Agglomerate erosion: A nonscaling solution to the fragmentation equation

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The erosion of agglomerates in viscous flows—a process proportional to surface area of the agglomerate—is analyzed by means of fragmentation theory. An analytical solution is obtained; this solution cannot be described by the usual scaling ansatz, nor does it have an asymptotic form. Fitting to available data is discussed.

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Dispersion of powdered or clustered solids in viscous liquids—often referred to as dispersive mixing—is of considerable industrial importance [1,2]. The number of fundamental studies is, however, small. It was, for example, only recently [3,4] that two mechanisms—rupture and erosion—were identified as critical, *rupture* referring to the breakage of a cluster into several fragments of comparable size, and *erosion* to the gradual shearing off of small fragments from larger clusters. It was further established that erosion dominates dispersion when hydrodynamic stresses are low, and that erosion may be important to dispersion *even* when rupture is present [5]. Both mechanisms—rupture and erosion—have been the subject of fragmentation theory; a good review of this area is provided by Redner [6]. However, in spite of its potential usefulness, fragmentation theory does not appear to be utilized in the dispersion literature. On the other hand, there have been relatively few attempts to develop a solution to the fragmentation equation when the process is dominated solely by erosion. There are therefore two interrelated objectives to this paper: (i) to illustrate the use of fragmentation theory in a problem of practical importance, where erosion is the primary mechanism; and (ii) to demonstrate that some aspects of fragmentation theory need further examination.

The irreversible, continuous breakup of solids in a well mixed batch system can be described by the linear fragmentation equation

$$\frac{\partial c(x,t)}{\partial t} = -a(x)c(x,t) + \int_x^\infty a(y)f(x|y)c(y,t)dy, \quad (1)$$

where $c(x,t)$ is the concentration of clusters of mass x at time t; $a(x)$ is the *overall rate* of fragmentation of clusters of mass x; and $f(x|y)$, referred to as the *relative rate* of breakup, is the expected number of fragments of size x produced from the breakage of a cluster of size y. The *breakup kernels* $a(x)$ and $f(x|y)$ determine the kinetics of the fragmentation process. The precise forms of $a(x)$ and $f(x|y)$ depend on the specific problem under investigation. However, general conclusions may be drawn even in the absence of the exact specification of the kernels.

Typically, *homogeneous* kernels are considered, in which the overall rate of breakup is characterized by $a(x) \sim x^\lambda$, where λ is known as the homogeneity index, and the relative rate of breakup has the form

$$f(x|y) = \frac{1}{y} b\left(\frac{x}{y}\right). \quad (2)$$

The function $b(r)$, which defines the average number of fragments $\int_0^1 b(r)dr$, produced on a single breakup event, obeys $\int_0^1 rb(r)dr = 1$, due to conservation of mass.

Frequently, the analysis of the fragmentation equation is simplified by use of the scaling ansatz

$$c(x,t) = s^\gamma \phi(\nu) \quad \text{with} \quad \nu = \frac{x}{s(t)}, \quad (3)$$

where $s(t)$ is the average cluster mass, and ϕ is referred to as the scaling distribution. The exponent γ is required by mass conservation to be -2. It is commonly assumed that the scaling distribution is approached asymptotically and is independent of the initial conditions. Hence, for long times, fragmentation is dependent on the ratio of cluster mass to the average cluster mass. From the scaling ansatz and the assumption of homogeneous breakup kernels, general forms may be determined for the tails of the distribution—limits of small mass, $x/s(t) \ll 1$, and large mass, $x/s(t) \gg 1$.

If the breakup kernels are left unspecified, only limiting forms of the size distribution may be obtained. However, a few solutions corresponding to specific breakup kernels have also been studied [7–9]. Many of these solutions have a physical basis; however, they do not seem to include erosion. For example, a recent study by Ziff [10] obtains explicit forms to the scaling distribution for homogeneous breakup kernels of the form

$$b(r) = \alpha q r^{\alpha-2} + \beta(1-q)r^{\beta-2}, \quad (4)$$

where q, α, and β are adjustable parameters. This kernel allows for an exact solution to the full transient fragmentation equation when $\alpha = \lambda$ and $q = \beta/(\beta-\lambda)$.

Here we consider another class of breakup kernels designed to model erosion. Powell and Mason [11] and Rwei, Manas-Zloczower, and Feke [4] note that the overall rate of erosion is proportional to the surface area. This dependence of the overall rate of breakup on surface area is consistent with homogeneous kernels, and, therefore, we may use $a(x) = kx^\lambda$ in (1) without loss in generality. Experiments [4] also indicate that the distribution of small fragments eroded off clusters is a relatively narrow Gaussian distribution. Thus we may approximate the relative rate of breakup as binary

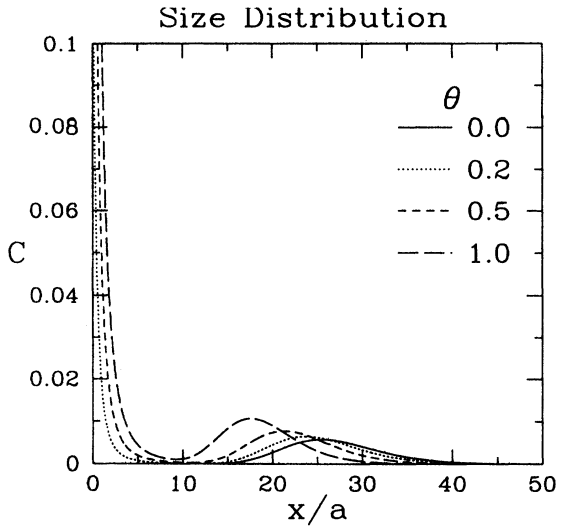

FIG. 1. Typical evolution of cluster size distribution during erosion. Here c is the concentration of clusters of mass x, a is the average size of the small fragments which are eroded off the larger clusters, the dimensionless time θ is $(Kt/a^{1-\lambda})$, and $\lambda=\frac{2}{3}$. The initial distribution is log normal with an average cluster size of $27a$ and a standard deviation of $5.4a$. The distribution of small fragments, $h(x)$ [Eq. (13)], is assumed to be log normal with a standard deviation of $2a$.

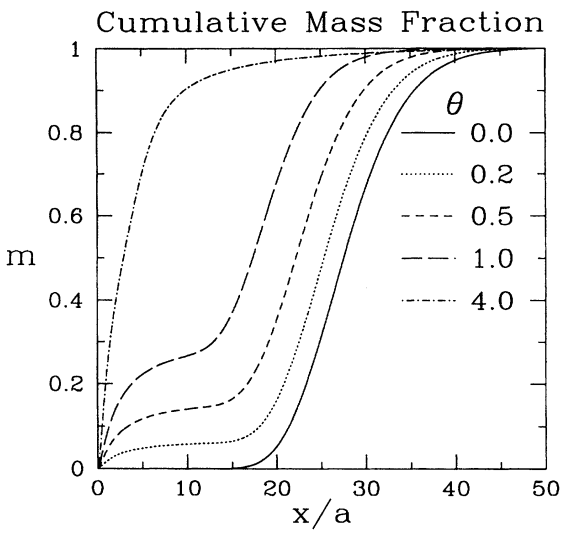

FIG. 2. Typical cumulative mass fraction of clusters less than mass x. The cumulative mass fraction of clusters, $m(x,t)$, is defined by $m(x,t)=(1/M_1)\int_0^x yc(y,t)dy$. All other parameters are the same as in Fig. 1.

breakup: a cluster of size ε and a cluster of size $y-\varepsilon$, with $y\gg\varepsilon$. With this assumption the relative rate of breakup becomes

$$f(x|y)=\delta[x-(y-\varepsilon)]+\delta(x-\varepsilon), \tag{5}$$

where y is the mass of the parent cluster, and ε the mass of the ultimate cluster or smallest possible cluster [this implies that clusters are made up of particles of mass ε; the size distribution is actually discrete; however, if $s(t)\gg\varepsilon$, we may approximate the distribution as a continuum.]

Substituting (5) into (1) yields

$$\frac{\partial c(x,t)}{\partial t}=k(x+\varepsilon)^{\lambda}c(x+\varepsilon,t)-kx^{\lambda}c(x,t)$$
$$+k\delta(x-\varepsilon)\int_{\varepsilon}^{\infty}y^{\lambda}c(y,t)dy. \tag{6}$$

The first term on the right hand side describes the rate of production of clusters of mass x due to the erosion of clusters of mass $x+\varepsilon$, whereas the second term is the rate of erosion of clusters of mass x; finally, the last term is the rate of production of mass ε clusters due to erosion of all clusters larger than ε. Since $\varepsilon\ll x$,

$$k[(x+\varepsilon)^{\lambda}c(x+\varepsilon,t)-x^{\lambda}c(x,t)]\approx\varepsilon k\frac{\partial}{\partial x}[x^{\lambda}c(x,t)], \tag{7}$$

and, therefore, the fragmentation equation becomes

$$\frac{\partial c(x,t)}{\partial t}=K\left\{\frac{\partial}{\partial x}[x^{\lambda}c(x,t)]+\frac{\delta(x-\varepsilon)}{\varepsilon}\int_{\varepsilon}^{\infty}y^{\lambda}c(y,t)dy\right\}, \tag{8}$$

where $K=\varepsilon k$ (this equation is somewhat similar to a model

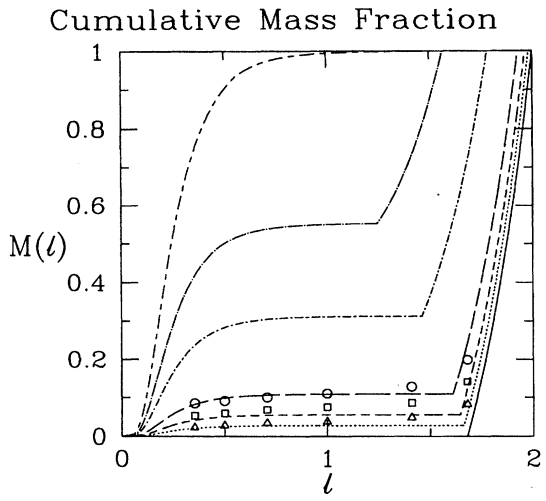

FIG. 3. Fit of experimental data from Neil and Bridgwater [14]. $M(l)$ is the cumulative mass fraction of clusters with a characteristic length less than l. The experimental data is for molecular sieve beads initially between 1.68 and 2.0 mm in diameter. We assume that the size distribution is initially uniform between 1.68 and 2.0 mm in diameter, $\lambda=\frac{2}{3}$, and $h(x)$ is log normal. The standard deviation and average of $h(x)$ and K are determined by least squares. The solid curve corresponds to the initial conditions. Other curves are parametrized as follows: $60s$, $120s$, $240s$, $750s$, $1500s$, and $5000s$. The experimental points are as follows: triangles $60s$, squares $120s$, and circles $240s$.

of erosion proposed by Pandya and Spielman [12]; however, it allows for an analytical solution).

After multiplying (9) by x^{λ} and substitution of $f(x,t)=x^{\lambda}c(x,t)$, we find that

$$\frac{\partial f(x,t)}{\partial t}=Kx^{\lambda}\frac{\partial f(x,t)}{\partial x} \quad \text{for } x>\varepsilon. \tag{9}$$

Here we use the fact that the erosion of large clusters is unaffected by the concentration of size ε clusters. Equation (9) may be solved with the method of characteristics [13], giving

$$f(x,t)=f(\eta,0) \quad \text{with } f(\eta,0)=\eta^{\lambda}c(\eta,0), \tag{10}$$

where $\eta\equiv[x^{1-\lambda}+(1-\lambda)Kt]^{1/(1-\lambda)}$. Hence we obtain

$$c(x,t)=\left(\frac{\eta}{x}\right)^{\lambda}c(\eta,0) \quad \text{for } x>\varepsilon. \tag{11}$$

From (11) and (8), we can determine the concentration of ultimate clusters, $c(x=\varepsilon,t)$. The cluster size distribution is, therefore, given by

$$c(x,t)=\begin{cases} K\int_0^t\left\{\left(\frac{\eta}{\varepsilon}\right)^{\lambda}\left.\frac{df(\eta,0)}{d\eta}\right|_{\eta=[\varepsilon^{1-\lambda}+(1-\lambda)Kt']^{1/(1-\lambda)}}+\frac{1}{\varepsilon}\int_{\varepsilon}^{\infty}\xi^{\lambda}c(\xi,0)dx\right\}dt' & \text{for } x=\varepsilon \\ \left(\frac{\eta}{x}\right)^{\lambda}c(\eta,0) & \text{for } x>\varepsilon, \end{cases} \tag{12}$$

where $\xi=(x^{(1-\lambda)}+(1-\lambda)Kt')^{1/(1-\lambda)}$. Here we note that the first term inside the integral with respect to time, which accounts for the second cluster of mass ε created by the erosion of a cluster of mass 2ε, is negligible while $s(t)\gg\varepsilon$.

An examination of Eq. (12) reveals that the solution is essentially bimodal, with $c(x,t)$ for large x dependent upon the initial conditions, and $c(x,t)$ for small x equal to the δ distribution about $x=\varepsilon$ multiplied by a factor which allows for mass conservation. If we assume that the distribution for large x is independent of the distribution for small x, then we may substitute a more realistic distribution $h(x)$ for the δ distribution for small x. Hence we determine that the cluster size distribution produced by erosion may be approximated by the bimodal distribution

$$c(x,t)=\left(\frac{[x^{1-\lambda}+(1-\lambda)Kt]^{1/(1-\lambda)}}{x}\right)^{\lambda}c([x^{1-\lambda}+(1-\lambda)Kt]^{1/(1-\lambda)},0)+g(t)h(x). \tag{13}$$

Here $g(t)$ is determined via mass conservation to be

$$g(t)=\frac{M_1-\int_0^{\infty}x^{1-\lambda}([x^{1-\lambda}+(1-\lambda)Kt]^{1/(1-\lambda)})^{\lambda}c([x^{1-\lambda}+(1-\lambda)Kt]^{1/(1-\lambda)},0)dx}{\int_0^{\infty}xh(x)dx}, \tag{14}$$

where M_1 is the total mass in the system. Typical evolution of the cluster size distribution and cumulative mass is shown in Figs. 1 and 2.

Experimental data of erosion processes are limited—no experiments provide cluster size distributions suitable for comparison with our model. Hence we fit data from the comparable process of attrition [14]. Neil and Bridgwater [14] determine, by sieving, the cumulative mass fraction of clusters $M(l)$, with a characteristic length less than l. Assume that $x\approx(\pi/6)\rho l^3$, where ρ is the density of the eroding solids, and that $h(x)$ is log normal [12]. Figure 3 shows the fitting to available experimental data. Deviation of the experimental data from the model is due to inefficiencies in the sieving process, the assumed relationship between x and l, and/or the presence of rupture. Better data are clearly needed before hard conclusions can be drawn (concentrations would be more enlightening than cumulative mass, and the reported size distributions are relatively narrow). It is nevertheless apparent that the relative constancy of $M(l)$ with l indicates the presence of a bimodal distribution.

This approximation of erosion highlights two important points. First, erosion cannot be described by the usual scaling ansatz (4). This is a result of the nonhomogeneous relative rate of breakup. Second, the size distribution does not have an asymptotic form, i.e., it is always dependent upon the initial conditions.

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