Perturbation theories of a discrete, integrable nonlinear Schrödinger equation

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We rederive the discrete inverse-scattering transform (IST) perturbation results for the time evolution of the parameters of a discrete nonlinear Schrödinger soliton from certain mathematical identities that can be viewed as conserved quantities in the discrete, integrable nonlinear Schrödinger equation in (1+1) dimension. This method significantly simplifies the derivation of the IST perturbation results. We also present a specific example for which the adiabatic IST perturbation results and the collective coordinate method results exactly coincide. This is achieved by establishing a correct Lagrangian formalism for soliton parameters via transforming dynamical variables that obey a deformed Poisson structure to ones that possess a canonical Poisson structure.

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I. INTRODUCTION

The discrete nonlinear Schrödinger (NLS) system arises in diverse physical situations [1]. Understanding its dynamics is an essential step towards illumination of the dynamics of general, extended, nonlinear discrete systems. The central question is how to describe nonlinear coherent excitations and their associated dynamics in discrete NLS systems. Since the NLS system in the Ablowitz-Ladik (AL) discretization, i.e.,

$$i\dot{\psi}_n = -\left(\psi_{n+1}+\psi_{n-1}\right) - \mu\left(\psi_{n+1}+\psi_{n-1}\right)\left|\psi_n\right|^2 \quad (1)$$

is completely integrable in (1+1) dimension [2,3], various discrete NLS equations derived from physical systems (see, e.g., [4,5]) can be usefully viewed as the AL NLS equation with a perturbation. Of these systems, the perturbed discrete NLS equation

$$\begin{aligned} i\dot{\psi}_n = &-\left(\psi_{n+1}+\psi_{n-1}\right) \\ &-\left[\mu\left(\psi_{n+1}+\psi_{n-1}\right)+2\nu\psi_n\right]\left|\psi_n\right|^2 \end{aligned} \quad (2)$$

is a prototype that has a very simple structure, i.e., a combination of the integrable and nonintegrable terms in a tunable way, thus enabling us to emphasize the effects of both nonintegrability and discreteness with conceptual simplicity [7,8]. The perturbation theory [4] based on an inverse scattering transform (IST) is useful in the study of the nonlinear localized structures [6] in these systems by answering the question of how the perturbation modifies the dynamics of an AL soliton. In what follows we will demonstrate that the IST perturbation results for the time evolution of the parameters of an AL soliton, under adiabatic approximation, can be simply derived from certain mathematical identities that are associated with the conservation laws in the AL system. This method enables us to obtain the IST perturbation results in a significantly simplified way. Furthermore, we point out that, for the prototypical NLS equation (2), in the adiabatic framework, the perturbation theories based on the IST and based on a collective coordinate method coincide exactly, which leads to an equivalent description for an effective point particle theory for the soliton motion in the lattice. As noted in Ref. [4], this equivalence is qualitative rather than being exact for most systems, so the case is particularly interesting.

II. REDERIVATION OF THE IST PERTURBATION RESULTS

The general perturbed AL NLS equation can be written as

$$i\dot{\psi}_n + \left(\psi_{n+1}+\psi_{n-1}\right)\left(1+|\psi_n|^2\right) = \epsilon f(\psi_n), \quad (3)$$

where $f^*(\psi_n) = f(\psi_n^*)$, n is a lattice index, $-\infty < n < +\infty$, and the scaling property of Eq. (3) allows μ to be set to unity. A detailed account of the IST perturbation theory for the evolution of the parameters of an AL soliton

$$\psi_{s,n} = \sinh\beta\,\mathrm{sech}\left[\beta(n-x)\right]\exp\left[i\alpha(n-x)+i\sigma\right] \quad (4)$$

in the adiabatic approximation can be found in Ref. [4]. Note that, for the unperturbed case $\epsilon = 0$, the soliton (4) has

$$\dot{x} = \frac{2\sinh\beta}{\beta}\sin\alpha, \quad (5)$$

$$\dot{\sigma} = 2\cos\alpha\cosh\beta + 2\alpha\sin\alpha\frac{\sinh\beta}{\beta}, \quad (6)$$

and $\alpha \in [-\pi,\pi]$ and $\beta \in (0,+\infty)$ are constants. Here we prove that the IST perturbation results can be simply derived from some mathematical identities. Since these identities can be viewed as conserved quantities in the unperturbed AL system (see the Appendix), the IST perturbation results are direct consequences of conservation laws in the AL system. We will invoke the following three identities that hold for arbitrary real x:

$$\sum_{n=-\infty}^{\infty} \operatorname{sech}[\beta(n-x)] \operatorname{sech}[\beta(n+1-x)] = \frac{2}{\sinh\beta}, \quad (7)$$

$$\sum_{n=-\infty}^{\infty} \ln\left\{1+\sinh^2\beta \operatorname{sech}^2[\beta(n-x)]\right\} = 2\beta, \quad (8)$$

$$\sum_{n=-\infty}^{\infty} (n-x) \ln\left\{1+\sinh^2\beta \operatorname{sech}^2[\beta(n-x)]\right\} = 0. \quad (9)$$

The proof of these identities is relegated to the Appendix. Using Identities (8) and (9), we have

$$2\beta = \sum_{n=-\infty}^{+\infty} \ln\left(1+|\psi_{s,n}|^2\right), \quad (10)$$

$$2\beta x = \sum_{n=-\infty}^{+\infty} n \ln\left(1+|\psi_{s,n}|^2\right). \quad (11)$$

Taking time derivatives of Eqs. (10) and (11), we arrive at

$$2\beta\dot{x} = \sum_{n=-\infty}^{+\infty} (n-x) \frac{d}{dt} \ln\left(1+|\psi_{s,n}|^2\right). \quad (12)$$

Next, we invoke the adiabatic approximation, which requires that the evolution of the parameters be constrained such that $\psi_{s,n}(t)$ [Eq. (4)] obeys the equation of motion (3). Substituting Eq. (3) for $\dot{\psi}_{s,n}$ leads to

$$2\beta\dot{x} = \sum_{n=-\infty}^{+\infty} -i(n-x)[\psi_{s,n}(\psi^*_{s,n+1}+\psi^*_{s,n-1}) - \psi^*_{s,n}(\psi^*_{s,n+1}+\psi^*_{s,n-1})] + \epsilon \sum_{n=-\infty}^{+\infty} \frac{2(n-x)\mathrm{Im}[f(\psi_{s,n})\psi^*_{s,n}]}{1+|\psi_{s,n}|^2}. \quad (13)$$

Substituting the form of the soliton (4) for $\psi_{s,n}$ and noting that

$$\frac{1}{1+\sinh^2\beta \operatorname{sech}^2[\beta(n-x)]} = \frac{\cosh^2[\beta(n-x)]}{\cosh[\beta(n+1-x)]\cosh[\beta(n-1-x)]}, \quad (14)$$

we derive

$$\dot{x} = \frac{2\sinh\beta}{\beta}\sin\alpha + \epsilon\frac{\sinh\beta}{\beta} \times \sum_{n=-\infty}^{+\infty} \frac{(n-x)\cosh[\beta(n-x)]}{\cosh[\beta(n+1-x)]\cosh[\beta(n-1-x)]} \times \mathrm{Im}\left\{f(\psi_{s,n})\exp[-i\alpha(n-x)-i\sigma]\right\}. \quad (15)$$

Similarly, the derivative of Eq. (10) with respect to time leads to

$$\dot{\beta} = \epsilon\sinh\beta \sum_{n=-\infty}^{\infty} \frac{\cosh[\beta(n-x)]}{\cosh[\beta(n+1-x)]\cosh[\beta(n-1-x)]} \times \mathrm{Im}\{f(\psi_{s,n})\exp[-i\alpha(n-x)-i\sigma]\}. \quad (16)$$

To derive the equation for the time evolution of the parameter α, we proceed as follows. From identity (7), we obtain

$$2\sinh\beta\exp(-i\alpha) = \sum_{n=-\infty}^{\infty} \psi_{s,n}\psi^*_{s,n+1}, \quad (17)$$

the time derivative of which yields

$$-2\cosh\beta\dot{\beta} + 2i\sinh\beta\dot{\alpha} = i\sinh\beta \sum_{n=-\infty}^{\infty} \operatorname{sech}[\beta(n+1-x)]\, f(\psi_{s,n})\exp[-i\alpha(n-x)-i\sigma] \quad (18)$$

$$-i\sinh\beta \sum_{n=-\infty}^{\infty} \operatorname{sech}[\beta(n-1-x)]\, f^*(\psi_{s,n})\exp[i\alpha(n-x)+i\sigma]. \quad (19)$$

Here the equation of motion (3) has been used again. The real part of the above equation also leads to the time evolution of β [Eq. (16)], while the imaginary part yields the evolution of α,

$$\dot{\alpha} = -\epsilon\sinh\beta \times \sum_{n=-\infty}^{\infty} \frac{\cosh[\beta(n-x)]\tanh[\beta(n-x)]}{\cosh[\beta(n+1-x)]\cosh[\beta(n-1-x)]} \times \mathrm{Re}\{f(\psi_{s,n})\exp[-i\alpha(n-x)-i\sigma]\}. \quad (20)$$

Equations (15), (16), and (20) are precisely the same as those for the evolution of the parameters of the soliton derived in the adiabatic IST perturbation theory [4].

III. THE IST PERTURBATION THEORY AND COLLECTIVE COORDINATE APPROACH

It is noted in Ref. [4] that, for the evolution of the parameters of the soliton, collective coordinate methods and the IST perturbation generally do not render the same results. Here we present an example for which these two approaches are equivalent, i.e., the equations of motion for the effective point particle are exactly the same from both approaches. As mentioned above, the discrete, one-dimensional NLS equation (2) has this desired property. This equation can be derived from the Hamiltonian

$$H = -\sum_n \left(\psi_n\psi_{n+1}^* + \psi_n^*\psi_{n+1}\right) - \frac{2\nu}{\mu}\sum_n |\psi_n|^2 + \frac{2\nu}{\mu^2}\sum_n \ln\left(1+\mu|\psi_n|^2\right) \tag{21}$$

with the deformed Poisson brackets

$$\{\psi_n, \psi_m^*\} = i\left(1+\mu|\psi_n|^2\right)\delta_{nm}, \tag{22}$$

$$\{\psi_n, \psi_m\} = \{\psi_n^*, \psi_m^*\} = 0. \tag{23}$$

In general,

$$\{B, C\} = i\sum_n \left(\frac{\partial B}{\partial \psi_n}\frac{\partial C}{\partial \psi_n^*} - \frac{\partial B}{\partial \psi_n^*}\frac{\partial C}{\partial \psi_n}\right)\left(1+\mu|\psi_n|^2\right). \tag{24}$$

The equation of motion is

$$\dot{\psi}_n = \{H, \psi_n\}. \tag{25}$$

Due to the scaling property of Eq. (2), the nonlinearity parameter $\mu > 0$ can always be scaled to unity. The dynamics of localized states in this system has been studied [7]. In the physics of localized states, an important issue is how the motion of a localized state is affected by the presence of the nonintegrable term $-2\nu|\psi_n|^2\psi_n$. Treating the term $-2\nu|\psi_n|^2\psi_n$ as a perturbation to the integrable Ablowitz-Ladik equation, within the adiabatic approximation, we have the IST perturbation results to the first order in ν:

$$\dot{x} = \frac{2\sinh\beta}{\beta}\sin\alpha, \tag{26}$$

$$\dot{\alpha} = \nu\frac{\partial}{\partial x}\sum_{s=1}^{\infty}\frac{4\pi^2 s\sinh^2\beta}{\beta^3\sinh\left(\frac{\pi^2 s}{\beta}\right)}\cos(2\pi s x), \tag{27}$$

$$\dot{\beta} = 0, \tag{28}$$

where, for the evaluation of $\dot{\alpha}$, use is made of the Poisson summation formula

$$\sum_{-\infty}^{\infty} f(n) = \int_{-\infty}^{+\infty} dx f(x)\left[1 + 2\sum_{s=1}^{\infty}\cos(2\pi s x)\right]. \tag{29}$$

Although Eq. (26) still has the form of Eq. (5) for the unperturbed case, now the variable α is no longer a constant.

It is evident that this set of the equations constitutes a Hamiltonian system

$$\dot{x} = \frac{\partial}{\partial \alpha}\mathcal{H}_{\rm eff}, \tag{30}$$

$$\dot{\alpha} = -\frac{\partial}{\partial x}\mathcal{H}_{\rm eff}, \tag{31}$$

where

$$\mathcal{H}_{\rm eff} = \mathcal{K} + \mathcal{V}(x), \tag{32}$$

$$\mathcal{K} = -\frac{2\sinh\beta}{\beta}\cos\alpha, \tag{33}$$

$$\mathcal{V}(x) = -\nu\sum_{s=1}^{\infty}\frac{4\pi^2 s\sinh^2\beta}{\beta^3\sinh\left(\frac{\pi^2 s}{\beta}\right)}\cos(2\pi s x). \tag{34}$$

Hence the motion of the soliton in the perturbed system can be viewed as a point particle described by the general coordinates (x, α) in an effective periodic potential $\mathcal{V}(x)$.

Next we turn to the derivation of the above results from an effective Lagrangian obtained from a collective coordinate approach. This approach will afford us a transparent interpretation of the origin of the effective kinetic energy of the particlelike excitation and the effective potential in which it moves. In order to use collective coordinate method, we first should write down the Lagrangian formulation for the system (2). We need a pair of canonical variables that obey the canonical Poisson bracket instead of the deformed one [Eqs. (22) and (23)]. This can be achieved by the transformation

$$\varphi_n = \psi_n\sqrt{\frac{\ln\left(1+\mu|\psi_n|^2\right)}{\mu|\psi_n|^2}}, \tag{35}$$

$$\varphi_n^* = \psi_n^*\sqrt{\frac{\ln\left(1+\mu|\psi_n|^2\right)}{\mu|\psi_n|^2}}. \tag{36}$$

The mapping is nonsingular and the inverse is

$$\psi_n = \varphi_n\sqrt{\frac{-1+\exp\left(\mu|\varphi_n|^2\right)}{\mu|\varphi_n|^2}}, \tag{37}$$

$$\psi_n^* = \varphi_n^*\sqrt{\frac{-1+\exp\left(\mu|\varphi_n|^2\right)}{\mu|\varphi_n|^2}}. \tag{38}$$

The new variable φ_n and φ_n^* are conjugate canonical variables that satisfy

$$\{\varphi_n, \varphi_m^*\} = i\delta_{n,m}, \tag{39}$$

$$\{\varphi_n, \varphi_m\} = \{\varphi_n^*, \varphi_m^*\} = 0. \tag{40}$$

Thus the Lagrangian can be written via the Legendre transform of H as

$$L = i\sum_n \frac{1}{2}\left(\dot{\varphi}_n\varphi_n^* - \dot{\varphi}_n^*\varphi_n\right) - H. \tag{41}$$

The collective coordinate approach postulates that the parameters of the soliton are independent variables and vary adiabatically. For the soliton (4),

$$\varphi_n = e^{i[\alpha(n-x)+\sigma]}\sqrt{\ln\left[1+\sinh^2\beta\,\mathrm{sech}^2\beta(n-x)\right]}, \tag{42}$$

$$\varphi_n^* = e^{-i[\alpha(n-x)+\sigma]}\sqrt{\ln\left[1+\sinh^2\beta\,\mathrm{sech}^2\beta(n-x)\right]}. \tag{43}$$

It is easy to evaluate that

$$\sum_{n=-\infty}^{\infty} \frac{i}{2}\left(\dot{\varphi}_n\varphi_n^* - \dot{\varphi}_n^*\varphi_n\right) \tag{44}$$

$$= \sum_{n=-\infty}^{\infty} -\left[\dot{\alpha}(n-x) - \alpha\dot{x} + \dot{\sigma}\right]$$
$$\times \ln\left[1 + \sinh^2\beta\,\mathrm{sech}^2\beta(n-x)\right] \tag{45}$$
$$= 2\beta\alpha\dot{x} - 2\beta\dot{\sigma}, \tag{46}$$

where we have used identities (8) and (9).

For the Hamiltonian (21), using the above soliton solution as an ansatz, we derive

$$-\sum_{n=-\infty}^{\infty}\left(\psi_n\psi_{n+1}^* + \psi_n^*\psi_{n+1}\right) = -4\sinh\beta\cos\alpha \tag{47}$$

and

$$-2\nu\sum_n |\psi_n|^2 = U(x) + C, \tag{48}$$

where

$$U(x) \equiv -\nu\sum_{s=1}^{\infty}\frac{8\pi^2 s\sinh^2\beta}{\beta^2\sinh\left(\frac{\pi^2 s}{\beta}\right)}\cos(2\pi s x), \tag{49}$$

and $C = -4\nu\sinh^2\beta/\beta$. Hence the Hamiltonian is

$$H = -4\sinh\beta\cos\alpha + U(x) + C_0, \tag{50}$$

where $C_0 = -4\nu(\sinh^2\beta/\beta - \beta)$. Therefore, for the soliton, the Lagrangian is

$$L = 2\alpha\beta\dot{x} - 2\beta\dot{\sigma} + 4\sinh\beta\cos\alpha - U(x) - C_0. \tag{51}$$

The Euler-Lagrange equation for σ yields

$$\dot{\beta} = 0, \tag{52}$$

which is consistent with the conservation of the norm $\mathcal{N} = \sum_{n=-\infty}^{+\infty}\ln(1+\mu|\psi_n|^2) = 2\beta$ in the system (2) [7] [see also Eq. (28)]. The other two Euler-Lagrange equations for α and x have the forms

$$\dot{x} = \frac{2\sinh\beta}{\beta}\sin\alpha, \tag{53}$$

$$\dot{\alpha} = -\frac{1}{2\beta}\frac{\partial}{\partial x}U(x). \tag{54}$$

Noticing that

$$\mathcal{V}(x) = \frac{1}{2\beta}U(x), \tag{55}$$

the equations of motion we obtain above via this collective coordinate method are exactly the same as those derived by the IST perturbation method [see Eqs. (26) and (27)]. We stress the importance of using the canonical variables φ_n and φ_n^* to derive the Lagrangian for the motion of the effective point particle. It is incorrect to use

$$L = i\sum_n \frac{1}{2}\left(\dot{\psi}_n\psi_n^* - \dot{\psi}_n^*\psi_n\right) - H, \tag{56}$$

which is not a proper Lagrangian since ψ_n and ψ_n^* possess a deformed Poisson structure. From this Lagrangian one would not be able to derive the same equations of motion as those from the IST perturbation theory.

From the collective coordinate approach, we readily conclude that the term $-\sum_n(\psi_n\psi_{n+1}^* + \psi_n^*\psi_{n+1})$ in the Hamiltonian is related to the kinetic energy of the effective point particle and $-2\nu\sum_n|\psi_n|^2$ is related to the effective potential in which the point particle moves. This potential generates the Peierls-Nabarro barrier for the translating motion of a soliton [7]. Obviously, for the Ablowitz-Ladik system ($\nu = 0$) the point particle does not experience any such potential barrier at all. For the solitons with $\beta \ll \pi^2$, keeping only the term in $s = 1$, the potential takes a simple approximate form

$$\mathcal{V}(x) = -\nu\frac{4\pi^2\sinh^2\beta}{\beta^3\sinh\left(\frac{\pi^2}{\beta}\right)}\cos(2\pi x). \tag{57}$$

As expected, this barrier becomes exponentially weak as the amplitude of a localized state becomes smaller since

$$\mathcal{V} \sim -\frac{\nu}{\beta}\exp\left(-\frac{\pi^2}{\beta}\right) \qquad \text{as } \beta \to 0. \tag{58}$$

IV. CONCLUSIONS

In the above derivation of the discrete IST perturbation results, we have shown that the dynamics of a soliton in a perturbed AL system is closely related to some conservation laws in the unperturbed AL system. Hence a clear physical interpretation of the IST perturbation under the adiabatic approximation emerges. Clearly, this is a consequence of the adiabatic assumption, i.e., a soliton retains its functional form [Eq. (4)] in the presence of perturbations. We have also presented a perturbed AL system for which the adiabatic IST perturbation theory and the collective coordinate method yield an equivalent description of an effective particle theory for the dynamics of a soliton in a lattice. Using this example, we have emphasized the importance of establishing a correct Lagrangian formalism using dynamical variables that obey the true Poisson structure rather than a deformed one. This equivalence again has its origin in the identities we invoked above, which can be regarded as conserved quantities in the unperturbed AL system. The question still remains open whether there is a general formal equivalence between the adiabatic IST perturbation theory and the collective coordinate method for the dynamics of a soliton in perturbed AL systems.

APPENDIX

Here we use a very simple theorem to convert some summations to an integral form. The theorem states that, if $\sum_{n=-\infty}^{\infty} f(n+\Delta)$ is invariant under a Δ translation, i.e., independent of Δ, $\Delta \in (-\infty, +\infty)$, then

$$\sum_{n=-\infty}^{\infty} f(n+\Delta) = \int_{-\infty}^{\infty} f(x)dx. \tag{A1}$$

Using this theorem and the constants of motion of the Ablowitz-Ladik equation, we can prove the following identities for arbitrary x:

$$\sum_{n=-\infty}^{\infty} \operatorname{sech}[\beta(n-x)] \operatorname{sech}[\beta(n+1-x)] = \frac{2}{\sinh\beta}, \tag{A2}$$

$$\sum_{n=-\infty}^{\infty} \ln\left\{1+\sinh^2\beta \operatorname{sech}^2[\beta(n-x)]\right\} = 2\beta, \tag{A3}$$

$$\sum_{n=-\infty}^{\infty} (n-x)\ln\left\{1+\sinh^2\beta \operatorname{sech}^2[\beta(n-x)]\right\} = 0. \tag{A4}$$

The first identity can be proven by noticing that the Hamiltonian for the Ablowitz-Ladik equation is

$$H = -\sum_{n=-\infty}^{\infty} \left(\psi_n\psi_{n+1}^* + \psi_n^*\psi_{n+1}\right).$$

Substituting the soliton solution

$$\begin{aligned}\psi_n &= \sinh\beta \operatorname{sech}[\beta(n-x)] \\ &\quad\times \exp\left[i\left(2t\cosh\beta\cos\alpha + \alpha n + \sigma_0\right)\right]\end{aligned} \tag{A5}$$

into the Hamiltonian, where $x = Vt + x_0$ with $V = (2\sinh\beta\sin\alpha)/\beta$ and x_0 and σ_0 are arbitrary constants, we obtain

$$\begin{aligned}H &= -2\sinh^2\beta\cos\alpha \\ &\quad\times \sum_{n=-\infty}^{\infty} \operatorname{sech}[\beta(n-x)] \operatorname{sech}[\beta(n+1-x)].\end{aligned} \tag{A6}$$

Since H is a constant of motion, we conclude that

$$\sum_{n=-\infty}^{\infty} \operatorname{sech}[\beta(n-x)] \operatorname{sech}[\beta(n+1-x)]$$

is independent of x. Using the above theorem we convert the sum to the integral and obtain

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} \operatorname{sech}[\beta(n-x)] \operatorname{sech}[\beta(n+1-x)] \\ &\quad= \frac{1}{\beta}\int_{-\infty}^{\infty} \frac{1}{\cosh(\beta x)\cosh\beta(x+1)} d\beta x \\ &\quad= \frac{1}{\beta\sinh\beta} \ln \frac{\cosh[\beta(x+1)]}{\cosh\beta x}\Big|_{-\infty}^{\infty} \\ &\quad= \frac{2}{\sinh\beta}.\end{aligned}$$

Additionally, we can compute the momentum for the Ablowitz-Ladik soliton, which is

$$P = i\sum_{n=-\infty}^{\infty} \left(\psi_n\psi_{n+1}^* - \psi_n^*\psi_{n+1}\right) \tag{A7}$$

$$= 4\sinh\beta\sin\alpha. \tag{A8}$$

The second identity can be similarly proven using the conserved norm of the Ablowitz-Ladik equation $\mathcal{N} = \sum_{n=-\infty}^{+\infty} \ln\left(1+\mu|\psi_n|^2\right)$. Hence $\mathcal{N} = 2\beta$ for the AL soliton (A5).

The proof of the third identity is more involved since the summation is not directly related to any conserved quantity of the Ablowitz-Ladik equation. First, it can readily be proven that any $\psi_n(t)$ that solves the Ablowitz-Ladik equation

$$i\dot{\psi}_n = -\left(\psi_{n+1}+\psi_{n-1}\right)\left(1+|\psi_n|^2\right) \tag{A9}$$

satisfies

$$\begin{aligned}&\frac{d}{dt}\sum_{n=-\infty}^{\infty} n\ln\left(1+|\psi_n|^2\right) \\ &\quad= i\sum_{n=-\infty}^{\infty} \left(\psi_n\psi_{n+1}^* - \psi_n^*\psi_{n+1}\right) = P.\end{aligned} \tag{A10}$$

Second, substituting the soliton solution (A5) into the above equation, we obtain

$$\begin{aligned}&\frac{d}{dx}\sum_{n=-\infty}^{\infty} n\ln\left\{1+\sinh^2\beta\operatorname{sech}^2[\beta(n-x)]\right\} \\ &\quad= \frac{1}{V}\frac{d}{dt}\sum_{n=-\infty}^{\infty} n\ln\left\{1+\sinh^2\beta\operatorname{sech}^2[\beta(n-x)]\right\} \\ &\quad= \frac{P}{V} = 2\beta,\end{aligned}$$

which leads to

$$\sum_{n=-\infty}^{\infty} n\ln\left\{1+\sinh^2\beta\operatorname{sech}^2[\beta(n-x)]\right\} = 2\beta x. \tag{A11}$$

Note that the integral constant is zero. This can easily be seen by setting $x = 0$.

Combining the identity (A3) and Eq. (A11) yields the desired result.

[1] See, e.g., Physica D **68** (1) (1993), special issue on future directions of nonlinear dynamics in physical and biological systems, edited by P. L. Christiansen, J. C. Eilbeck, and R. D. Parmentier.
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