Normal approach to the linearized Fokker-Planck equation for the inverse-square force

Yongbin Chang and Ding Li
Institute of Plasma Physics, Chinese Academy of Sciences, P.O. Box 1126, Hefei 230031, People's Republic of China
(Received 3 August 1995)

It is found that the customary approach to Fokker-Planck coefficients for the inverse-square force has three defects. First, a small scattering angle cannot guarantee a small Taylor expansion argument. Second, a cutoff on the scattering angle did not fulfill Debye cutoff theory because it cannot exclude distant (weak) collisions with small relative velocity nor include close (effective) collisions with large relative velocity. Third, a singularity attributed to zero relative velocity had been overlooked. These defects had been vaguely covered up by the artificial treatment of replacing a variable relative velocity in a Coulomb logarithm by the constant thermal velocity. Therefore, the customary approach is questionable because one cannot regard the replacement as some kind of "average" or "approximation." In this paper, the difference between small-angle scattering and small-momentum-transfer collisions of the inverse-square force has been clarified. The probability function $P(\mathbf{v},\Delta\mathbf{v})$ for Maxwellian scatters is derived by choosing velocity transfer $\Delta\mathbf{v}$, which is the true measure of collision strength, as an independent variable. With the help of the probability function, Fokker-Planck coefficients can be obtained by the normal original Fokker-Planck approach. The previous unproved treatment of the replacement of the relative velocity is naturally avoided, and the completed linearized Fokker-Planck coefficients are generated as a uniform expression.

PACS number(s): 52.25.Dg, 05.40.+j, 51.10.+y, 02.50.Ga

I. INTRODUCTION

The original Fokker-Planck equation was derived from the random motion model, in which a stochastic process is described by a transition probability $P(\mathbf{v},\Delta\mathbf{v})$, where the velocity $\mathbf{v}$ and its transfer $\Delta\mathbf{v}$ are independent variables [1–4]. The Fokker-Planck equation has also been widely used to evaluate the collision term of the inverse-square type of force. In stellar dynamics, Chandrasekhar first discussed this theory for stochastic effect of gravity [5,6]. The applications of this equation to plasma physics were first developed by Landau [7], by Spitzer [8], and later by Rosenbluth, MacDonald, and Judd [9]. Although the approach to Fokker-Planck coefficients has undergone some improvement from Landau to Rosenbluth, MacDonald, and Judd, their approaches, as well as those of other authors [10–13], still deviated from the original Fokker-Planck theory because the velocity increment $\Delta\mathbf{v}$ was replaced by scattering angle θ as a measurement of collision strength. This aberration finally resulted in an unclear small-angle cutoff $\theta_{\min}$ and a tough integration difficulty. The cutoff on scattering angle at $\theta_{\min}$ could not cut off the distant (weak) collisions with very small relative velocity; meanwhile, it unfairly excluded the close (effective) collisions with very large relative velocity. The integration difficulty had to be artificially treated by replacing varied relative velocity in the Coulomb logarithm by the thermal velocity of field particles. Some authors have noted the defect [14]. However, no sound justification has ever been given for this treatment.

In principle, the completed Fokker-Planck equation is expressed as the following infinite series [12,13,15]:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}=\sum_{N=1}^{\infty}\frac{(-1)^N}{N!}\frac{\partial^N}{\partial\mathbf{v}^N}\cdot(\langle\Delta\mathbf{v}^N\rangle f)\ , \tag{1}$$

where $(\partial f/\partial t)_{\text{coll}}$ is the time rate of the change of the distribution function f due to collisions, $\Delta\mathbf{v}^N$ is the Nth order dyadic of velocity increment for the test particle, and the Fokker-Planck coefficients $\langle\Delta\mathbf{v}^N\rangle$ should be calculated directly from the probability function $P(\mathbf{v},\Delta\mathbf{v})$ as follows [3,4]:

$$\langle\Delta\mathbf{v}^N\rangle=\int\Delta\mathbf{v}^N P(\mathbf{v},\Delta\mathbf{v})d\Delta\mathbf{v}\ . \tag{2}$$

However, the exact form of $P(\mathbf{v},\Delta\mathbf{v})$ has never been derived even for the Maxwellian distribution $f_M(\mathbf{v}_F)$. All of the previous authors alternately had to use the following five-fold integral for the Fokker-Planck coefficients [3,4,10]:

$$\langle\Delta\mathbf{v}^N\rangle=\int\Delta\mathbf{v}^N f_M(\mathbf{v}_F)g\sigma_R\sin\theta\,d\theta\,d\varphi\,d\mathbf{v}_F\ , \tag{3}$$

where $\sigma_R=(ZZ_Fe^2/4\pi\varepsilon_0\mu)^2/(4g^4\sin^4\theta/2)$ is the Rutherford cross section, in which $\mu=mm_F/(m+m_F)$ is the reduced mass, θ is the scattering angle in the center-of-mass system, and φ is the azimuthal angle around the relative velocity $\mathbf{g}=\mathbf{v}-\mathbf{v}_F$, in which $\mathbf{v}$ and $\mathbf{v}_F$, respectively, are the velocity of test particle and field particle; Z and Z_F, respectively, are the charge number of the test and field particle. The velocity increment had to be expressed by field particle velocity and scattering angles, that is, $\Delta\mathbf{v}=\Delta\mathbf{v}(\mathbf{v}_F,\theta,\varphi)$. By the customary approach of directly calculating Eq. (3), the divergent difficulty for $N<2$ appeared before Eq. (3) has been integrated over the field velocity $\mathbf{v}_F$. Therefore, the cutoff on the scattering angle had to be introduced for the first twofold integral. In order to facilitate the remained integral over the velocity $\mathbf{v}_F$, all previous authors [5–13] had to replace the varied g by a thermal velocity $v_{\rm th}$ in the Coulomb logarithm. Rosenbluth, MacDonald, and Judd pointed out that such treatment is made mainly because no better way could be found [9]. Furthermore, the previous approach is inefficient for calculating the higher order Fokker-Planck coefficients ($N>2$) because the integrals over the scattering angles for *every order* of the coefficients have to be done differently and repeatedly. The complex calculation for the coefficients from $N=3$ [10] to $N=4$ [11] increases quickly. To our knowledge no one has tried to calculate the coefficients over $N=5$.

Based on the original physics consideration, this paper presents another approach to the linearized Fokker-Planck equation. It is found that the customary replacement about the relative velocity g can be naturally avoided by keeping the momentum transfer $m\Delta\mathbf{v}$ as an independent variable. Consequently, the explicit form of the probability function $P(\mathbf{v},\Delta\mathbf{v})$ can be obtained provided the distribution function of field particles is Maxwellian. After variables transform from $(\mathbf{v}_F,\theta,\varphi)$ to $(\Delta\mathbf{v},\theta,\varphi')$, the integral over the scattering angle θ and φ', the azimuthal angle around $\Delta\mathbf{v}$, can be exactly carried out so that the cutoff on the scattering angle at $\theta_{\rm min}$ is no longer necessary. Then all the Fokker-Planck coefficients can be derived from normal probability function $P(\mathbf{v},\Delta\mathbf{v})$ by the direct integral over $\Delta\mathbf{v}$ from Eq. (2). In other words, this approach is actually the normal Fokker-Planck approach. The Fokker-Planck collision operator is generated to an exact expression of infinite series. The Chandrasekher function [16] can be recovered from the first and second terms of the general series.

The rest of the paper will be organized as follows. In Sec. II, the different points between the scattering angle and velocity increment are briefly clarified. Then, the probability function is derived in Sec. III for Maxwellian scatters. In Sec. IV, the completed Fokker-Planck coefficients are calculated and expressed as a uniform expression by the normal approach. Finally, the conclusion is addressed in Sec. V.

II. DIFFERENCE BETWEEN SCATTERING ANGLE AND VELOCITY INCREMENT

In their treatment, the previous authors [5–13] were limited by the misguided convention that collision event of small- (large-) angle scattering is equivalent to a collision event of small- (large-) momentum transfer, and therefore to a weak (strong) or distant (close) collision event. In the previous treatment, the velocity increment $\Delta\mathbf{v}$ was confused with and incorrectly replaced by the scattering angle θ when the collision strength was defined. Such a replacement is valid only if $g=|\mathbf{v}-\mathbf{v}_F|$ is constant. This is the case of Rutherford scattering for an α particle, in which all α particles (test particles) have same velocity, and the velocities of the target nucleus (field particles) are almost zero. In plasma physics, however, the field particle's velocity $\mathbf{v}_F$ can vary from zero to infinity. Hence the relative velocity g also varies from zero to infinity even if the test particle velocity $\mathbf{v}$ is a constant. The magnitude of the velocity transfer $\Delta\mathbf{v}$, which is proportional to $g\sin(\theta/2)$, cannot be determined by θ alone because of the variation of g. Suppose a test particle with velocity $\mathbf{v}$ is impacted by many field particles with different velocities $\mathbf{v}_F$. Based only on the magnitude of the scattering angle θ, it is impossible to decide which of these collisions belongs to ineffective (very distant) collisions that should be cutoff and which are effective collisions that should be included. In other words, the Debye cutoff could not be fulfilled by a cutoff at the scattering angle alone. The variation of the relative velocity or the velocity of field particles should be taken into account.

The impact parameter $b\propto 1/g^2\tan(\theta/2)$, from which the relation $b\propto \sin\theta/\Delta v^2$ can be derived, is well known. From these two relations, it is found that the cutoff on scattering angle at $\theta_{\rm min}$ cannot exclude the distant collision events when g approaches 0, but a cutoff on velocity increment at $\Delta v_{\rm min}$ can, regardless of the variation of the relative velocity. Obviously, it is Δv rather than θ that determines the collision strength when relative velocities are not constant. Therefore, a cutoff on the velocity increment at $\Delta v_{\rm min}$ should be introduced instead of the cutoff on scattering angle at $\theta_{\rm min}$.

It will be proved in this paper that the singularity point of the Fokker-Planck integral exists at $\Delta v=0$ rather than $\theta=0$, and that a cutoff on g at $g_{\rm min}$ is essential to make the integral infinite. The usual $\theta_{\rm min}$ cutoff is ineffective because it fails to remove the singularity point. In fact, the Fokker-Planck integral is still divergent after the $\theta_{\rm min}$ cutoff when g approaches 0. The customary finite result was obtained by replacing the varied g by the thermal velocity $v_{\rm th}$ in the Coulomb logarithm. Therefore, the divergent property at $g=0$ after the $\theta_{\rm min}$ cutoff was vaguely covered up.

III. DERIVATION OF PROBABILITY FUNCTION FOR MAXWELLIAN SCATTERS

The difference between $\Delta\mathbf{v}$ and θ also manifests itself in other aspects. In Fokker-Planck theory, $\Delta\mathbf{v}$ was employed as an argument for Taylor expansion, which correctly reflects $\Delta\mathbf{v}$ as the real measurement of the collision strength. It is clear from the relation $\Delta v=(2\mu/m)g\sin(\theta/2)$ that θ cannot replace $\Delta\mathbf{v}$ as the argument for Taylor expansion, because small θ cannot ensure the argument Δv small when g is quite large. The original Fokker-Planck equation was derived on the random motion model, in which a stochastic process is described by a transition probability $P(\mathbf{v},\Delta\mathbf{v})$. Naturally, it is necessary to keep $\Delta\mathbf{v}$ as an independent variable when the normal approach is used to derive the linearized Fokker-Planck equation for the case of inverse-square force. In order to keep $\Delta\mathbf{v}$ as an independent variable, one can make the variables transform from $(\mathbf{v}_F,\theta,\varphi)$ to $(\Delta\mathbf{v},\theta,\varphi')$ in Eq. (3) as follows:

$$\mathbf{v}_F=\mathbf{v}+\frac{1}{a}\left[\Delta\mathbf{v}+\Delta\mathbf{v}\times\left[\frac{\Delta\mathbf{v}}{\Delta v_{\perp E}}\times\mathbf{E}\cos\varphi'+\frac{\Delta v}{\Delta v_{\perp E}}\mathbf{E}\sin\varphi'\right]\cot\frac{\theta}{2}\right], \tag{4}$$

and

$$\varphi=\arccos\frac{\Delta v_{\parallel E}+\Delta v_{\perp E}\tan\frac{\theta}{2}\cos\varphi'}{\left[\Delta v^2\sin^2\varphi'+\left[\Delta v_{\perp E}\tan\frac{\theta}{2}+\Delta v_{\parallel E}\cos\varphi'\right]^2\right]^{1/2}}, \tag{5}$$

where $a=2\mu/\mathrm{m}$ is the mass ratio constant; $\mathbf{E}$ is an arbitrary constant unit vector; $\perp E$ and $\parallel E$, respectively, denote the directions perpendicular and parallel to $\mathbf{E}$; and $\times$ represents the cross product.

Taking $\Delta\mathbf{v}$, θ, and φ' as independent variables, Eq. (3) can be expressed as

$$\langle\Delta\mathbf{v}^N\rangle=\int\Delta\mathbf{v}^N f_M(\mathbf{v}_F)g\sigma_R\sin(\theta)\left|\frac{\partial(\mathbf{v}_F,\varphi)}{\partial(\Delta\mathbf{v},\varphi')}\right| \times d\theta\, d\varphi' d\Delta\mathbf{v} \tag{6}$$

Now the integral over the solid scattering angle becomes independent of $\Delta\mathbf{v}^N$, so that it needs to be integrated just once for all Fokker-Planck coefficients. It is notable that this approach greatly simplifies the calculation procedure. As mentioned above, the previous approach required integrals over the scattering angle for *every order* of the coefficients. Obviously, the normal approach developed in this paper is more concise than previous approaches. Therefore, all Fokker-Planck coefficients can be derived and expressed as a unified form, as will be shown in Sec. IV.

With the help of Eqs. (4) and (5), the following relations for the variable transform can be obtained as

$$g=(\Delta v/a)\csc(\theta/2), \tag{7}$$

$$v_F^2=v^2+\frac{2}{a}(\mathbf{v}\cdot\Delta\mathbf{v})+\left[\frac{\Delta v}{a}\right]^2\csc^2\frac{\theta}{2}-\frac{2}{a}|\mathbf{v}\times\Delta\mathbf{v}|\cot\frac{\theta}{2}\sin\varphi', \tag{8}$$

and

$$\left|\frac{\partial(\mathbf{v}_F,\varphi)}{\partial(\Delta\mathbf{v},\varphi')}\right|=\left[\frac{\csc(\theta/2)}{a}\right]^3. \tag{9}$$

Substitute Eqs. (7)–(9) into Eq. (6), and compare the result with Eq. (2); the probability function can be expressed as

$$P(\mathbf{v},\Delta\mathbf{v})=\frac{\omega\exp(-u^2-2\mathbf{u}\cdot\mathbf{u}_\delta)}{2\pi^{3/2}\Delta v^3}\int_0^\pi\exp\left[-u_\delta^2\csc^2\frac{\theta}{2}\right]\cot\frac{\theta}{2}\csc^2\frac{\theta}{2}d\theta\int_0^{2\pi}\exp\left[2|\mathbf{u}\times\mathbf{u}_\delta|\cot\frac{\theta}{2}\sin\varphi'\right]d\varphi', \tag{10}$$

where $\mathbf{u}=\mathbf{v}/v_{\text{th}}$ and $\mathbf{u}_\delta=\Delta\mathbf{v}/av_{\text{th}}$, respectively, are dimensionless variables for $\mathbf{v}$ and $\Delta\mathbf{v}$, in which $v_{\text{th}}=\sqrt{2kT/m_F}$ is the thermal velocity of the field particle; $\omega=v_F v_{\text{th}}\pi b_0^2$ has the order of the close collision frequency of a test particle; n_F is the density of the field particle; and $b_0=(m_F/2\mu)ZZ_Fe^2/4\pi\varepsilon_0 kT$ is the classical distance of closest approach (Landau length) [17]. Take the integral over φ' and the scattering angle θ; the probability function is obtained as

$$P(\mathbf{v},\Delta\mathbf{v})=\frac{\omega\exp(-u^2-2\mathbf{u}\cdot\mathbf{u}_\delta)}{\pi^{3/2}(av_{\text{th}})^3}\times\sum_{n=0}^{\infty}\frac{\Gamma(n+1,u_\delta^2)}{n!u_\delta^{2n+5}}J_n(2|\mathbf{u}\times\mathbf{u}_\delta|)|\mathbf{u}\times\mathbf{u}_\delta|^n, \tag{11}$$

where $\Gamma(n+1,z)$ is the $(n+1)$th order incomplete gamma function, and J_n is the first kind of Bessel function defined in Ref. [18]. It is not surprising that the integral over scattering angle θ is not divergent. Because Fokker-Planck coefficients for $N>2$ are convergent integrals, it is clear that the integral over θ, which is the same for all coefficients, is not divergent even for $N<2$ coefficients. Actually, $\Delta\mathbf{v}$ is an independent parameter, and $\Delta\mathbf{v}\neq\mathbf{0}$ in the integral over θ, so that small-angle scattering does not mean distant collision events, but close collision events with very large g since $b\propto\sin\theta/\Delta v^2$ and $g\propto\Delta v/\sin(\theta/2)$. After integrating over the azimuthal angle φ' and the scattering angle θ, the divergent property of Eq. (3) for $N<2$ is still retained in the simplified integral, which is similar to Eq. (2). The probability function Eq. (11) is one of the main results of this paper. Many results, such as the collision frequency, can be derived from the probability function [19,20].

IV. NORMAL APPROACH TO LINEARIZED FOKKER-PLANCK EQUATION

Introducing $(\mathbf{e}_\parallel,\mathbf{e}_{\perp 1},\mathbf{e}_{\perp 2})$ as an orthogonal triplet of unit vectors with $\mathbf{e}_\parallel=\mathbf{v}/v$, then $\mathbf{u}_\delta$ can be expressed as

$$\mathbf{u}_\delta=u_{\delta\parallel}\mathbf{e}_\parallel+u_{\delta\perp 1}\mathbf{e}_{\perp 1}+u_{\delta\perp 2}\mathbf{e}_{\perp 2} =u_\delta(\mathbf{e}_\parallel\cos\chi+\mathbf{e}_{\perp 1}\sin\chi\cos\psi+\mathbf{e}_{\perp 2}\sin\chi\sin\psi), \tag{12}$$

in which χ is the angle between $\mathbf{u}_\delta$ and $\mathbf{u}$, and ψ is the azimuthal angle of $\mathbf{u}_\delta$ around $\mathbf{u}$. Here, $\perp 1$ and $\perp 2$ and $\|$, respectively, denote the directions perpendicular and parallel to $\mathbf{v}$. Choosing a spherical coordinate system with $\mathbf{u}$ as the polar axis, we then have $d\mathbf{u}_\delta = u_\delta^2 \sin\chi d\psi d\chi du_\delta$. Naturally, the Nth order term of coefficient $\langle \Delta \mathbf{v}^N \rangle$ is simplified by calculating the following threefold dimensionless integral:

$$\langle \Delta \mathbf{v}^N \rangle = (a v_{\text{th}})^N \langle \mathbf{u}_\delta^N \rangle \ , \tag{13}$$

where

$$\langle \mathbf{u}_\delta^N \rangle = \int p(\mathbf{u}, \mathbf{u}_\delta) \mathbf{u}_\delta^N u_\delta^2 \sin\chi \, d\psi \, d\chi \, du_\delta \ , \tag{14}$$

in which

$$p(\mathbf{u}, \mathbf{u}_\delta) = \frac{\omega \exp(-2uu_\delta \cos\chi)}{\pi^{3/2} \exp(u^2)} \times \sum_{n=0}^{\infty} \frac{\Gamma(n+1, u_\delta^2)}{n! u_\delta^{2n+5}} J_n(2uu_\delta \sin\chi)(uu_\delta \sin\chi)^n \ . \tag{15}$$

After substituting Eq. (12) into Eq. (14) and using the integral formula Eq. (A1) which is derived in the Appendix, the nonzero terms of the Nth order dyadic of $\langle \mathbf{u}_\delta^N \rangle$ are as follows:

$$\langle u_{\delta\|}^{N-2(J+K)} u_{\delta\perp 1}^{2J} u_{\delta\perp 2}^{2K} \rangle = \langle u_\delta^N \cos^{N-2(J+K)}\chi \sin^{2(J+K)}\chi \cos^{2J}\psi \sin^{2K}\psi \rangle$$
$$= 2B(J+1/2, K+1/2) \int p(\mathbf{u}, \mathbf{u}_\delta) u_\delta^{N+2} \cos^{N-2(J+K)}\chi \sin^{2(J+K)+1}\chi \, d\chi \, du_\delta \ , \tag{16}$$

where $B(x,y)$ is the beta function defined in Ref. [18]. Using the binomial theorem, one can obtain the expansion

$$\cos^{N-2(J+K)}\chi \sin^{2(J+K)}\chi = \cos^{N-2(J+K)}\chi \sum_{i=0}^{J+K} C_{J+K}^i (-1)^{J+K-i} \cos^{2(J+K-i)}\chi$$
$$= (-1)^{J+K} \sum_{i=0}^{J+K} C_{J+K}^i (-1)^i \cos^{N-2i}\chi \ . \tag{17}$$

Substituting Eqs. (15) and (17) into Eq. (16), and utilizing Eq. (A2), the integral over χ can be carried out, and Eq. (16) becomes

$$\langle u_{\delta\|}^{N-2(J+K)} u_{\delta\perp 1}^{2J} u_{\delta\perp 2}^{2K} \rangle = \frac{2\omega B(J+1/2, K+1/2)}{(-1)^{J+K} \pi \exp(u^2)} \times \int du_\delta \sum_{n=0}^{\infty} \frac{\Gamma(n+1, u_\delta^2)}{n! u_\delta^{2n-N+3}} \sum_{i=0}^{J+K} C_{J+K}^i (-1)^i$$
$$\times (-uu_\delta)^{2-N+2i} \sum_{l=L-i}^{N-2i-1} \frac{\sqrt{\pi} C_{M-i}^{l-L+i} \Gamma(3/2+L-i)(uu_\delta)^{2n+2l}}{\Gamma(3/2-M+i+l)\Gamma(5/2+n+l)} \ , \tag{18}$$

where $L = [(N-1)/2]$ and $M = [N/2]$, respectively, are the integer parts of $(N-1)/2$ and $N/2$. Let $l = j - i$, and change the order of the summations and the argument of the integral; then Eq. (18) can be arranged as

$$\langle u_{\delta\|}^{N-2(J+K)} u_{\delta\perp 1}^{2J} u_{\delta\perp 2}^{2K} \rangle = \frac{\omega B(J+1/2, K+1/2)}{(-1)^{J+K} \pi} \times \sum_{i=0}^{J+K} C_{J+K}^i \sum_{j=L}^{N-i-1} \frac{\Gamma(3/2+L-i)}{\Gamma(3/2-M+j)} C_{M-i}^{j-L+i}$$
$$\times (-1)^{N+i} \exp(-u^2) u^{2j-N+2} \sum_{n=0}^{\infty} \frac{u^{2n} \int \Gamma(n+1, u_\delta^2) u_\delta^{2(j-1)} du_\delta^2}{n! \Gamma(n+j-i+5/2)} \ . \tag{19}$$

For $N \geq 3$, it is easy to find $L \geq 1$ and then $j \geq 1$, so that the integral in Eq. (19) is not divergent and can be calculated as

$$\int \Gamma(n+1, u_\delta^2) u_\delta^{2(j-1)} du_\delta^2 = (n+j)!/j \ . \tag{20}$$

However, when $N \leq 2$, one has $L = 0$ and then $j = 0$ so that the integral in Eq. (19) becomes divergent. It is clear from the onefold integral of Eq. (19) that the singular point exists at $u_\delta = 0$ rather than $\theta = 0$. Therefore, the cutoff should be made at $u_\delta = u_{\delta\min}$. Because the cutoff is not necessary to be introduced until the lastfold integral, the usual unproved treatment of replacing g by v_{th} has been naturally avoided. This cutoff at $u_\delta = u_{\delta\min}$ actually implies a cutoff on relative velocity g. It is easy to result in $g > g_{\min} = v_{\rm th} u_{\delta\min}$ from $u_\delta > u_{\delta\min}$ with the help of the relation $u_\delta = (g/v_{\rm th})\sin(\theta/2)$. This shows that the implicit cutoff on the relative velocity at $g = g_{\min}$ is necessary and essential to make the divergent integral finite. Hence a pure scattering angle cutoff at $\theta = \theta_{\min}$ without $g > g_{\min}$ cannot remove the singular point. When the small angle $\theta_{\min}$ was misguidedly used as the cutoff parameter, the replacement of g by $v_{\rm th}$ was inevitable, and actually played a role on cutoff at $g_{\min}$. After the scattering angle cutoff at $\theta = \theta_{\min}$, the singular point at $g = 0$ [corresponding to $\Delta v = ag\sin(\theta/2) = 0$] is actually removed by a replacement about the relative velocity. The cutoff on the small screening angle happened to be "valid" for the special case $\theta \ge \theta_{\min}$ and $g = v_{\rm th}$ because it actually ensures the cutoff at $\Delta v_{\min} = a v_{\rm th}\sin(\theta_{\min}/2)$. In order to include the special case, the cutoff $u_{\delta\min}$ may be taken at

$$u_{\delta\min} = \Delta v_{\min}/a v_{\rm th} = 1/\Lambda \ , \tag{21}$$

where the Coulomb constant $\Lambda = \lambda_D/b_0$, and Debye length $\lambda_D = \sqrt{\varepsilon_0 kT/ne^2 Z_{\rm eff}}$ [9]. It is necessary to emphasize that there is no artificial limitation on relative velocity $\mathbf{g}$ here. The integral in Eq. (19) for $j = 0$ is therefore

$$\int_{\Lambda^{-2}}^{\infty} \Gamma(n+1, u_\delta^2) u_\delta^{-2} du_\delta^2 \approx n!2\ln\Lambda \ . \tag{22}$$

Equations (20) and (22) can be merged as

$$\int \Gamma(n+1, u_\delta^2) u_\delta^{2(j-1)} du_\delta^2 = (n+j)!\mathcal{J} \ , \tag{23}$$

where $\mathcal{J} = 1/j$ for $j \neq 0$, and $\mathcal{J} = 2\ln\Lambda$ for $j = 0$. Substituting Eq. (23) into Eq. (19), one obtains

$$\begin{aligned}\langle u_{\delta\parallel}^{N-2(J+K)} u_{\delta\perp 1}^{2J} u_{\delta\perp 2}^{2K} \rangle = {} & \frac{\omega B(J+1/2, K+1/2)}{(-1)^{J+K}\pi} \\ & \times \sum_{i=0}^{J+K} C_{J+K}^{i} \sum_{j=L}^{N-i-1} \frac{\Gamma(3/2+L-i)}{\Gamma(3/2-M+j)} C_{M-i}^{-L+j} \\ & \times (-1)^{N+i} u^{2j-N+2} \exp(-u^2) \sum_{n=0}^{\infty} \frac{u^{2n}(n+j)!\mathcal{J}}{n!\Gamma(n+j-i+5/2)} \ .\end{aligned} \tag{24}$$

The summation of Eq. (24) with respect to n can be expressed as a finite form with the help of the relation

$$[x^j e^x \gamma^*(t,x)]^{(j)} = \frac{d^j}{dx^j} \sum_{n=0}^{\infty} \frac{x^{n+j}}{\Gamma(n+t+1)} = \sum_{n=0}^{\infty} \frac{x^n (n+j)!}{n!\Gamma(n+t+1)} \ , \tag{25}$$

where (j) indicates the jth derivative with respect to x, and $\gamma^*(t,x)$ is the analytic incomplete gamma function defined as 6.5.29 in Ref. [18].

Then, from Eq. (13), the Nth order Fokker-Planck coefficient $\langle \Delta v^N \rangle$ is finally obtained as

$$\langle \Delta v_\parallel^{N-2(J+K)} \Delta v_{\perp 1}^{2J} \Delta v_{\perp 2}^{2K} \rangle = \frac{B(J+1/2, K+1/2)}{(-1)^{J+K}\pi} \sum_{i=0}^{J+K} C_{J+K}^{i} \sum_{j=L}^{N-i-1} \frac{\Gamma(3/2+L-i)}{\Gamma(3/2-M+j)} C_{M-i}^{-L+j} F_{N,i,j}(u^2) \tag{26}$$

where the function $F_{N,i,j}(x)$ is defined as

$$\begin{aligned}F_{N,i,j}(x) = {} & (-1)^{N+i}\mathcal{J}\omega(a v_{\rm th})^N x^{j-N/2+1} \\ & \times e^{-x}[x^j e^x \gamma^*(3/2-i+j, x)]^{(j)} \ .\end{aligned} \tag{27}$$

Equations (26) and (27) can recover all the previous results [3,4,11,12] for the Fokker-Planck coefficients with Maxwellian scatters. For example, when $N = 1$ and 2, the usual form of the first two-order Fokker-Planck coefficients can be obtained from Eqs. (26) and (27) as

$$\langle \Delta v_\parallel \rangle = F_{1,0,0}(u^2) \ , \tag{28a}$$

$$\langle \Delta v_\parallel^2 \rangle = F_{2,0,0}(u^2)/2 + F_{2,0,1}(u^2) \ , \tag{28b}$$

$$\begin{aligned}\langle \Delta v_{\perp 1}^2 \rangle = \langle \Delta v_{\perp 2}^2 \rangle = {} & -[F_{2,0,0}(u^2)/2 + F_{2,0,1}(u^2) \\ & + F_{2,1,0}(u^2)]/2 \ ,\end{aligned} \tag{28c}$$

$$\langle \Delta v^2 \rangle = \langle \Delta v_\parallel^2 \rangle + \langle \Delta v_{\perp 1}^2 \rangle + \langle \Delta v_{\perp 2}^2 \rangle = -F_{2,1,0}(u^2) \ . \tag{28d}$$

It is not difficult to verify that

$$F_{2,0,0}(u^2) = -a v_{\rm th} F_{1,0,0}(u^2)/u \ , \tag{29a}$$

$$F_{1,0,0}(u^2) = -4\omega a v_{\rm th} \ln\Lambda G(u) \ , \tag{29b}$$

$$F_{2,1,0}(u^2) = -2\omega a^2 v_{\rm th}^2 \ln\Lambda \Phi(u)/u \ , \tag{29c}$$

where $\Phi(u)$ is the error function, and $G(u)$ is the Chandrasekhar function [16]. After substituting Eqs. (29a)–(29c) into Eqs. (28a)–(28d), the friction and diffusion coefficients are found to be in agreement with the usual results in Refs. [3,4]. The nondominant part ($j \neq 0$) of the second order Fokker-Planck coefficient, $F_{2,0,1}(u^2)$, is also in agreement with the result in Refs. [10,11].

V. CONCLUSION

In summary, the correct physical concept of collision strength has been clarified, and the normal approach to the linearized Fokker-Planck equation has been developed based on this correct concept. The difference between the small-angle scattering and small-momentum transfer collisions for the inverse-square force has been elaborated. From the viewpoint of physics, the collision strength is determined by momentum transfer $m\Delta v$ rather than scattering angle θ when g is not a constant. The customary cutoff at $\theta_{\min}$ can neither exclude the distant (weak) collision events with large θ and small g, nor include the close (strong) collision events with small θ and large g. However, the cutoff at $\Delta v_{\min}$ cannot only exclude such distant (weak) collisions, but also include such close (strong) collisions. Furthermore, the cutoff at $\Delta v_{\min}$ is even better than the cutoff on the impact parameter at $b_{\max}$ in a sense, because the cutoff at $\Delta v_{\min}$ actually implies a cutoff on g at $g_{\min}$. For example, even if the impact parameter $b=0$, two ions cannot get closer than the Debye length if their g is smaller than $g_{\min}$. Such distant collisions with small b, which cannot be excluded by the cutoff at $b_{\max}$, should be excluded according to the Debye shielding theory, and really can be excluded by the cutoff at $\Delta v_{\min}$. From the viewpoint of mathematics, the singularity point of the Fokker-Planck integral exists at $\Delta v=0$ rather than at $\theta=0$. Therefore, the cutoff at $\theta_{\min}$ cannot remove the singularity point of the Fokker-Planck integral at all, because the cutoff should be made on the velocity increment at $\Delta v_{\min}$. It is demonstrated that the implicit cutoff on g at $g_{\min}$ is necessary and essential to $\Delta v_{\min}$. After the cutoff on the scattering angle at $\theta_{\min}$ was misguidedly used, the integral was still divergent at $g=0$. Therefore, the replacement of g by v_{th} was inevitable, and actually played a role of cutoff on g. It actually transferred the divergent integral into the finite one by change. Obviously, the previous approach is questionable, since one cannot regard such a replacement as some kind of "average" [4] or "approximation" [14].

The probability function $P(\mathbf{v},\Delta\mathbf{v})$ for Maxwellian scatters has been obtained based on the original physical concept of collision strength. The customary replacement of g by thermal velocity v_{th} is naturally avoided. The derivation of Fokker-Planck coefficients is greatly simplified with the help of the probability function, since it is no longer necessary to integrate repeatedly over the solid angle for *every order* of the coefficients. Finally, the completed Fokker-Planck coefficients have been expressed in a uniform expression as Eqs. (26) and (27).

ACKNOWLEGMENTS

The authors are grateful to Professor Jixing Liu, Professor Bingren Shi, and Professor Maoquan Wang for their interest and discussion. We would particularly like to thank Academician Yuping Huo for his constant encouragement.

APPENDIX

In deriving Fokker-Planck coefficients, we encounter two integrals

$$\int_0^{2\pi}\sin^m\psi\cos^n\psi\,d\psi = \begin{cases} 2B(J+1/2,K+1/2) & \text{when } m=2J, n=2K \\ 0 & \text{otherwise} \end{cases} \tag{A1}$$

where $B(J+1/2,K+1/2)$ is the beta function, and

$$\int_0^{\pi}\exp(-a\cos\chi)J_n(a\sin\chi)\sin^{n+1}\chi\cos^N\chi\,d\chi = \left[-\frac{a}{2}\right]^{2-N}\sum_{l=L}^{N-1}\frac{\sqrt{\pi}C_M^{l-L}\Gamma(3/2+L)(a/2)^{n+2l}}{\Gamma(3/2-M+l)\Gamma(5/2+n+l)}, \tag{A2}$$

where J_n is the first kind of Bessel function, and Γ is the Gamma function; and M and L are the integer parts of $N/2$ and $(N-1)/2$, respectively.

Consider (4.3.127) and (6.2.1) in Ref. [18]; there is no difficult to verify Eq. (A1).

We define the integral on the left hand side of Eq. (A2) as a function $S(N,n,a)$. When $N=2M$, the magnitude of the integral does not change if the exponential function $\exp(-a\cos\chi)$ is changed into a hyperbolic function $2\cosh(a\cos\chi)$; meanwhile the upper limit π is changed into $\pi/2$. Thus one has

$$S(2M,n,a)=2\int_0^{\pi/2}\cosh(a\cos\chi)J_n(a\sin\chi)\times\sin^{n+1}\chi\cos^{2M}\chi\,d\chi. \tag{A3}$$

By expanding the hyperbolic cosine function $\cosh(a\cos\chi)$ (Ref. [18], 4.5.63), and using the integral formula (Ref. [18], 11.4.11), the integral over χ is found to be

$$S(2M,n,a)=\sum_{m=0}^{\infty}\frac{\Gamma(m+M+1/2)2^{2m}}{(2m)!(a/2)^{M-m+1/2}}J_{n+M+m+1/2}(a). \tag{A4}$$

After expanding Bessel function $J_{n+M+m+1/2}(a)$ (Ref. [18], 9.1.10), we obtain

$$S(2M,n,a)=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(m+M+1/2)2^{2m}}{\Gamma(n+m+M+j+3/2)(2m)!j!}\times\left[\frac{a}{2}\right]^{n+2(m+j)} \tag{A5}$$

Reorder the summation of Eq. (A5) by the transform $k=m+j$ as

$$S(2M,n,a)=\sum_{k=0}^{\infty}\frac{\sqrt{\pi}(a/2)^{n+2k}}{\Gamma(n+k+M+3/2)}\times\sum_{j=0}^{k}\frac{(-1)^j\Gamma(k-j+M+1/2)}{j!(k-j)!\Gamma(k-j+1/2)}. \tag{A6}$$

When $N=2M+1$, Eq. (1) can similarly be changed into

$$S(2M+1,n,a)=2\int_0^{\pi/2}\sinh(a\cos\chi)J_n(a\sin\chi)\times\sin^{n+1}\chi\cos^{2M+1}\chi\,d\chi. \tag{A7}$$

Expanding the hyperbolic sine function $\sinh(a\cos\chi)$ and repeating the procedure from Eq. (A3) to Eq. (A6), one finds

$$S(2M+1,n,a) = -\frac{a}{2}\sum_{k=0}^{\infty}\frac{\sqrt{\pi}(a/2)^{n+2k}}{\Gamma(n+k+M+5/2)}\times\sum_{j=0}^{k}\frac{(-1)^j\Gamma(k-j+M+3/2)}{j!(k-j)!\Gamma(k-j+3/2)}. \tag{A8}$$

Equations (A6) and (A8) can be combined as

$$S(N,n,a)=\left[-\frac{a}{2}\right]^{\sigma}\sum_{k=0}^{\infty}\frac{\sqrt{\pi}(a/2)^{n+2k}}{\Gamma(n+k+M+\sigma+3/2)}\sum_{j=0}^{k}\frac{(-1)^j\Gamma(k-j+M+\sigma+1/2)}{j!(k-j)!\Gamma(k-j+\sigma+1/2)} \tag{A9}$$

where M is the integer part of $N/2$; $\sigma=1$ when N is odd; and $\sigma=0$ when N is even.

We define the function $h(N,i)$ by the generating fraction

$$\frac{\Gamma(k-j+M+\sigma+1/2)}{\Gamma(k-j+\sigma+1/2)}=\sum_{i=0}^{M}h(N,i)\frac{(k-j)!}{(k-j-i)!} \tag{A10}$$

On one hand, the left hand side of Eq. (A10) is the coefficient of the Mth derivative of the function $x^{k-j+M+\sigma-1/2}$, namely

$$(x^{k-j+M+\sigma-1/2})^{(M)}=\frac{\Gamma(k-j+M+\sigma+1/2)}{\Gamma(k-j+\sigma+1/2)}x^{k-j+\sigma-1/2} . \tag{A11}$$

On the other hand, the Mth derivative of the function $x^{k-j+M+\sigma-1/2}$ can be calculated as

$$\begin{aligned}(x^{k-j+M+\sigma-1/2})^{(M)}=(x^{M+\sigma-1/2}x^{k-j})^{(M)}&=\sum_{i=0}^{M}C_M^i(x^{M+\sigma-1/2})^{(M-i)}(x^{k-j})^{(i)}\\&=\sum_{i=0}^{M}C_M^i\frac{\Gamma(M+\sigma+1/2)}{\Gamma(\sigma+i+1/2)}\frac{(k-j)!}{(k-j-i)!}x^{k-j+\sigma-1/2} .\end{aligned} \tag{A12}$$

Comparing Eq. (A12) with Eq. (A11), one can obtain

$$\frac{\Gamma(k-j+M+\sigma+1/2)}{\Gamma(k-j+\sigma+1/2)}=\sum_{i=0}^{M}C_M^i\frac{\Gamma(M+\sigma+1/2)}{\Gamma(\sigma+i+1/2)}\frac{(k-j)!}{(k-j-i)!} . \tag{A13}$$

One result of the expression of $h(N,i)$ defined by Eq. (A10) from Eq. (A13) is

$$h(N,i)=\frac{C_M^i\Gamma(M+\sigma+1/2)}{\Gamma(\sigma+i+1/2)} . \tag{A14}$$

With the help of Eqs. (A10) and (A14), the summation about j in Eq. (A9) can be derived as

$$\begin{aligned}\sum_{j=0}^{k}\frac{(-1)^j\Gamma(k-j+M+\sigma+1/2)}{j!(k-j)!\Gamma(k-j+\sigma+1/2)}&=\sum_{i=0}^{M}h(N,i)\sum_{j=0}^{k-i}\frac{(-1)^j}{j!(k-i-j)!}\\&=\sum_{i=0}^{M}h(N,i)\frac{1}{(k-i)!}\sum_{j=0}^{k-i}\frac{(-1)^j(k-i)!}{j!(k-i-j)!}\\&=\sum_{i=0}^{M}h(N,i)\delta_{k,i}\\&=\sum_{i=0}^{M}\frac{C_M^i\Gamma(M+\sigma+1/2)}{\Gamma(\sigma+i+1/2)}\delta_{k,i} ,\end{aligned} \tag{A15}$$

where $\delta_{k,i}=1$ if $k=i$, and $\delta_{k,i}=0$ if $k\neq i$. Substituting Eq. (A15) into Eq. (A9), one obtains

$$S(N,n,a)=\left[-\frac{a}{2}\right]^{\sigma}\sum_{i=0}^{M}\frac{\sqrt{\pi}C_M^i\Gamma(M+\sigma+1/2)(a/2)^{n+2i}}{\Gamma(\sigma+i+1/2)\Gamma(n+i+M+\sigma+3/2)} . \tag{A16}$$

It is easily verified that $M+\sigma=L+1$ if we recall that M and L are the integer parts of $N/2$ and $(N-1)/2$, respectively. Then Eq. (A16) becomes

$$S(N,n,a)=\left[-\frac{a}{2}\right]^{L-M+1}\sum_{i=0}^{M}\frac{\sqrt{\pi}C_M^i\Gamma(L+3/2)(a/2)^{n+2i}}{\Gamma(L-M+i+3/2)\Gamma(n+i+L+5/2)} . \tag{A17}$$

Let $i=l-L$; with the help of the relation $L+M=N-1$, Eq. (A17) becomes

$$S(N,n,a)=\left[-\frac{a}{2}\right]^{2-N}\sum_{l=L}^{N-1}\frac{\sqrt{\pi}C_M^{l-L}\Gamma(3/2+L)(a/2)^{n+2l}}{\Gamma(3/2-M+l)\Gamma(5/2+n+l)} . \tag{A18}$$

Finally, from Eq. (A18) we get Eq. (A2).

[1] A. D. Fokker, Ann. Phys. (Leipzig) **43**, 812 (1914).
[2] M. Planck, Preuss. Akad. Wiss. Phys. Math. **K1**, 324 (1916).
[3] L. C. Woods, *Principles of Magnetoplasma Dynamics* (Clarendon, Oxford, 1987), Chap. 3.
[4] D. C. Montgomery and D. A. Tidman, *Plasma Kinetic Theory* (McGraw-Hill, New York, 1964), Chap. 2.
[5] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[6] S. Chandrasekhar, Astrophys. J. **97**, 255 (1943).
[7] L. D. Landau, J. Phys. J. Exp. Phys. **7**, 203 (1937).
[8] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1962), Chap. 5.
[9] M. N. Rosenbluth, W. MacDonald, and D. Judd, Phys. Rev. **107**, 1 (1957).
[10] Chi-Kang Li and Richard D. Petrasso, Phys. Rev. Lett **70**, 3063 (1993).
[11] E. C. Shoub, Phys. Fluids **30**, 1340 (1987).
[12] J. Hubbard, 1961 Proc. R. Soc. London Ser. A **260**, 114 (1961).
[13] J. Hubbard, 1962 Proc. R. Soc. London Ser. A **261**, 371 (1962).
[14] G. E. Cooper and P. T. Herman, Phys. Fluids **16**, 118 (1973).
[15] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).
[16] F. L. Hinton, in *Handbook of Plasma Physics,* edited by A. A. Galeev and R. N. Sudan (McGraw-Hill, New York, 1983), Vol. 1, Chap. 1.5, p. 147.
[17] D. R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983), p. 11.
[18] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Function with Formulas, Graphs, and Mathematical Tables* (U.S. Government Printing Office, Washington, DC, 1966).
[19] Y. B. Chang, Phys. Fluids B **4**, 313 (1992).
[20] Y. B. Chang, Y. P. Huo, and G. Y. Yu, Phys. Fluids B **4**, 3621 (1992).