\chi_1,\chi_2)\cdots S^{(\gamma_J-3),[q_J,q']}_{(1)I_J}(\mathbf{k},\chi_{J-1},\zeta')$$
$$=\left(\prod_{j=1}^{J}S^{(\gamma_J-3[q_j,q'_j])}_{\{\}\{\}}(\mathbf{k})\right)$$
$$\times\left[\prod_{j=1}^{J-1}\int d\chi_j\rho(\chi_j)A^{[q'_j]}_{\{\}}(\mathbf{X}_j)A^{[q_{j+1}]}_{\{\}}(\mathbf{X}_j)\right].\qquad\text{(B6)}$$

$S^{(\gamma)}_{\{J\}I}(\mathbf{r},\zeta,\zeta')$ is of order $|\mathbf{k}|^{\gamma-3}$, with

$$\gamma-3=\sum_{j=1}^{J}(\gamma_j-3)=-3J+\sum_{j=1}^{J}P_j+q+q'$$
$$+\sum_{j=1}^{J-1}(q'_j+q_{j+1}),\qquad\text{(B7)}$$

where $q_j\geqslant P_j$ and $q'_j\geqslant P_j$. (We have used the notation $q_1\equiv q$ and $q'_J\equiv q'$.) $\int d\chi_j\rho(\chi_j)A^{[q'_j]}_{\{\}}(\mathbf{X}_j)A^{[q_{j+1}]}_{\{\}}(\mathbf{X}_j)$ does not vanish only if q'_j+q_{j+1} is even. Since $q'_j\geqslant P_j$ and $q_{j+1}\geqslant P_{j+1}$, the even values that q'_j+q_{j+1} can take are $P_j+P_{j+1}+\theta(P_j+P_{j+1})+2N$, where N is a positive integer, and $\theta(n)=0$ if n is even, while $\theta(n)=1$ if n is odd. As a consequence, when the values of the q_j's (with $j=2,\dots,J$) and those of the q'_j's (with $j=1,\dots,J-1$) vary, the power γ given by (B7) may take the values

$$\gamma(\{P_j\},q,q')=P(\{P_j\},q,q')+q+q',\qquad\text{(B8)}$$

where $P(\{P_j\},q,q')=\tilde{P}(\{P_j\},q,q')+2N$, with N any positive integer, and

$$\tilde{P}(\{P_j\},q,q')=3-3J+\sum_{j=1}^{J}P_j+\sum_{j=1}^{J-1}[P_j+P_{j+1}$$
$$+\theta(P_j+P_{j+1})],\qquad\text{(B9)}$$

with $q\geqslant P_1$ and $q'\geqslant P_J$. Since the allowed values for P_1 and P_J depend on q and q', so do the allowed values for $\tilde{P}$. The minimal value $\tilde{P}_{\min}$ for $\tilde{P}(\{P_j\},q,q')$ corresponds to $P_j=1$ for all $j=1,\dots,J$ (namely, all the nonanalytic terms come from W_3's). In this case $P_j+P_{j+1}+\theta(P_j+P_{j+1})=2$ and $\tilde{P}_{\min}(q,q')=1$ for any values of q and q'. If $q=q'=1$, then $P_1=1$ and $P_J=1$, so that the next value for $\tilde{P}(\{P_j\},q=1,q'=1)$ is given by $P_j=1$ for all j except one j_0, which is different from both 1 and J, and for which $P_{j_0}=2$. When P_{j_0} is increased from 1 to 2, then both $P_{j_0-1}+P_{j_0}+\theta(P_{j_0-1}+P_{j_0})$ and $P_{j_0}+P_{j_0+1}+\theta(P_{j_0}+P_{j_0-1})$ jump from 2 to 4 and $\tilde{P}(\{P_j\},1,1)$ jumps from $\tilde{P}_{\min}(q,q')=1$ to the value 6. Subsequently, according to (B8), when the P_j's vary, $P(1,1)=1,3,5,6,\dots$. On the contrary, if $q'\geqslant2$, while q takes any value (1 included), then $1\leqslant P_j\leqslant2$ and the smallest allowed value for $\tilde{P}(\{P_j\},q,q')$ that is greater than $\tilde{P}_{\min}(q,q')$ is given by $P_J=2$ and $P_j=1$ for $j\neq J$ because only $P_{J-1}+P_J+\theta(P_{J-1}+P_J)$ jumps from 2 to 4 and $\tilde{P}(\{P_j\},q,q')$ jumps from 1 to 4. Subsequently, $P(q,q')$ may take the values $P(q,q')=1,3,4,\dots$ as soon as $q'\geqslant2$ or $q\geqslant2$.

We notice that, if the convolution (with $J\geqslant2$) does not contain any W bond, then $P_j\geqslant2$ for all $j=1,\dots,J$, while $q\geqslant2$ and $q'\geqslant2$ as well. The minimal value for $\tilde{P}(\{P_j\},q,q')$ corresponds to $P_j=2$ for all $j=1,\dots,J$. In this case $P_j+P_{j+1}+\theta(P_j+P_{j+1})=4$ and $\tilde{P}=3J-1\geqslant5$, so that $P(q,q')$ takes the values $P(q,q')=5,7,8,\dots$ when $J\geqslant2$.

As a conclusion, the algebraic tails of a convolution of W or F_{R6} bonds with subdiagrams of $\tilde{\Pi}_{Wc}$ that have either a W or an F_{R6} bond at both ends have the structure (3.26) where $P(q,q')$ may take the values 1,3,5,6,... if $q=q'=1$, while P may take the values 1,3,4,... if $q\geqslant2$ or $q'\geqslant2$. Moreover, we notice that, if the convolution does not contain any W bond (except in the subdiagrams $\tilde{\Pi}_{Wc}$), then $q\geqslant2$ and $q'\geqslant2$, while $P(q,q')$ may take only the values 5,7,8,... .

Eventually, the allowed values for $P(q,q')$ in (3.27) are

$$P(q,q')=\begin{cases}1,3,5,6,\dots, & \text{if } q=q'=1\\ 1,3,4,\dots, & \text{if } (q,q')=(1,2)\text{ or }(2,1)\\ 1,2,\dots, & \text{if } q\geqslant2,\ q'\geqslant2.\end{cases}\qquad\text{(B10)}$$

Moreover, we notice that $P=1$ only in the case of a W bond or in the case of a convolution involving at least one W bond. If there is no W bond in the convolution, then $q\geqslant2$ and $q'\geqslant2$ and the allowed values for $P(q,q')$ are only

$$P_{Wc}(q,q')=2,5,7,8,\dots\ .\qquad\text{(B11)}$$

For instance, F_{R6} decays as $1/r^6$, while the convolution $F_{R6}(\mathbf{r},\chi_i,\chi_{i+1})*F_{R6}(\mathbf{r},\chi_{i+1},\chi_{i+2})$ involves a $1/r^9$ tail originating from $[W_3(\mathbf{r},\chi_i,\chi_{i+1})]^2*[W_3(\mathbf{r},\chi_{i+1},\chi_{i+2})]^2$, whose Fourier transform involves a $S^{(3)}(\mathbf{k},\chi_i,\chi_{i+1})S^{(3)}(\mathbf{k},\chi_{i+1},\chi_{i+2})$.

APPENDIX C

In this appendix we determine the exponents of the algebraic tails of various functions involving the algebraic tails T of a $\tilde{\Pi}_{Wc}$ diagrams, before and after integration over the shapes of both their root points and their internal points. An algebraic tail T has the structure (3.29) before integration over the shapes of its end points. We call γ^*_T the values of γ_T that survive after integration over $\mathbf{X}_a$ and $\mathbf{X}_b$ for a given (L,Q_a,Q_b) and γ^* denotes the values that γ^*_T takes when L, Q_a, and Q_b vary, namely, the γ^*'s are the allowed values for any $\tilde{\Pi}_{Wc}$ diagram.

Since the allowed values of δ_T in (3.33) depend on the values of $\{(q_l,q'_l)\}_{l=1,\dots,L}$, we have to distinguish four cases (I)–(IV) for a $\tilde{\Pi}_{Wc}$ diagram. In case (I), $q_l=q'_l=1$ for all $l=1,\dots,L$, so that, according to (B10), $\Sigma^L_{l=1}P_l$ may take only the values $L,L+2,L+4,L+5,\dots$, namely,

$$\delta_{T(\mathrm{I})}=0,2,4,5,\dots\qquad\text{(C1)}$$

and

$$\gamma_{T(\mathrm{I})}(\{q_l=1\},\{q'_l=1\},L)=3L+Q_a+Q_b+\delta_{T(\mathrm{I})}.\qquad\text{(C2)}$$

In case (II), there exists at least one index l_0 such that $(q_{l_0},q'_{l_0})=(1,2)$ or (2,1), while (q_l,q'_l) is equal to (1,1), (2,1), or (1,2) for all $l\neq l_0$. Then, according to (B10), $\Sigma_l P_l$ may take only the values $L,L+2,L+3,...,$

$$\delta_{T(\mathrm{II})}=0,2,3,\ldots, \tag{C3}$$

while $\Sigma_{l=1}^{L}(q_l+q'_l)\geqslant 2L+1$. In case (III), there exists at least one l such that $q_l\geqslant 2$ and $q'_l\geqslant 2$; then $\Sigma_{l=1}^{L}P_l$ may take any integer value greater than or equal to L,

$$\delta_{T(\mathrm{III})}=0,1,\ldots, \tag{C4}$$

while $\Sigma_{l=1}^{L}q_l\geqslant L+1$ and $\Sigma_{l=1}^{L}q'_l\geqslant L+1$. If the convolution contains no W bond [case (IV)], then $q\geqslant 2$ and $q'\geqslant 2$, while, according to (B11),

$$\gamma_{T(\mathrm{IV})}=2+q+q'+Q_a+Q_b+\delta_{T(\mathrm{IV})}, \tag{C5}$$

where $\delta_{T(\mathrm{IV})}\equiv P_{Wc}(q,q')-2$ may take the values 0,3,5,6,..., according to (B11).

First, we study the decay of $\int D(\mathbf{X}_a)D(\mathbf{X}_b)\widetilde{\Pi}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)$. After integration over the shapes of the end points $\mathscr{L}_a$ and $\mathscr{L}_b$, the algebraic tail (3.29) is not canceled only if $\int D(\mathbf{X}_a)\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)\neq 0$ and $\int D(\mathbf{X}_b)\mathscr{A}_{\{\}}^{[Q_b+\Sigma_l q'_l]}(\mathbf{X}_b)\neq 0$. Since $\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)$ is a tensor of rank $Q_a+\Sigma_{l=1}^{L}q_l$, $\int D(\mathbf{X}_a)\mathscr{A}_{\{\}}^{[Q_a+\Sigma_l q_l]}(\mathbf{X}_a)$ may be nonzero only if $Q_a+\Sigma_{l=1}^{L}q_l$ is even and the same property also holds for $Q_b+\Sigma_{l=1}^{L}q'_l$. Now, we have to distinguish the previous cases (I)–(IV).

In case (I), $Q_a+\Sigma_{l=1}^{L}q_l=Q_a+L$ and $Q_b+\Sigma_{l=1}^{L}q'_l=Q_b+L$. According to (C2), $\gamma_{T(\mathrm{I})}=3L+Q_a+Q_b+\delta_{T(\mathrm{I})}$ and $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)T(\mathbf{r}_{ab},\chi_a,\chi_b)$ does not vanish only if Q_a, Q_b, and L have the same parity. If Q_a and Q_b are even [case (Iα)], the only values of L that contribute are $L=2,4,6,...$; then $\gamma^*_{T(\mathrm{I})}-\delta_{T(\mathrm{I})}$ takes the values $6+Q_a+Q_b,12+Q_a+Q_b,...$, so that, according to (C1), when L varies,

$$\gamma^*_{T(\mathrm{I}\alpha)}=6+Q_a+Q_b,\ \ 8+Q_a+Q_b,\ \ 10+Q_a+Q_b,\ \ 11+Q_a+Q_b,\ldots, \tag{C6}$$

with $Q_a+Q_b=0,2,4,...$. Then $\gamma^*_{(\mathrm{I}\alpha)}=6,8,10,11,...$. If Q_a and Q_b are odd [case (Iβ)], the only values of L that contribute are $L=3,5,7,...$ and $\gamma^*_{T(\mathrm{I})}-\delta_{T(\mathrm{I})}$ takes the values $9+Q_a+Q_b,15+Q_a+Q_b,...$, so that, according to (C1), when L varies,

$$\gamma^*_{T(\mathrm{I}\beta)}=9+Q_a+Q_b,\ \ 11+Q_a+Q_b,\ \ 13+Q_a+Q_b,\ \ 14+Q_a+Q_b,\ldots, \tag{C7}$$

with $Q_a+Q_b=1,3,5,...$. Then $\gamma^*_{(\mathrm{I}\beta)}=10,12,14,15,...$. Eventually, in the case (I), $\gamma^*_{(\mathrm{I})}$ takes the values 6,8,10,11,... .

In case (II), either $Q_a+\Sigma_{l=1}^{L}q_l\geqslant Q_a+L+1$ while $Q_b+\Sigma_{l=1}^{L}q'_l\geqslant Q_b+L$ [subcase (IIα)] or $Q_a+\Sigma_{l=1}^{L}q_l\geqslant Q_a+L$ while $Q_b+\Sigma_{l=1}^{L}q'_l\geqslant Q_b+L+1$ [subcase (IIβ)]. We recall that $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)T(\mathbf{r}_{ab},\chi_a,\chi_b)$ does not vanish only if $Q_a+\Sigma_{l=1}^{L}q_l$ and $Q_b+\Sigma_{l=1}^{L}q'_l$ are even. In subcase (IIα), $Q_a+\Sigma_{l=1}^{L}q_l$ takes the even values

$$Q_a+L+1+\theta(Q_a+L+1)+2N \tag{C8}$$

while $Q_b+\Sigma_{l=1}^{L}q'_l$ takes the even values $Q_b+L+\theta(Q_b+L)+2N'$, with the notation $\theta(n)=0$ if n is even and $\theta(n)=1$ if n is odd. (N and N' are integers, $N\geqslant 0$, and $N'\geqslant 0$.) Thus, according to (3.33), when the (q_l,q'_l)'s vary, after integration over $\mathbf{X}_a$ and $\mathbf{X}_b$, $\gamma^*_{T(\mathrm{II}\alpha)}-\delta_{T(\mathrm{II})}$ may take only the values

$$\gamma^*_{T(\mathrm{II}\alpha)}-\delta_{T(\mathrm{II})}=3L+1+Q_a+Q_b+\theta(Q_a+L+1)+\theta(Q_b+L)+2N. \tag{C9}$$

According to (C3), $\gamma^*_{T(\mathrm{II}\alpha)}=\gamma^*_{T(\mathrm{II}\alpha)\min},\gamma^*_{T(\mathrm{II}\alpha)\min}+2,\gamma^*_{T(\mathrm{II}\alpha)\min}+3,\ldots$, with

$$\gamma^*_{T(\mathrm{II}\alpha)\min}=3L+1+Q_a+Q_b+\theta(Q_a+L+1)+\theta(Q_b+L) \tag{C10}$$

and $Q_a\geqslant 0$ as well as $Q_b\geqslant 0$. $\gamma^*_{T(\mathrm{II}\alpha)\min}(L=2)=7+Q_a+Q_b+\theta(Q_a+1)+\theta(Q_b)\geqslant 8$, $\gamma^*_{T(\mathrm{II}\alpha)\min}(L=3)=10+Q_a+Q_b+\theta(Q_a)+\theta(Q_b+1)\geqslant 11$, and $\gamma^*_{T(\mathrm{II}\alpha)\min}(L+1,Q_a,Q_b)>\gamma^*_{T(\mathrm{II}\alpha)\min}(L,Q_a,Q_b)$. In subcase (II$\beta$), the roles of Q_a and Q_b are exchanged and

$$\gamma^*_{T(\mathrm{II}\beta)\min}(L)=3L+1+Q_a+Q_b+\theta(Q_a+L)+\theta(Q_b+L+1). \tag{C11}$$

Eventually, $\gamma^*_{T(\mathrm{II})}$ takes the values

$$\gamma^*_{T(\mathrm{II})}=\gamma^*_{T(\mathrm{II})\min},\ \ \gamma^*_{T(\mathrm{II})\min}+2,\ \ \gamma^*_{T(\mathrm{II})\min}+3,\ldots, \tag{C12}$$

with

$$\gamma^*_{T(\mathrm{II})\min}=\inf[7+Q_a+Q_b+\theta(Q_a+1)+\theta(Q_b),\ \ 7+Q_a+Q_b+\theta(Q_a)+\theta(Q_b+1)]\geqslant 8 \tag{C13}$$

because $Q_a\geqslant 0$ and $Q_b\geqslant 0$. Thus $\gamma^*_{(\mathrm{II})}=8,10,11,...$.

In case (III), $Q_a+\Sigma_{l=1}^{L}q_l\geqslant Q_a+L+1$ and $Q_b+\Sigma_{l=1}^{L}q'_l\geqslant Q_b+L+1$, so that $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)T(\mathbf{r}_{ab},\chi_a,\chi_b)$ does not vanish only if $Q_a+\Sigma_{l=1}^{L}q_l$ takes the even values

$$Q_a+L+1+\theta(Q_a+L+1)+2N \tag{C14}$$

and if $Q_b+\Sigma_{l=1}^{L}q'_l$ takes the even values $Q_b+L+1+\theta(Q_b+L+1)+2N'$. Then, according to (3.33) and (C4), $\gamma^*_{T(\mathrm{III})}$ takes any integer value that is greater than or equal to $Q_a+Q_b+3L+2+\theta(Q_a+L+1)+\theta(Q_b+L+1)$. The smallest value corresponds to $L=2$,

$$\gamma^*_{T(\mathrm{III})}(L=2)-\delta_{T(\mathrm{III})}=8+Q_a+Q_b+\theta(Q_a+1)+\theta(Q_b+1)\geqslant 10, \tag{C15}$$

because $Q_a\geqslant 0$ and $Q_b\geqslant 0$. Eventually, $\gamma^*_{(\mathrm{III})}$ takes any integer value greater than or equal to 10.

In the case of a single convolution without any W bond [case (IV)], $L=1$, $q\geqslant 2$, and $q'\geqslant 2$, the even values taken by $Q_a+q\geqslant Q_a+2$ are

$$Q_a+2+\theta(Q_a)+2N \tag{C16}$$

and those taken by $Q_b+q'\geqslant Q_b+2$ are $Q_b+2+\theta(Q_b)+2N'$. So the allowed values of $\gamma^*_{T(\mathrm{IV})}$ given by (C5) are

$$\gamma^*_{T(\mathrm{IV})}-\delta_{T(\mathrm{IV})}=6+Q_a+Q_b+\theta(Q_a)+\theta(Q_b)+2N. \tag{C17}$$

Since $Q_a\geqslant 0$ and $Q_b\geqslant 0$, $\gamma^*_{(\mathrm{IV})}=6,8,9,\ldots$.

As a conclusion $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)\tilde{\Pi}_{Wc}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)$ decays only as $1/r^6,1/r^8,1/r^9,1/r^{10},\ldots$. The tails $1/r^6$ corresponds only to case (I) with $L=2$ and to the case (IV), both with $Q_a=Q_b=0$, while the tail $1/r^8$ is given only by cases (I), (II) and (IV). The $1/r^9$ tail comes from convolutions of case (IV), such as $[W_3]^2*[W_3]^2$. We notice that, from (3.32) and from the conclusion in the paragraph before (B.10), we can also retrieve the results of Sec. III B about the convolution chains with possible W bonds: according to dimensional analysis, such a single convolution chain decreases at least as $1/r^5,1/r^7$.

Second, we turn to the possible algebraic tails of the inverse Fourier transform of $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}T(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$. The inverse Fourier transform of the integrand decays as $1/r^{\gamma_T+n_a}$. Howerver, $\int D(\mathbf{X}_a)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}\mathscr{A}^{[Q_a+\Sigma_l q_l]}_{\{\}}(\mathbf{X}_a)\neq 0$ only if $n_a+Q_a+\Sigma^L_{l=1}q_l$ is even and, after integration over the shapes $\mathbf{X}_a$ and $\mathbf{X}_b$, some tails disappear. The discussion is similar to that performed for $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)T(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$ with n_a+Q_a in place of Q_a.

In case (I), only the n_a+Q_a's and the Q_b's that have the same parity contribute. In case (Iα), both n_a+Q_a and Q_b are even and $[n_a+\gamma_T]^*$ takes the values given by (C6) with n_a+Q_a in place of Q_a. The even values taken by n_a+Q_a, with $Q_a\geqslant 0$, are

$$n_a+\theta(n_a)+2N, \tag{C18}$$

while Q_b takes the even values $Q_b=0,2,4,\ldots$. Thus the possible values for $[n_a+\gamma]^*_{(I\alpha)}$ are

$$[n_a+\gamma]^*_{(\mathrm{I}\alpha)}=6+n_a+\theta(n_a),\;\; 8+n_a+\theta(n_a),\;\; 10+n_a+\theta(n_a),\;\; 11+n_a+\theta(n_a),\ldots\ . \tag{C19}$$

In case (Iβ), both n_a+Q_a and Q_b are odd and $[n_a+\gamma_T]^*$ takes the values given by (C7) with n_a+Q_a in place of Q_a. The odd values taken by n_a+Q_a, with $Q_a\geqslant 0$, are

$$n_a+\theta(n_a+1)+2N, \tag{C20}$$

while Q_b takes the odd values $1,3,5,\ldots$. Thus the possible values for $[n_a+\gamma]^*_{(I\beta)}$ are

$$[n_a+\gamma]^*_{(\mathrm{I}\beta)}=9+n_a+\theta(n_a+1),\;\; 11+n_a+\theta(n_a+1),\;\; 13+n_a+\theta(n_a+1),\;\; 14+n_a+\theta(n_a+1),\ldots\ . \tag{C21}$$

Eventually, in case (I), by inspection of (C19) and (C21) in the cases where n_a is either even or odd, we find that $[n_a+\gamma]^*_{(\mathrm{I})}$ takes the values

$$[n_a+\gamma]^*_{(\mathrm{I})}=6+n_a+\theta(n_a),\;\; 8+n_a+\theta(n_a),\;\; 10+n_a+\theta(n_a),\;\; 11+n_a+\theta(n_a),\ldots\ . \tag{C22}$$

In case (II) we have to consider two subcases, according to whether $Q_a+\Sigma^L_{l=1}q_l\geqslant L+1$ while $Q_b+\Sigma^L_{l=1}q'\geqslant L$ [subcase (IIα)] or $Q_a+\Sigma^L_{l=1}q_l\geqslant L$ while $Q_b+\Sigma^L_{l=1}q'\geqslant L+1$ [subcase (IIβ)]. In subcase (IIα), $n_a+Q_a+\Sigma^L_{l=1}q_l\geqslant n_a+Q_a+L+1$ takes the even values given by (C8), with n_a+Q_a in place of Q_a, and in subcase (IIβ) the roles of Q_a and Q_b are exchanged. According to (C12), $[n_a+\gamma_T]^*_{(\mathrm{II})}$ takes the values

$$[n_a+\gamma_T]^*_{(\mathrm{II})}=[n_a+\gamma_T]^*_{(\mathrm{II})\min},\;\; [n_a+\gamma_T]^*_{(\mathrm{II})\min}+2,\;\; [n_a+\gamma_T]^*_{(\mathrm{II})\min}+3,\ldots\ , \tag{C23}$$

with

$$[n_a+\gamma_T]^*_{(\mathrm{II})\min}=\inf[7+n_a+Q_a+Q_b+\theta(n_a+Q_a+1)+\theta(Q_b),\;\; 7+n_a+Q_a+Q_b+\theta(n_a+Q_a)+\theta(Q_b+1)]. \tag{C24}$$

Eventually, when Q_a and Q_b vary,

$$[n_a+\gamma]^*_{(\mathrm{II})\min}=\inf[7+n_a+\theta(n_a+1),8+n_a+\theta(n_a)]=7+n_a+\theta(n_a+1) \tag{C25}$$

and

$$[n_a+\gamma]^*_{(\mathrm{II})}=7+n_a+\theta(n_a+1),\;\; 9+n_a+\theta(n_a+1),\;\; 10+n_a+\theta(n_a+1),\ldots\ . \tag{C26}$$

In case (III), the discussion is the same as for $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)T(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)$, with n_a+Q_a in place of n_a. According to (C15), the allowed values for $[n_a+\gamma_T]^*$ are $8+n_a+Q_a+\theta(n_a+Q_a+1)+Q_b+\theta(Q_b+1)+N$. Since $n_a+Q_a+\theta(n_a+Q_a+1)\geqslant n_a+\theta(n_a+1)$ and $Q_b+\theta(Q_b+1)\geqslant 1$,

$$[n_a+\gamma]^*_{(\mathrm{III})}=9+n_a+\theta(n_a+1)+N\geqslant 10. \tag{C27}$$

In case (IV), $[n_a+\gamma_T]^*$ takes the values given by (C17) with n_a+Q_a in place of Q_a and, since $Q_a\geqslant 0$ and $Q_b\geqslant 0$,

$$[n_a+\gamma]^*_{(\mathrm{IV})}=6+n_a+\theta(n_a),\;\; 8+n_a+\theta(n_a),\;\; 9+n_a+\theta(n_a),\ldots\ . \tag{C28}$$

As a conclusion, by inspection of (C22) and (C26)–(C28) in the cases where n_a is either even or odd, we find that the inverse Fourier transform of $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}\tilde{\Pi}_{Wc}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)$ may decay as $1/r^{[n_a+\gamma]^*}$ with

$$[n_a+\gamma]^*=6+n_a+\theta(n_a),\;\; 8+n_a+\theta(n_a),\;\; 9+n_a+\theta(n_a),\ldots\ . \tag{C29}$$

Since $n_a+\theta(n_a)\geqslant 2$, this falloff behaves at least as $1/r^8$. The $1/r^{6+n_a+\theta(n_a)}$ tail comes from the cases (I) and (IV). For instance, in the case $n_a=1$ or 2, the allowed algebraic tails are $1/r^8, 1/r^{10}, 1/r^\gamma$ with $\gamma\geqslant 11$. In the case $n_a=3$ or 4, they are $1/r^{10}, 1/r^{12}, 1/r^\gamma$ with $\gamma\geqslant 13$.

Subsequently, since $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)[\exp(i\mathbf{k}\cdot\mathbf{X}_a)-1]\widetilde{\Pi}_{Wc}(\mathbf{r}_{ab},\mathbf{X}_a,\mathbf{X}_b)$ may be seen as a series of terms $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}\widetilde{\Pi}_{Wc}(\mathbf{r}_{ab},\mathbf{X}_a,\mathbf{X}_b)$, with $n_a\geqslant 1$, its algebraic tails are $1/r^8, 1/r^{10}, 1/r^{11},\ldots$.

Third, the inverse Fourier transform of the nonanalytic term $(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}(\mathbf{k}\cdot\mathbf{X}_b)^{m_b}T(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$ decays as $1/r^{\gamma_T+n_a+m_b}$, but after integration over the shapes of the root points, the only algebraic tails that survive are such that $n_a+Q_a+\Sigma_{l=1}^L q_l$ and $m_b+Q_b+\Sigma_{l=1}^L q_l'$ are even. The discussion is similar to that carried out for $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}T(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$, with m_b+Q_b in place of Q_b.

In case (I), only the n_a+Q_a's and the m_b+Q_b's that have the same parity contribute. If n_a+Q_a and m_b+Q_b are even, then, according to (C19), the values taken by $[n_a+m_b+\gamma]^*_{(\mathrm{I}\alpha)}$ are such that

$$[n_a+m_b+\gamma]^*_{(\mathrm{I}\alpha)}-n_a-\theta(n_a)-m_b-\theta(m_b)=6,8,10,11,\ldots\ . \tag{C30}$$

If n_a+Q_a and m_b+Q_b are odd, then, according to (C21), the values taken by $[n_a+m_b+\gamma]^*_{(\mathrm{I}\beta)}$ are

$$[n_a+m_b+\gamma]^*_{(\mathrm{I}\beta)}-n_a-\theta(n_a+1)-m_b-\theta(m_b+1)=9,11,13,14,\ldots\ . \tag{C31}$$

Eventually, in case (I), after inspection of (C30) and (C31) for n_a's and m_b's with various parities, $[n_a+m_b+\gamma]^*$ can be written as

$$[n_a+m_b+\gamma]^*_{(\mathrm{I})}=[n_a+m_b+\gamma]^*_{(\mathrm{I})\min}+\delta_{(\mathrm{I})}(m_a,m_b), \tag{C32}$$

with

$$[n_a+m_b+\gamma]^*_{(\mathrm{I})\min}=6+n_a+\theta(n_a)+m_b+\theta(m_b)\geqslant 10 \tag{C32$'$}$$

and

$$\delta_{(\mathrm{I})}(n_a,m_b)=\begin{cases}0,2,4,5,\ldots & \text{if } n_a \text{ and } m_b \text{ are even}\\ 0,2,3,\ldots & \text{if } n_a \text{ and } m_b \text{ do not have the same parity}\\ 0,1,\ldots & \text{if } n_a \text{ and } m_b \text{ are odd.}\end{cases} \tag{C33}$$

For instance, the algebraic tails may have the exponent $[n_a+m_b+\gamma]^*_{(\mathrm{I})}=10,11,\ldots$ in the case $n_a=m_b=1$, $[n_a+m_b+\gamma]^*_{(\mathrm{I})}=10,12,13,\ldots$ if (n_a,m_b) is equal to (1,2) or (2,1), and $[n_a+m_b+\gamma]^*_{(\mathrm{I})}=10,12,14,15,\ldots$ in the case $n_a=m_b=2$.

In case (II), by replacing Q_b by m_b+Q_b in (C24), we get a result similar to (C26),

$$[n_a+m_b+\gamma]^*_{(\mathrm{II})}=[n_a+m_b+\gamma]^*_{(\mathrm{II})\min},\ [n_a+m_b+\gamma]^*_{(\mathrm{II})\min}+2,\ [n_a+m_b+\gamma]^*_{(\mathrm{II})\min}+3,\ldots, \tag{C34}$$

with, since $\inf[7+n_a+\theta(n_a+1)+m_b+\theta(m_b),7+n_a+\theta(n_a)+m_b+\theta(m_b+1)]=7+n_a+m_b+\theta(n_a+m_b+1)$,

$$[n_a+m_b+\gamma]_{(\mathrm{II})\min}=7+n_a+m_b+\theta(n_a+m_b+1)\geqslant 10. \tag{C35}$$

Moreover, in case (II) the $1/r^{10}$ tail may appear only if $(n_a,m_b)=(1,1)$ or (1,2) or (2,1).

In case (III), $[n_a+m_b+\gamma_T]^*_{(\mathrm{III})}$ takes any integer value greater than or equal to $3L+2+n_a+Q_a+m_b+Q_b+\theta(n_a+Q_a+L+1)+\theta(m_b+Q_b+L+1)$ and, when L varies, the lowest value is obtained for $L=2$. Since $n_a+Q_a+\theta(n_a+Q_a+1)\geqslant n_a+\theta(n_a+1)$,

$$[n_a+m_b+\gamma]^*_{(\mathrm{III})}=8+n_a+\theta(n_a+1)+m_b+\theta(m_b+1)+N\geqslant 10. \tag{C36}$$

In case (III), the $1/r^{10}$ tail may appear only if $n_a=m_b=1$.

In case (IV) of convolutions with no W bond, the even values taken by n_a+Q_a+q, with $Q_a+q\geqslant 2$, are $n_a+2+\theta(n_a)+2N$, while the even values taken by m_b+Q_b+q', with $Q_b+q'\geqslant 2$, are $m_b+2+\theta(m_b)+2N'$. So, according to (C5),

$$[n_a+m_b+\gamma]^*_{(\mathrm{IV})}-n_a-m_b-\theta(n_a)-\theta(m_b)=6,8,9,\ldots\ . \tag{C37}$$

Eventually, after inspection of (C32), (C34), (C36), and (C37), $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}(\mathbf{k}\cdot\mathbf{X}_b)^{m_b}\widetilde{\Pi}_{Wc}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$ decays at least as $1/r^\gamma$ with

$$[n_a+m_b+\gamma]^*=[n_a+m_b+\gamma]^*_{\min}+\delta(m_a,m_b), \tag{C38}$$

where

$$[n_a+m_b+\gamma]^*_{\min}=6+n_a+\theta(n_a)+m_b+\theta(m_b)\geqslant 10 \tag{C39}$$

and

$$\delta(n_a,m_b)=\begin{cases}0,2,3,\ldots & \text{if } n_a \text{ and } m_b \text{ are even or do not have the same parity}\\ 0,1,\ldots & \text{if } n_a \text{ and } m_b \text{ are odd,}\end{cases} \tag{C40}$$

If $(n_a,m_b)=(1,1)$, the allowed tails are $1/r^{10},1/r^{11},...$, and if $(n_a,m_b)=(1,2)$, (2,1) or (2,2), tails $1/r^{10},1/r^{12},1/r^{13},...$ appear.

Subsequently, $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}[\exp(i\mathbf{k}\cdot\mathbf{X}_b)-1]\tilde{\Pi}_{Wc}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$ has a series of algebraic tails $1/r^{[n_a+\gamma]^*}$ given by (C39), with $m_b=1,2,\ldots,$

$$[n_a+\gamma]^*=8+n_a+\theta(n_a)+\delta(n_a) \tag{C41}$$

with

$$\delta(n_a)=\begin{cases}0,2,3,\ldots & \text{if } n_a \text{ is even}\\ 0,1,\ldots & \text{if } n_a \text{ is odd.}\end{cases} \tag{C42}$$

If $n_a=1$, the allowed tails are $1/r^{10},1/r^{11},\ldots$, and if $n_a=2$, tails $1/r^{10},1/r^{12},1/r^{13},\ldots$ appear.

Eventually, the allowed tails for $\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)[\exp(i\mathbf{k}\cdot\mathbf{X}_a)-1][\exp(i\mathbf{k}\cdot\mathbf{X}_b)-1]\tilde{\Pi}_{Wc}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$ are $1/r^{10},1/r^{11},\ldots$.

APPENDIX D

In this appendix we study the asymptotic behavior of the contributions to $C^Q|_{\text{nonexch}}(r)$ from $h_{(C)}$, $h_{(D)}$, and $h_{(E)}$ defined in (4.21). We insert the decomposition (4.29) into the expression (4.28) of $C_{(C)}(r)$. According to Appendix C, the inverse Fourier transform of $\int D(\mathbf{X}'_a)\int D(\mathbf{X}'_b)\tilde{\Pi}_{Wc}(\mathbf{k},\chi'_a,\chi'_b)[\exp(i\mathbf{k}\cdot\mathbf{X}'_b)-1]$ decays at least as $1/r^8$, $1/r^{10}$, $1/r^{11},\ldots$ and the first nonanalytic terms in $\int D(\mathbf{X}'_a)\int D(\mathbf{X}'_b)H^{n-}(\mathbf{k},\chi'_a,\chi'_b)[\exp(i\mathbf{k}\cdot\mathbf{X}'_b)-1]$ are of order $|\mathbf{k}|^5$, $|\mathbf{k}|^7$, $|\mathbf{k}|^9,\ldots$. Subsequently, according to (4.28), the first nonanalytic terms in the contribution from $H^{n-}(\mathbf{k})$ to $C^Q_{(C)}(\mathbf{k})$ are of order $|\mathbf{k}|^7$, $|\mathbf{k}|^9$, $|\mathbf{k}|^{10},\ldots$ and the tails of the contribution from $H^{n-}(\mathbf{r})$ to $C_{(C)}(r)$ fall off as $1/r^{10}$, $1/r^{12}$, $1/r^{13},...$.

On the other hand, in a very similar way to what has been done for the particle-particle correlation in Sec. III B, the contribution to $C_{(C)}(\mathbf{k})$ from a chain of $I+1$ graphs K linked by I bonds W, where $\mathcal{L}'_a$ is a non-Coulomb-root point [see (4.29)], can be written as a series of contributions from a chain of $I+1$ graphs K linked by I bonds $w^{[m_i,n_i]}$. Before summation over the species α and the sizes p of the loops and before integration over the times τ, each of these contributions is proportional to

$$\begin{aligned}\mathfrak{C}_I[C^Q_{(C)}](\mathbf{k})=&\frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}_2}\frac{1}{(\mathbf{k}^2)^I}\mathfrak{K}^{n-[m_1]}_{a',1}(\mathbf{k})\\&\times\mathfrak{K}^{[n_1,m_2]}_{1,2}(\mathbf{k})\mathfrak{K}^{[n_2,m_3]}_{2,3}(\mathbf{k})\cdots\\&\times\mathfrak{K}^{[n_{I-1},m_I]}_{I-1,I}(\mathbf{k})\int D(\mathbf{X}'_I)\int D(\mathbf{X}'_b)\\&\times[\mathbf{k}\cdot\mathbf{X}'_b(\tau'_I)]^{n_I}K(\mathbf{k},\chi'_I,\chi'_b)\\&\times\int_0^{p'_b}d\tau[e^{i\mathbf{k}\cdot\mathbf{X}'_b(\tau)}-1]\frac{\kappa^2}{\kappa^2+\mathbf{k}^2}.\end{aligned} \tag{D1}$$

According to rotational invariance arguments, the small-$\mathbf{k}$ expansion of $\int D(\mathbf{X}'_I)\int D(\mathbf{X}'_b)[\mathbf{k}\cdot\mathbf{X}'_b(\tau'_I)]^{n_I}K(\mathbf{k},\chi'_I,\chi'_b)\times\{\exp[i\mathbf{k}\cdot\mathbf{X}'_b(\tau)]-1\}$ starts at the order $|\mathbf{k}|^{1+n_I+\theta(1+n_I)}$ and involves even powers of $|\mathbf{k}|$ up to the first nonanalytic term, which is of order $|\mathbf{k}|^{5+n_I+\theta(n_I)}$ according to (C.42)]. (The corresponding inverse Fourier transform decays at least as $1/r^{10}$.) The structure of the small-$\mathbf{k}$ expansion of $\mathfrak{K}^{n-[m_1]}_{a',1}$ and $\mathfrak{K}^{[n_i,m_{i+1}]}_{i',i+1}$ is given in (3.34) and (3.35) (because the fact that $\mathcal{L}'_a$ is a non-Coulomb-root point does not change the structure). The dimension of the first term in the small-$\mathbf{k}$ expansion of (D1) is

$$\begin{aligned}D_{\mathfrak{C}_I[C^Q_{(C)}]}(\{m_i\},\{n_i\})=&2-2I+[m_1+\theta(m_1)]\\&+[1+n_I+\theta(1+n_I)]\\&+\sum_{i=1}^{I-1}[n_i+m_{i+1}\\&+\theta(n_i+m_{i+1})]\geqslant 4.\end{aligned} \tag{D2}$$

The order of the first singular term is $|\mathbf{k}|^{m_1+\theta(m_1)+3}$ in $\mathfrak{K}^{n-[m_1]}_{a',1}(\mathbf{k})$, according to (3.34), $|\mathbf{k}|^{n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+3}$ in $\mathfrak{K}^{[n_i,m_{i+1}]}_{i',i+1}$, according to (3.35), and $|\mathbf{k}|^{n_I+\theta(n_I)+5}$ in $\mathfrak{K}^{[n_I]}_{I',b}(\mathbf{k})$, according to (C42). Thus the first terms in (D1) are analytic up to the order in $|\mathbf{k}|$ that is the infimum of the three values obtained from (D2) by replacing one (and only one) of the terms in square brackets by the order of the first singular term in the corresponding $\mathfrak{K}$. In other words, the three values to be considered are given by (D2) with either $m_1+\theta(m_1)+3$ in place of $m_1+\theta(m_1)$, $5+n_I+\theta(n_I)$ in place of $1+n_I+\theta(1+n_I)$, or $n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+3$ in place of $n_i+m_{i+1}+\theta(n_i+m_{i+1})$. We notice that

$$[1+n_I+\theta(1+n_I),5+n_I+\theta(n_I)]=\begin{cases}(2,7) & \text{if } n_I=1\\(4,7) & \text{if } n_I=2\\(4,9) & \text{if } n_I=3\\(6,9) & \text{if } n_I=4.\end{cases} \tag{D3}$$

Henceforth, $5+n_I+\theta(n_I)\geqslant[1+n_I+\theta(1+n_I)]+3$ and the first singular term in (D1) is of order $D_{\mathfrak{C}_I[C^Q_{(C)}]}+3\geqslant 7$: $C^Q_{(C)}$ falls off at least as $1/r^{10}$. According to (C41), the next tails are $1/r^{11}$, $1/r^{12},\ldots$. In other words, the sum of the convolution chains in the decomposition of $h^{n-}*F^{mc}$ decays at least as $1/r^8$ ($1/r^9,\ldots$) and their convolution with Σ_D on the left, which determines their contribution (4.28) to $C^Q_{(C)}$ decays at least as $1/r^{10}$ ($1/r^{11},\ldots$).

The slowest algebraic tail corresponds to the case where $n_i=m_{i+1}=1$ for all $i=1,\ldots,I-1$ and $(m_1,n_I)=(1,1)$ or $(m_1,n_I)=(2,1)$. Then the small-$\mathbf{k}$ expansion of (D1) reads

$$\mathfrak{C}_I[C^Q_{(C)}](\mathbf{k}) = \frac{\mathbf{k}^2}{\kappa^2+\mathbf{k}^2}\frac{1}{(\mathbf{k}^2)^I}[A^{[1]\mathrm{or}[2]}_{a,1}|\mathbf{k}|^2 + B^{[1]\mathrm{or}[2]}_{a,1}|\mathbf{k}|^4 + \cdots + S^{(5)}_{a,1}(|\mathbf{k}|) + \cdots + S^{(7)}_{a,1}(|\mathbf{k}|) + S^{(8)}_{a,1}(|\mathbf{k}|) + \widetilde{O}(|\mathbf{k}|^8)]$$
$$\times[A^{[1]}_{I',b}|\mathbf{k}|^2 + B^{[1]}_{I',b}|\mathbf{k}|^4 + \cdots + S^{(7)}_{I',b}(|\mathbf{k}|) + S^{(8)}_{I',b}(|\mathbf{k}|) + \widetilde{O}(|\mathbf{k}|^8)]$$
$$\times\prod_{i=1}^{I-1}[A^{[1,1]}_{i',i+1}|\mathbf{k}|^2 + B^{[1,1]}_{i',i+1}|\mathbf{k}|^4 + \cdots + S^{(7)}_{i',i+1}(|\mathbf{k}|) + S^{(8)}_{i',i+1} + (|\mathbf{k}|)\widetilde{O}(|\mathbf{k}|^8)]. \quad \text{(D4)}$$

The first singular terms are of order $|\mathbf{k}|^7$, $|\mathbf{k}|^9$, $|\mathbf{k}|^{10}$,... . According to (C42), in the case where $(m_1, n_I) = (1,2)$ or (2,2), the structure of $\mathfrak{C}_I[C^Q_{(C)}](\mathbf{k})$ is the same as in (D4), with the only difference that $A^{[1]}_{I',b} = 0$. Then, the first singular terms are of order $|\mathbf{k}|^9$, $|\mathbf{k}|^{10}$,... . The case of $C^Q_{(D)}$ is derived from the case of $C^Q_{(C)}$ by exchanging the roles of $\mathscr{L}_a$ and $\mathscr{L}_b$ and the results are the same: $C^Q_{(C)}$ and $C^Q_{(D)}$ decay as $1/r^{10}$, $1/r^{12}$, $1/r^{13}$,... .

Now, we turn to the contribution $C^Q_{(E)}(r)$ given by (4.30). According to Appendix B, the inverse Fourier transform of

$$\int D(\mathbf{X}'_a)\int D(\mathbf{X}'_b)[\exp(i\mathbf{k}\cdot\mathbf{X}'_a)-1]\widetilde{\Pi}_{Wc}(\mathbf{k},\chi'_a,\chi'_b)$$
$$\times[\exp(i\mathbf{k}\cdot\mathbf{X}'_b)-1]$$

decays at least as $1/r^{10}$, $1/r^{11}$,... and the first nonanalytic terms in

$$\int D(\mathbf{X}'_a)\int D(\mathbf{X}'_b)[\exp(i\mathbf{k}\cdot\mathbf{X}'_a)-1]H(\mathbf{k},\chi'_a,\chi'_b)$$
$$\times[\exp(i\mathbf{k}\cdot\mathbf{X}'_b)-1]$$

are of order $|\mathbf{k}|^7$, $|\mathbf{k}|^8$,... . Subsequently, the contribution from H to $C_{(E)}(r)$ falls off as $1/r^{10}$, $1/r^{11}$,... . The contribution to $C^Q_{(E)}(\mathbf{k})$ from a chain of $I+1$ graphs K linked by I bonds W where both $\mathscr{L}'_a$ and $\mathscr{L}'_b$ are non-Coulomb-root points can be analyzed along the same lines as the contribution to $C^Q_{(C)}(\mathbf{k})$. We get a formula analogous to (D1)

$$\mathfrak{C}_I[C^Q_{(E)}](\mathbf{k}) = \frac{1}{(\mathbf{k}^2)^I}\left(\frac{\kappa^2}{\kappa^2+\mathbf{k}^2}\right)^2 \mathfrak{K}^{[n_1,m_2]}_{1,2}(\mathbf{k})\cdots\mathfrak{K}^{[n_{I-1},m_I]}_{I-1,I}(\mathbf{k})\int_0^{p'_a}d\tau\int D(\mathbf{X}'_a)\int D(\mathbf{X}_1)[e^{i\mathbf{k}\cdot\mathbf{X}'_a(\tau)}-1]$$
$$\times[\mathbf{k}\cdot\mathbf{X}_1(\tau_1)]^{m_1}K^{n-}(\mathbf{k},\chi'_a,\chi_1)\int_0^{p'_b}d\tau'\int D(\mathbf{X}'_I)\int D(\mathbf{X}'_b)[\mathbf{k}\cdot\mathbf{X}'_b(\tau'_I)]^{n_I}[e^{-i\mathbf{k}\cdot\mathbf{X}'_b(\tau')}-1]K(\mathbf{k},\chi'_I,\chi'_b). \quad \text{(D5)}$$

The dimension of the first term in the small-$\mathbf{k}$ expansion of $C^Q_{(E)}$ is

$$D_{\mathfrak{C}_I[C^Q_{(E)}]}(\{m_i\},\{n_i\}) = -2I + [1+m_1+\theta(1+m_1)] + [1+n_I+\theta(1+n_I)] + \sum_{i=1}^{I-1}[n_i+m_{i+1}+\theta(n_i+m_{i+1})] \geqslant 2. \quad \text{(D6)}$$

As in the discussion about $C^Q_{(C)}$, the first terms in the small-$\mathbf{k}$ expansion of (D5) are analytic up to the order in $|\mathbf{k}|$ that is the infimum of the three values (D6) with either $5+m_1+\theta(m_1)\geqslant 7$ in place of $1+m_1+\theta(1+m_1)\geqslant 2$, $5+n_I+\theta(n_I)\geqslant 7$ in place of $1+n_I+\theta(1+n_I)\geqslant 2$, or $n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+3\geqslant 7$ in place of $n_i+m_{i+1}+\theta(n_i+m_{i+1})\geqslant 2$. So the first singular term in (D5) is at least of order 7 [the next term is of order 8 according to (C42)] and the contribution to $C^Q_{(E)}$ from the convolution chains falls off as $1/r^{10}$, $1/r^{11}$,... .

The slowest algebraic tail corresponds to $n_i = m_{i+1} = 1$ for all $i = 1,\ldots,I$ and reads

$$\mathfrak{C}_I[C^Q_{(E)}](\mathbf{k}) = \frac{1}{|\mathbf{k}|^{2I}}[A^{[1]}_{a,1}|\mathbf{k}|^2 + B^{[1]}_{a,1}|\mathbf{k}|^4 + \cdots + S^{(7)}_{a,1}(|\mathbf{k}|) + S^{(8)}_{a,1}(|\mathbf{k}|) + \widetilde{O}(|\mathbf{k}|^8)]$$
$$\times[A^{[1]}_{I',b}|\mathbf{k}|^2 + B^{[1]}_{I',b}|\mathbf{k}|^4 + \cdots + S^{(7)}_{I',b}(|\mathbf{k}|) + S^{(8)}_{I',b}(|\mathbf{k}|) + \widetilde{O}(|\mathbf{k}|^8)]$$
$$\times\prod_{i=1}^{I-1}[A^{[1,1]}_{i',i+1}|\mathbf{k}|^2 + B^{[1,1]}_{i',i+1}|\mathbf{k}|^4 + \cdots + S^{(7)}_{i',i+1}(|\mathbf{k}|) + S^{(8)}_{i',i+1}(|\mathbf{k}|) + \widetilde{O}(|\mathbf{k}|^8)]. \quad \text{(D7)}$$

The first singular terms are of order $|\mathbf{k}|^7$, $|\mathbf{k}|^8$,... and $C^Q_{(E)}$ decays as $1/r^{10}$, $1/r^{11}$,... .

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