Capillary gravity waves caused by a moving disturbance: Wave resistance

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The dispersive property of capillary gravity waves is responsible for the complicated wave pattern generated at the free surface of a calm liquid by a disturbance moving with a velocity V greater than the minimum phase speed $c^{\min}=(4g\gamma/\rho)^{1/4}$ (ρ is the liquid density, γ is the liquid-air surface tension, and g is the acceleration due to gravity). The disturbance may be produced by a small object immersed in the liquid or by the application of an external surface pressure distribution. The waves generated by the moving disturbance continually remove energy to infinity, and, consequently, the disturbance experiences a drag called the *wave resistance*. The wave resistance corresponding to a surface pressure distribution symmetrical about a point was analyzed by Havelock in the particular case of pure gravity waves (i.e., $\gamma=0$) for which the minimum phase speed reduces to zero. Here, we investigate the more general case of capillary gravity waves using a linearized theory. We also analyze the integral depression of the liquid, the momentum carried by the liquid, and the effective mass of the disturbance for velocities V smaller than $c^{\min}$. These results may possibly lead to a new method of probing soft surfaces.

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I. SOME PRELIMINARY IDEAS

A. Probing liquid surfaces

The surface of *soft systems* (e.g., liquids, gels, or smectic phases) is often rather difficult to probe, because many powerful techniques (e.g., electron microscopy) impose severe conditions (presence of a high vacuum, introduction of staining agents, etc.). These conditions perturb dramatically the original surface.

The atomic force microscopes (AFM) are essentially free of these problems: here we can bring a very sharp tip close to a soft surface. Usually, what is measured is the force between the two partners. Our general aim here is to propose another family of detection methods. When we move the tip at some uniform speed V parallel to the surface (Fig. 1), the tip induces a local distortion of the soft system. We may probe this distortion by optical means, but the resolution is poor. We may also use mechanical information: (a) a measure of the horizontal force on the probe, which is cleary related to the dissipation in the soft medium, or to the creation of surface waves, (b) more delicately, we may expect that the *mass* of the probe is renormalized by the coupling.

These effects are complex, and it is of importance to ascertain their magnitude. The simplest soft system is a nonviscous liquid. In previous notes [1,2], we considered a model system with a point like particle carrying an electric charge e moving at a small distance (10–1000 nm) from the liquid surface (see Fig. 1 of Ref. [2]). This, in itself, is utterly unrealistic, because the velocities V of interest are small, and an electron or a proton moving at such low velocities has a very short mean free path if we operate at the liquid vapor pressure (even if we think of a polymer liquid, which in itself has an exponentially small vapor pressure, oligomer contaminants in the liquid are enough to provide a vapor phase). However, the charged particle system was not useless, for the following reasons:

(a) Instead of using a single charge, we can think of using a beam of rapid electrons, and to move the beam laterally at a low speed (Fig. 2). Then we do not require a very high vacuum.

(b) We can think of charging the AFM tip, provided that we remain below a certain spark threshold.

(c) Ultimately, the point charge system was interesting: it brought in a number of physical features. A "bump" is expected to show up in the liquid, and the number of molecules involved is quite large, even for a single electron charge and relatively large distances from charge to surface. Originally, we had an incorrect belief: namely, that the mass in the bump would show up as a (large) effective mass correction for the probe [1]. After some time we realized that it is not so, and we constructed explicit formulas for the effective mass [2] [see also Sec. III, (2)].

(d) It is also quite clear that the exact nature of the probe, and of the surface-probe interaction, is not essential: many local perturbations show common features.

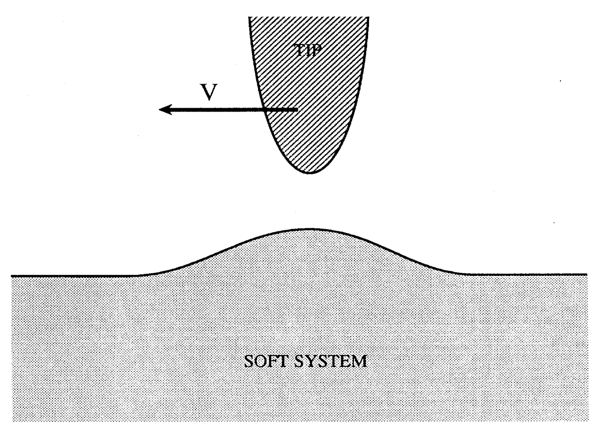

FIG. 1. An AFM tip moving above the surface of a soft system at a constant velocity V.

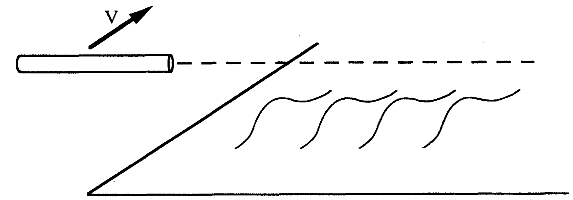

FIG. 2. A beam of rapid electrons moved laterally at a low speed V. The beam (represented by the dotted line) induced a local distorsion of the soft system.

One crucial point, however, was not deeply discussed in Refs. [1] and [2]: namely, the friction force due to the creation of a wake at the surface of the fluid. Wakes have been studied extensively in the hydrodynamic literature, mainly for large objects such as ships, etc. But we lack a detailed analysis of the wake friction for small objects where *capillary waves* are essential. This analysis is the aim of the present paper.

B. Capillary gravity waves

Consider a body of liquid in equilibrium in a gravitational field and having a plane free surface. If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid. This motion will be propagated over the whole surface in the form of waves, which are called *capillary gravity waves* [3]. These waves are driven by a balance between the liquid inertia and its tendency, under the action of the force of gravity and under that of surface tension forces, to return to a state of stable equilibrium.

Consider a small-amplitude sinusoidal plane wave propagating along the liquid-air interface in the negative x direction with a wave speed c. The liquid is taken to be incompressible and inviscid. The vertical displacement of the disturbed surface ζ may be written as

$$z=\zeta(x,t)=a\cos(kx+\omega t), \tag{1.1}$$

where ω is the (circular) frequency and k is the wave number. By "small amplitude" we mean that the amplitude a of the oscillations is small compared to the wavelength $\lambda=2\pi/k$. For a liquid of infinite depth, the relation between ω and k (i.e., the dispersion relation) is given by [4,5]

$$\omega^2=gk+\gamma\frac{k^3}{\rho}, \tag{1.2}$$

where ρ is the liquid density, γ is the liquid-air surface tension, and g is the acceleration due to gravity.

Equation (1.2) may also be written as a dependence of wave speed $c=\omega/k$ on the wave number

$$c=\left(\frac{g}{k}+\frac{\gamma k}{\rho}\right)^{1/2}. \tag{1.3}$$

For long wavelengths such that $k\ll\kappa=(\rho g/\gamma)^{1/2}$ (where κ^{-1} is the capillary length), the effects of the surface tension are negligible and the wave is a pure gravity wave: $c=(g/k)^{1/2}$. In the opposite case of the short wavelengths ($k\gg\kappa$), the effect of gravity may be neglected and we have a pure capillary wave: $c=(\gamma k/\rho)^{1/2}$. An important feature of Eq. (1.3) is that it implies a minimum phase speed of

$$c^{\min}=(4g\gamma/\rho)^{1/4} \tag{1.4}$$

reached at $k^{\min}=\kappa$. For water with $\gamma=73$ mN m^{-1} and $\rho=10^3$ kg m^{-3}, the minimum phase speed is $c^{\min}=0.23$ m s^{-1} and the corresponding wavelength is $\lambda^{\min}=2\pi/\kappa=1.7\ 10^{-2}$ m. Equation (1.3) indicates that there are two possible values of k for any prescribed value of c greater than $c^{\min}$.

The excess energy (which we shall call simply the energy) in the small-amplitude sinusoidal wave (1.1) is divided equally between (i) the kinetic energy and (ii) the energy associated with the restoring forces (gravity and surface tension). The energy per unit horizontal area E is easily shown to be [3]

$$E=\tfrac{1}{2}\rho c^2ka^2. \tag{1.5}$$

It is well known that the sinusoidal wave (1.1) transports energy not at the wave speed $c=\omega/k$ (also called the phase velocity) but at the group velocity defined by $c_g=d\omega/dk$. From Eq. (1.2) we obtain

$$c_g=\frac{g+3\gamma k^2/\rho}{2(gk+\gamma k^3/\rho)^{1/2}}. \tag{1.6}$$

Equations (1.3) and (1.6) indicate that the group velocity is smaller than the phase velocity for $k<\kappa$, while the reverse is true for $k>\kappa$. For long wavelengths ($k\ll\kappa$), Eq. (1.6) reduces to $c_g=(g/k)^{1/2}/2=c/2$. In the opposite case of the short wavelengths ($k\gg\kappa$), we have $c_g=(3/2)(\gamma k/\rho)^{1/2}=3c/2$.

C. Wave resistance

The dispersive property of capillary gravity waves is responsible for the complicated wave pattern generated at the free surface of a still liquid by a moving disturbance [4–7]. The disturbance may be produced by a small object (such as a fishing line) immersed in the liquid or by the application of an external surface pressure distribution $P_{\rm ext}$ [7]. Consider, for example, the waves generated by a pressure distribution localized along a line parallel to the y axis and traveling over the surface in the negative x direction with speed V. A wavecrest propagating in the negative x direction at a phase velocity c can be stationary relative to the moving disturbance if and only if $c=V$. Equation (1.3) shows that there are no steady waves generated by the disturbance if the velocity V is smaller than the minimum phase speed $c^{\min}$. If, however, $V>c^{\min}$, there are two values of k for which $c=V$ (say k_1 and k_2, with $k_1<k_2$). These values are given by

$$k_1=\kappa\left(\frac{V}{c^{\min}}\right)^2\left\{1-\left[1-\left(\frac{c^{\min}}{V}\right)^4\right]^{1/2}\right\} \tag{1.7}$$

and

$$k_2=\kappa\left(\frac{V}{c^{\min}}\right)^2\left\{1+\left[1-\left(\frac{c^{\min}}{V}\right)^4\right]^{1/2}\right\}. \tag{1.8}$$

The smaller (i.e., k_1) satisfies $k<\kappa^{-1}$ and represents a relatively long-wavelength wave for which the group velocity is less than the speed V of the disturbance. Accordingly, this wave is found behind the moving disturbance. The larger value (i.e., k_2) satisfies $k>\kappa^{-1}$ and represents a relatively short-wavelength wave for which the group velocity is greater than V. Accordingly, this wave is found in front of the moving disturbance. The waves generated by the moving disturbance continually remove energy to infinity. Consequently, for $V>c^{\min}$, the disturbance will experience a drag called the *wave resistance*. The wave resistance, denoted here as R, can be calculated as follows [5]: The energy E_d per unit area in the waves downstream of the disturbance is moving away from it at the velocity $V-c_g^d$, where c_g^d is the group velocity for these longer waves with wave speed $c=V$ [c_g^d is obtained by substituting $k=k_1$ into Eq. (1.6)]. Similarly, the energy E_u per unit area in the wave upstream of the disturbance is moving away from it at the velocity c_g^u-V [c_g^u is obtained by substituting $k=k_2$ into Eq. (1.6)]. Therefore, the power D that the moving disturbance must expend to generate both sets of waves is (per unit of length along the y direction)

$$D=[V-c_g^d]E_d+[c_g^u-V]E_u. \tag{1.9}$$

Since the rate at which work is being done by the moving disturbance is $D=RV$, the force R resisting the disturbance motion is (per unit of length along the y direction) [5]

$$R=\left[\frac{V-c_g^d}{V}\right]E_d+\left[\frac{c_g^u-V}{V}\right]E_u. \tag{1.10}$$

An explicit expression for R may be obtained by substituting Eqs. (1.3), (1.5) and (1.6) into Eq. (1.10), yielding

$$\begin{aligned}R&=\left[\frac{1-k_1^2\kappa^{-2}}{2(1+k_1^2\kappa^{-2})}\right]\frac{1}{2}\rho V^2 k_1 a_d^2\\&\quad+\left[\frac{k_2^2\kappa^{-2}-1}{2(1+k_2^2\kappa^{-2})}\right]\frac{1}{2}\rho V^2 k_2 a_u^2\\&=\tfrac{1}{4}\gamma\kappa^2[(1-k_1^2\kappa^{-2})a_d^2+(k_2^2\kappa^{-2}-1)a_u^2],\end{aligned} \tag{1.11}$$

where a_d is the amplitude of the waves downstream of the disturbance and a_u is the amplitude of the waves upstream of the disturbance. These amplitudes depend on the wave numbers k_1 and k_2 and on the exact form of the pressure distribution [6]. Consider, for instance, the case of a pressure distribution P_{ext} concentrated on a mathematical line. It is then of the form $P_{\text{ext}}=p\,\delta(x+Vt)$, where δ denotes the Dirac δ function. It will be shown below (Sec. II A) that in this case the amplitudes a_d and a_u are equal and given by [6–8]

$$a_d=a_u=\frac{2p}{\gamma(k_2-k_1)}. \tag{1.12}$$

The wave resistance for $V>c^{\min}$ is therefore [cf. Eq. (1.11)]

$$R=\frac{p^2}{\gamma}\frac{k_2+k_1}{k_2-k_1}=\frac{p^2}{\gamma}\frac{1}{[1-(c^{\min}/V)^4]^{1/2}} \tag{1.13}$$

(remember that $R=0$ for $V<c^{\min}$, since in that case there are no steady waves generated by the disturbance). In the limit $V\gg c^{\min}$, Eq. (1.13) reduces to $R=p^2/\gamma$. In this high velocity limit, R is independent of gravity. As the velocity V approaches $c^{\min}$ (from above), the amplitudes a_d and a_u and, consequently, the wave resistance (1.13), become unbounded. The reason for this unbounded response is the following: As V approaches $c^{\min}$, the group velocities c_g^d and c_g^u tend to V and the energy transferred to the moving pressure distribution cannot be radiated away.

In the examples discussed above, the surface pressure distribution was uniform in one direction (the y direction). Accordingly, the generated waves were two-dimensional (2D) waves with straight, parallel crests (perpendicular to the direction of the disturbance motion) and the wave resistance was given by the simple expression given in Eq. (1.10). The main concern of the present paper is the wave resistance corresponding to a surface pressure distribution localized near a point, rather than distributed along a line. The wave pattern is then more complete and involves oblique waves that propagate at nonzero angles to the direction of the disturbance motion (we will refer to such waves as 3D waves) [5]. The wave resistance corresponding to a surface pressure distribution symmetrical about a point was analyzed by Havelock [9] in the particular case of pure gravity waves (i.e., $\gamma=0$) for which the minimum phase speed $c^{\min}$ reduces to zero [see Eq. (1.4)]. In this article we will explore the more general case of the capillary gravity waves. This study was done within the framework of the linear theory of capillary gravity waves [see Sec. III, (1)].

II. WAVE MOTION CREATED BY A MOVING PRESSURE SYSTEM

Consider an incompressible, inviscid, infinitely deep liquid whose free surface is unlimited. We take the xy plane as the equilibrium surface of the fluid and the z axis as the direction perpendicular to the equilibrium surface. Let $z=\zeta(x,y,t)$ denote the displacement of the free surface from its equilibrium position. Let us assume that the fluid motion is irrotational. Consequently, the liquid velocity can be expressed as $v=\text{grad}\varphi$, where φ is called the velocity potential. In the presence of an external pressure distribution $P_{\text{ext}}(x,y,t)$, the velocity potential φ is determined by solving Laplace's equation

$$\Delta\varphi=0, \tag{2.1}$$

along with the boundary conditions

$$\rho g\frac{\partial\varphi}{\partial z}+\rho\frac{\partial^2\varphi}{\partial t^2}-\gamma\frac{\partial}{\partial z}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)=-\frac{\partial}{\partial t}P_{\text{ext}} \quad \text{for } z=0, \tag{2.2}$$

$$\frac{\partial\varphi}{\partial z}\to 0 \quad \text{for } z\to-\infty. \tag{2.3}$$

We now study the wave motion created by an external surface pressure distribution that moves with speed V in the negative x direction. The external pressure distribution is then of the form

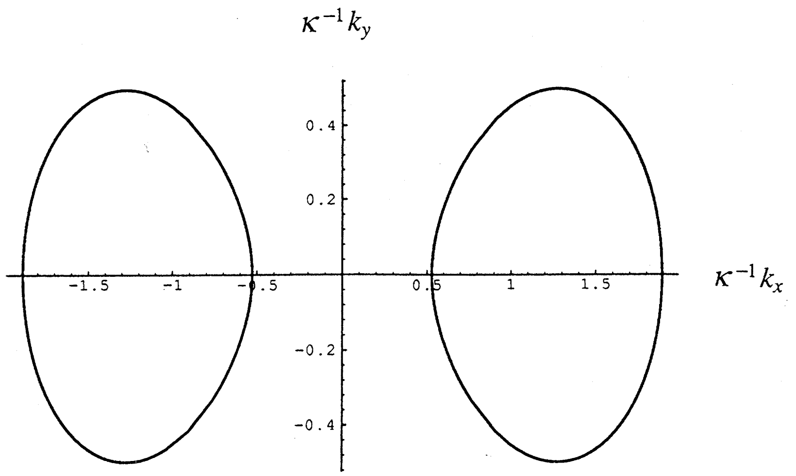

FIG. 3. The curve C defined by Eq. (2.11) for $V/c^{\min}=1.1$.

$$P_{\text{ext}}(x,y,t)=P(x+Vt,y). \tag{2.4}$$

Having in mind Eq. (2.1) and the boundary condition (2.3), we shall seek a velocity potential of the form

$$\varphi(x,y,z,t)=\int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi} A(k_x,k_y)\exp i[k_x(x+Vt) + k_y y]\exp[kz], \tag{2.5}$$

where $k=(k_x^2+k_y^2)^{1/2}$. The amplitude $A(k_x,k_y)$ is obtained by substituting Eq. (2.5) into the free surface boundary condition (2.2), and is

$$[\gamma k(k^2+\kappa^2)-\rho V^2 k_x^2]A(k_x,k_y)=-ik_xV\hat{P}(k_x,k_y), \tag{2.6}$$

where $\hat{P}(k_x,k_y)$ denotes the Fourier transform of the function $P(x,y)$

$$P(x,y)=\int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\hat{P}(k_x,k_y)\exp i[k_x x+k_y y]. \tag{2.7}$$

A. Surface displacement

The vertical displacement of the liquid-air interface may be obtained by combining the kinematic relation at the free surface $\partial\zeta/\partial t=(\partial\varphi/\partial z)_{z=0}$ [3] and Eq. (2.5), and is

$$\zeta(x,y,t)=\int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\hat{\zeta}(k_x,k_y)\exp i[k_x(x+Vt)+k_y y], \tag{2.8}$$

where

$$\hat{\zeta}(k_x,k_y)=\frac{k}{ik_xV}A(k_x,k_y). \tag{2.9}$$

According to Eq. (2.6), the Fourier component $\hat{\zeta}(k_x,k_y)$ satisfies

$$\left[\gamma(k^2+\kappa^2)-\rho V^2\frac{k_x^2}{k}\right]\hat{\zeta}(k_x,k_y)=-\hat{P}(k_x,k_y). \tag{2.10}$$

As long as the stream velocity V is smaller than $c^{\min}$, the bracket on the left hand side of Eq. (2.10) is positive. For $V>c^{\min}$, however, the equation

$$\gamma(k^2+\kappa^2)-\rho V^2\frac{k_x^2}{k}=0 \tag{2.11}$$

gives solutions that define a curve C in the (k_x,k_y) plane. It is worth noticing that Eq. (2.11) corresponds to the dispersion relation (1.2) for $\omega=k_xV$. The curve C is sketched in Fig. 3. It is symmetrical about both the k_x and the k_y axes. Using polar coordinates, Eq. (2.11) can be rewritten as

$$k^2-k\frac{\rho V^2}{\gamma}\cos^2\theta+\kappa^2=0, \tag{2.12}$$

where $k_x=k\cos\theta$ and $k_y=k\sin\theta$. Let χ be the angle defined by

$$\cos\chi=c^{\min}/V,\quad 0\leqslant\chi<\pi/2. \tag{2.13}$$

For a given θ in the range $0\leqslant\theta\leqslant\chi$, Eq. (2.12) gives two solutions, $k_-(\theta)$ and $k_+(\theta)$:

$$k_-(\theta)=\kappa\left(\frac{V}{c^{\min}}\right)^2\{\cos^2\theta-(\cos^4\theta-\cos^4\chi)^{1/2}\},$$

$$k_+(\theta)=\kappa\left(\frac{V}{c^{\min}}\right)^2\{\cos^2\theta+(\cos^4\theta-\cos^4\chi)^{1/2}\}. \tag{2.14}$$

Since the motion of the pressure distribution is steady, we can consider the physical properties of the system at any time including, in particular, $t=0$. According to Eqs. (2.8) and (2.10), the surface displacement is given by

$$\zeta(x,y,t=0)=-\int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\frac{\hat{P}(k_x,k_y)}{\left[\gamma(k^2+\kappa^2)-\rho V^2\frac{k_x^2}{k}\right]}\times\exp i[k_x x+k_y y]. \tag{2.15}$$

[In order to simplify the notations, we shall simply write $\zeta(x,y)$ instead of $\zeta(x,y,t=0)$]. The integral (2.15) cannot be evaluated unambiguously because the poles of the integrand are on the domain of integration. This ambiguity is removed by imposing the *radiation condition* that there be no wave coming in from infinity [5]. There are several mathematical procedures equivalent to this radiation condition. One of them is to consider that the amplitude of the disturbance has increased slowly to its present value in the interval $-\infty\leqslant t\leqslant 0$:

$$P_{\text{ext}}(x,y,t)=e^{\epsilon t}P(x+Vt,y). \tag{2.16}$$

Here, ϵ is a small positive number that will ultimately be allowed to tend to zero. The solution (2.15) is now replaced by

$$\zeta(x,y)=\lim_{\epsilon\to 0}\zeta_\epsilon(x,y), \tag{2.17a}$$

where

$$\zeta_\epsilon(x,y)=-\int \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\frac{\hat{P}(k_x,k_y)}{\left[\gamma(k^2+\kappa^2)-\frac{\rho}{k}(k_xV-i\epsilon)^2\right]}\times\exp i[k_x x+k_y y]. \tag{2.17b}$$

In order to illustrate how Eqs. (2.17) can be used, it is instructive to consider the case of 2D waves. The surface displacement is then given by $\zeta(x)=\lim_{\epsilon\to 0}\zeta_\epsilon(x)$ where

$$\zeta_\epsilon(x)=-\frac{1}{\pi}\text{Re}\left\{\int_0^{+\infty}dk_x\frac{\hat{P}(k_x)}{[\gamma(k_x^2+\kappa^2)-\rho k_xV^2+2i\epsilon\rho V]}\times\exp i[k_x x]\right\} \tag{2.18}$$

(Re stands for "real part of"). The pole of the integrand that was located at k_1 [Eq. (1.7)] has now been shifted above the k_x axis and lies at $k_1+\Delta$, where $\Delta=2i\epsilon\rho V\gamma^{-1}/(k_2-k_1)$. Similarly, the pole that was located at k_2 [Eq. (1.8)] has now been shifted below the k_x axis and lies at $k_2-\Delta$. Note that the fact that k_1 (k_2) has been shifted *above (below)* the k_x axis is a direct consequence of the inequality $c_g^d<V$ ($c_g^u>V$). Since the poles are now out of the domain of integration, the integral (2.18) can be easily evaluated by using contour integral techniques [10]. For a pressure distribution of the form $P(x)=p\,\delta(x)$, for instance, we obtain [6,7]

$$\zeta(x)=-\frac{2p}{\gamma(k_2-k_1)}\sin(k_1x)+F(x) \tag{2.19}$$

for positive values of x, and

$$\zeta(x)=-\frac{2p}{\gamma(k_2-k_1)}\sin(k_2x)+F(x) \tag{2.20}$$

for negative values of x, where

$$F(x)=\frac{p}{\pi\gamma(k_2-k_1)}\int_0^{+\infty}dm\left(\frac{m}{m^2+k_1^2}-\frac{m}{m^2+k_2^2}\right)\times\exp(-m|x|). \tag{2.21}$$

The disturbance of the level represented by the function $F(x)$ is very small for values of x that exceed, say, half the greater wavelength $\lambda_1=2\pi/k_1$. Beyond this distance, the surface is covered on the downstream side of the disturbance by a simple sinusoidal wave of wavelength $\lambda_1=2\pi/k_1$ and amplitude $a_d=2p[\gamma(k_2-k_1)]^{-1}$, and on the upstream side of the disturbance by a simple sinusoidal wave of wavelength $\lambda_2=2\pi/k_2$ and amplitude $a_u=2p[\gamma(k_2-k_1)]^{-1}$. In the absence of capillary forces (i.e., $\gamma=0$), Eqs. (2.19)–(2.21) reduce to [6,7]

$$\zeta(x>0)=-\frac{2p}{\rho V^2}\sin\left(\frac{g}{V^2}x\right)+\frac{p}{\pi\rho V^2}\int_0^{+\infty}dm\frac{m}{m^2+(g/V^2)^2}\exp(-mx) \tag{2.22}$$

and

$$\zeta(x<0)=\frac{p}{\pi\rho V^2}\int_0^{+\infty}dm\frac{m}{m^2+(g/V^2)^2}\exp(+mx), \tag{2.23}$$

respectively. Using Eqs. (2.22) and (2.23), Lamb [6] calculated the "integral depression" of the surface $\Omega=\int dx\,\zeta(x)$ and found that $\Omega=-p(\rho g)^{-1}$ [11], exactly as if the fluid were at rest.

The fact that Ω is independent of the velocity V is in fact a general result that is not restricted to the particular case of 2D pure gravity waves with a pressure distribution concentrated on a mathematical line [i.e., $\gamma=0$ and $P=p\,\delta(x)$]. This can been seen as follows: According to Eq. (2.8), the integral depression of the surface can be written as

$$\Omega=\int\int dx\,dy\,\zeta(x,y)=\lim_{k_x\to 0}\lim_{k_y\to 0}\hat{\zeta}(k_x,k_y). \tag{2.24}$$

Now, using Eq. (2.10), we obtain

$$\Omega=-\frac{\hat{P}(k_x=0,\ k_y=0)}{\rho g} \tag{2.25}$$

exactly as if the fluid were at rest [$\hat{P}(k_x=0,\ k_y=0)$ corresponds to the total force acting on the fluid surface along the negative z direction] [1,2]. It is remarkable that Ω is independent of the velocity V of the disturbance. This result is valid over the all range $0\leqslant V<+\infty$.

B. Wave resistance

We now investigate the wave resistance experienced by the disturbance, using the method proposed by Havelock in his study of pure gravity waves [9]. According to Havelock, we may imagine a rigid cover fitting the surface everywhere. The assigned pressure system $P(x,y)$ is applied to the liquid surface by means of this cover; hence the wave resistance is simply the total resolved pressure in the x direction. This leads to

$$R=-\int dx\, dy\, P(x,y)\left\{\frac{\partial}{\partial x}\zeta(x,y)\right\}. \tag{2.26}$$

By using Eqs. (2.17), the wave resistance (2.26) can be written as

$$R=\lim_{\epsilon\to 0}\int\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\frac{ik_x|\hat{P}(k_x,k_y)|^2}{\gamma(k^2+\kappa^2)-\rho V^2\frac{k_x^2}{k}+2i\epsilon\rho V\frac{k_x}{k}}. \tag{2.27}$$

Let us consider a pressure system symmetrical around the origin

$$P(x,y)=g(r),\quad r=(x^2+y^2)^{1/2}. \tag{2.28}$$

The Fourier transform $\hat{P}(k_x,k_y)$ is then a function only of k and can be written as $\hat{P}(k_x,k_y)=G(k)$, where

$$G(k)=\int_0^{+\infty}dr\, rg(r)\int_0^{2\pi}d\theta\,\exp[-ikr\cos\theta]$$
$$=2\pi\int_0^{+\infty}dr\, rg(r)J_0(kr). \tag{2.29}$$

Here, J_0 denotes the Bessel function of the first kind of zeroth order [12]. Using polar coordinates and remembering that curve C [see Eq. (2.11)] is symmetrical about the k_x and the k_y axes, we can rewrite Eq. (2.27) as

$$R=4\int_0^{\chi}d\theta\cos\theta\frac{i}{(2\pi)^2\gamma}[\quad], \tag{2.30}$$

where

$$[\quad]=\lim_{\epsilon\to 0}\int_0^{+\infty}dk\frac{[kG(k)]^2}{[k-k_+(\theta)][k-k_-(\theta)]+\frac{2i\epsilon\rho V}{\gamma}\cos\theta} \tag{2.31}$$

and $k_-(\theta)$ and $k_+(\theta)$ are given by Eq. (2.14). The pole $k=k_-(\theta)$ $[k=k_+(\theta)]$ in (2.31) gives a contribution

$$i\pi\frac{\{k_-(\theta)G[k_-(\theta)]\}^2}{k_-(\theta)-k_+(\theta)}\left(-i\pi\frac{\{k_+(\theta)G[k_+(\theta)]\}^2}{k_+(\theta)-k_-(\theta)}\right). \tag{2.32}$$

Hence

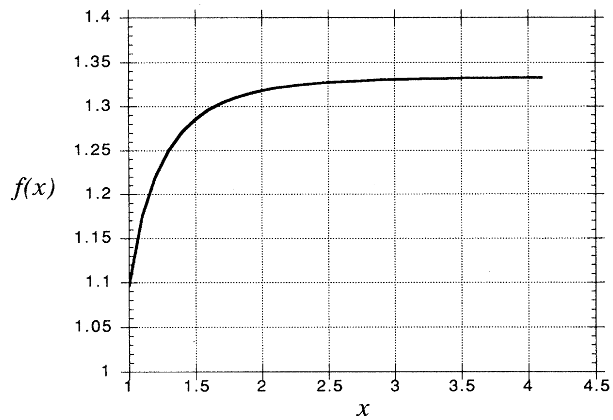

FIG. 4. The function $f(x)$ defined by Eq. (2.37).

$$R=\frac{1}{\pi\gamma}\int_0^{\chi}d\theta\cos\theta\times\frac{\{k_+(\theta)G[k_+(\theta)]\}^2+\{k_-(\theta)G[k_-(\theta)]\}^2}{k_+(\theta)-k_-(\theta)}. \tag{2.33}$$

Equation (2.33) is our central result. It gives the wave resistance as a function of the velocity of the pressure distribution.

If we now consider the case of a very localized pressure

$$P(x,y)=p\,\delta(x)\,\delta(y), \tag{2.34}$$

Eq. (2.33) then becomes

$$R=\frac{p^2}{\pi\gamma}\kappa\left(\frac{V}{c^{\min}}\right)^2\int_0^{\chi}d\theta\cos\theta\frac{2\cos^4\theta-\cos^4\chi}{(\cos^4\theta-\cos^4\chi)^{1/2}}. \tag{2.35}$$

Since $\cos\chi=c^{\min}/V$, Eq. (2.35) can be rewritten as

$$R=\frac{p^2}{\pi\gamma}\kappa\left(\frac{V}{c^{\min}}\right)^2 f\left(\frac{V}{c^{\min}}\right), \tag{2.36}$$

where the function $f(x)$ is defined by

$$f(x)=\int_0^{\mathrm{Arccos}(1/x)}d\theta\cos\theta\frac{2\cos^4\theta-x^{-4}}{(\cos^4\theta-x^{-4})^{1/2}} \tag{2.37}$$

and is sketched in Fig. 4 [note that $f(x)$ is only defined for $x\geqslant 1$]. Expanding $f(x)$ near $x=1$, we find

$$f(x)=\frac{\pi}{2\sqrt{2}}\{1+\tfrac{3}{4}(x-1)+O[(x-1)^2]\}. \tag{2.38}$$

For $x\to+\infty$, $f(x)$ reaches a constant value

$$\lim_{x\to+\infty}f(x)=2\int_0^{\pi/2}d\theta\cos^3\theta=4/3. \tag{2.39}$$

The behavior of the wave resistance Eq. (2.36) as a function of $V/c^{\min}$ is depicted in Fig. 5. As the disturbance ve-

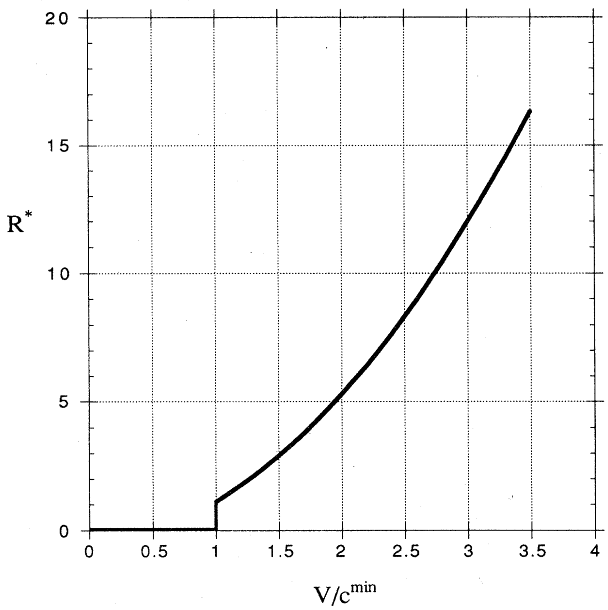

FIG. 5. $R^* = \pi\gamma(p^2\kappa)^{-1} R$ as a function of $V/c^{\min}$, with R being the wave resistance Eq. (2.36).

locity V approaches $c^{\min}$ from above, R takes the value $p^2\kappa/(2^{3/2}\gamma)$ [see Eqs. (2.36) and (2.38)]. Since R must be zero for $V<c^{\min}$ (no steady waves being generated by the disturbance in that case), we find that the wave resistance is discontinuous at $V=c^{\min}$. In the limit $V \gg c^{\min}$, Eq. (2.36) reduces to

$$R=\frac{2}{3}\frac{p^2\rho V^2}{\pi\gamma^2}, \quad V \gg c^{\min} \tag{2.40}$$

and the wave resistance is independent of gravity [13].

III. CONCLUDING REMARKS

(1) We would like to stress that our study was done within the framework of the *linear* theory of capillary gravity waves. When the response of the liquid becomes unbounded, it is clear that the linearized theory cannot be valid. For example, in the 2D case studied in Sec. I B, various corrections to the results (1.12) and (1.13) should show up when V tends to $c^{\min}$. For the very small objects considered here, we suspect that the cutoff will be due to viscosity, rather than to nonlinear effects; but this shall require a separate study.

(2) Our discussion was mainly concerned with case $V>c^{\min}$. The opposite case $V<c^{\min}$, for which no steady waves are generated (the disturbance of the level being confined around the coordinates $x=-Vt$ and $y=0$), is nevertheless interesting. Consider, for example, the momentum $\mathcal{Q}$ of the fluid along the direction of motion (i.e., along the negative x direction)

$$\mathcal{Q}=-\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\int_{-\infty}^{\zeta}dz\;\rho v_x \tag{3.1}$$

and let us assume that $V<c^{\min}$. Since $v_x=\partial\varphi/\partial x$, the x component of the velocity can be written as [cf. Eqs. (2.5) and (2.9)]

$$v_x(x,y,z,t)=\int\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\hat{v}_x(k_x,k_y)\exp i[k_x(x+Vt)+k_y y]\exp[kz], \tag{3.2}$$

where

$$\hat{v}_x(k_x,k_y)=ik_xA(k_x,k_y)=-\frac{k_x^2}{k}V\hat{\zeta}(k_x,k_y). \tag{3.3}$$

Substituting (3.2) into Eq. (3.1) and integrating with respect to z, we obtain

$$\mathcal{Q}=\rho V\int\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\frac{k_x^2}{k^2}\hat{\zeta}(k_x,k_y)\int dx\,dy\,\exp i[k_x x+k_y y]\exp[k\zeta(x,y)]. \tag{3.4}$$

We now expand $\exp[k\zeta(x,y)]$ in powers of $k\zeta(x,y)$. It is important to realize that the zeroth order term does not contribute to $\mathcal{Q}$. This point can be checked by using periodic boundary conditions along the x coordinate and noticing that no velocity field is associated with the $k=0$ mode [see Eq. (3.3): $\hat{v}_x(k=0)=0$]. Going to the next order, we find

$$\mathcal{Q}=\rho V\int\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\frac{k_x^2}{k}|\hat{\zeta}(k_x,k_y)|^2. \tag{3.5}$$

Inserting Eq. (2.10) into Eq. (3.5), and assuming that $P(x,y)$ is of the form of (2.28), we obtain

$$\mathcal{Q}=\frac{\rho V\kappa^{-1}}{4\pi\gamma^2}\int_0^{+\infty}du\,\frac{u^2G^2(u\kappa)}{(u^2+1)^{1/2}[u^2-2(V/c^{\min})^2u+1]^{3/2}}. \tag{3.6}$$

Finally, the fluid momentum Eq. (3.6) can be rewritten as

$$\mathcal{Q}=\frac{\rho V\kappa^3\Omega^2}{16}\left[\frac{4}{\pi}\int_0^{+\infty}du\times\frac{u^2}{(u^2+1)^{1/2}[u^2-2(V/c^{\min})^2u+1]^{3/2}}\left(\frac{G(u\kappa)}{G(0)}\right)^2\right] \tag{3.7}$$

by virtue of Eq. (2.25). Equation (3.7) describes the variations of the fluid momentum $\mathcal{Q}$ with the disturbance velocity V. For $V \ll c^{\min}$, the fluid momentum varies linearly. For larger values of V, however, $\mathcal{Q}$ deviates from linearity and diverges like $[1-(V/c^{\min})]^{-1}$ as V approaches $c^{\min}$ [see Sec. III, (1)].

Suppose now that the disturbance is accelerated by some external force F (along the negative x direction) [14]. In this process, the liquid momentum will also be increased. Hence, the force F must be equal to the time derivative of the total momentum of the system, which is the sum of the momentum mV of the disturbance (m being the mass of the disturbance) and the momentum $\mathcal{Q}$ of the liquid:

$$F=m\,dV/dt+d\mathcal{Q}/dt. \tag{3.8}$$

Equation (3.8) can be rewritten as

$$F=[m+\mathcal{Q}'(V)]dV/dt, \tag{3.9}$$

where $\mathcal{Q}'(V)$ denotes the derivative of $\mathcal{Q}$ [Eq. (3.7)] with respect to V. The coefficient of dV/dt is called the *effective mass* of the disturbance. It consists of the actual mass of the disturbance m and the *induced mass*, which, according to Eq. (3.9), is

$$\mathcal{Q}'(V)=\frac{\rho\kappa^3\Omega^2}{16}[\quad], \tag{3.10}$$

where

$$[\quad]=\frac{4}{\pi}\int_0^{+\infty}du\,\frac{u^2[u^2+4(V/c^{\min})^2u+1]}{(u^2+1)^{1/2}[u^2-2(V/c^{\min})^2u+1]^{5/2}}\times\left(\frac{G(u\kappa)}{G(0)}\right)^2. \tag{3.11}$$

For $V\ll c^{\min}$, the induced mass is constant [15]. For larger values of V, $\mathcal{Q}'(V)$ becomes velocity dependent and diverges like $[1-(V/c^{\min})]^{-2}$ as V approaches $c^{\min}$ [see Sec. III, (1)].

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[8] In the case of a pressure distribution $P_{\rm ext}=pb\{\pi[b^2+(x+Vt)^2]\}^{-1}$ diffused over a band of sensible breadth, the amplitudes a_d and a_u are given by $a_d=2p[\gamma(k_2-k_1)]^{-1}\exp(-bk_1)$ and $a_u=2p[\gamma(k_2-k_1)]^{-1}\times\exp(-bk_2)$, respectively [6], and the wave resistance is given by $R=p^2[\gamma(k_2-k_1)]^{-1}[k_1\exp(-2bk_1)+k_2\times\exp(-2bk_2)]$.

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[10] See, for instance, G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic, New York, 1985).

[11] In order to properly calculate Ω, one has first to calculate $\int dx\,\zeta_\epsilon(x)$ and then to take the limit $\epsilon\to 0$.

[12] See, for instance, *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1972).

[13] In the case of a pressure distribution $g(r)=pd/[2\pi(d^2+r^2)^{3/2}$ diffused over a circular region of radius d (with $d\ll\kappa^{-1}$), one expects R to be given by Eq. (2.40) for $c^{\min}\ll V\ll[3^{1/2}\gamma/4\rho d)]^{1/2}$ and by $R=p^2/(8\pi\rho d^2V^2)$ for $V\gg[3^{1/2}\gamma/(4\rho d)]^{1/2}$.

[14] The following analysis is reminiscent of the development given in Landau and Lifshitz concerning the drag force in the potential flow past a body, see Ref. [3], pp. 26–31.

[15] Note that $\mathcal{Q}'(V\ll c^{\min})$ is not simply proportional to the bump volume: $\mathcal{Q}'(V\ll c^{\min})\neq\rho\Omega$.