Dissipative effects on the localization of a charged oscillator in a magnetic field

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We consider the effect of dissipation on a charged, quantum harmonic oscillator coupled to a heat bath, in the presence of an external magnetic field. General conclusions about the equal-time position autocorrelation functions can be reached by using only those properties of the generalized susceptibility tensor imposed by fundamental physical principles. Explicit calculations are made for an Ohmic heat bath at zero temperature, enabling us to show that (unlike the problem without the magnetic field where dissipation always leads to enhanced localization) when the magnetic field is stronger than a certain critical value, weak dissipation actually delocalizes the oscillation of the charged particle.

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I. INTRODUCTION

The problem of dissipative effects on localization has been investigated by many people in connection with the study of dissipative quantum phase coherence [1–6]. It has been shown recently [7] by calculating explicitly the equal-time position autocorrelation functions for a specific model of a one-dimensional quantum harmonic oscillator in both Ohmic and black-body radiation heat baths at arbitrary temperatures, that increasing dissipation always results in enhanced localization, in agreement with previous work on the subject [6]. These results are not unexpected. Here we extend these considerations to include the presence of an external field, specifically a uniform magnetic field, for a three-dimensional charged quantum oscillator in a heat bath. (The field produced by the particle itself by virtue of its interaction with the heat bath is accounted for by the dissipative and fluctuation terms in the equations of motion.) We find that the interplay between the dissipation and the external field not only complicates the problem but also gives rise to unexpected results.

The organization of this paper is as follows. In Sec. II we first introduce the general formalism and notation used in this paper. We then establish several useful properties about the generalized susceptibility tensor $\alpha_{\rho\sigma}(\omega)$ obtained from the generalized quantum Langevin equation (GLE) for an isotropic harmonic oscillator. Since the symmetrized position correlation functions can be expressed in terms of the generalized susceptibility tensor by using the fluctuation-dissipation theorem, we are able to prove two general theorems concerning the equal-time position autocorrelation functions (dispersions) that are true for any physical heat baths. In Sec. III we shall restrict our consideration, for simplicity, to the specific case of Ohmic heat bath. We calculate in detail the equal-time position autocorrelation functions and their derivatives with respect to the frictional parameter of the Ohmic heat bath at zero temperature. In Sec. IV, we summarize the analysis, discuss possible physical interpretations of our results, and present our conclusions.

II. POSITION AUTOCORRELATION FUNCTION

For a particle of charge e and mass m in a three-dimensional (3D) harmonic potential well with spring constant K, in the presence of a uniform static magnetic field $\vec{B}$, and coupled to a heat bath at zero temperature, the equal-time position correlation functions can be derived using the GLE with the result [see Eq. (2.17) in Ref. [8] and set $t=t'$]

$$\frac{1}{2}\langle r_\rho r_\sigma + r_\sigma r_\rho\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega\, \mathrm{Im}[\alpha^s_{\rho\sigma}(\omega+i0^+)]\coth\left[\frac{\hbar\omega}{2kT}\right], \quad (2.1)$$

where the Greek indices ρ and σ stand for different spatial components of the position operator $\vec{r}$, and $\alpha^s_{\rho\sigma}(\omega)$ is the symmetric part (with respect to the indices ρ and σ) of the generalized susceptibility tensor $\alpha_{\rho\sigma}(\omega)$ [Eqs. (2.14) and (3.28) of Ref. [8]]. Explicitly,

$$\alpha_{\rho\sigma}(\omega)=\frac{\left[\lambda^2\delta_{\rho\sigma}-\left[\omega\frac{e}{c}\right]^2 B_\rho B_\sigma-\varepsilon_{\rho\sigma\eta}B_\eta\lambda i\omega\frac{e}{c}\right]}{\lambda\left[\lambda^2-\left[\omega\frac{e}{c}\right]^2\vec{B}^2\right]} \quad (2.2)$$

and

$$\alpha^s_{\rho\sigma}(\omega)=\frac{\left[\lambda^2\delta_{\rho\sigma}-\left[\omega\frac{e}{c}\right]^2 B_\rho B_\sigma\right]}{\lambda\left[\lambda^2-\left[\omega\frac{e}{c}\right]^2\vec{B}^2\right]}\ , \tag{2.3}$$

with

$$\lambda(\omega)=-m\omega^2+K-i\omega\tilde{\mu}(\omega)\ , \tag{2.4}$$

where $\tilde{\mu}(\omega)$ is the spectral distribution of the heat bath [9], and $\delta_{\rho\sigma}$ is the Kronecker delta function, while $\varepsilon_{\rho\sigma\eta}$ is the Levi-Civita symbol.

The c-number-generalized susceptibility tensor $\alpha_{\rho\sigma}(\omega)$ uniquely determines the dynamics of linear systems. As with the Fourier transform of the memory function $\tilde{\mu}(\omega)$ [9], $\alpha_{\rho\sigma}(\omega)$ obeys several important properties required by general physical principles. First of all, $\alpha_{\rho\sigma}(\omega)$ satisfies the reality condition [8]

$$\alpha^*_{\rho\sigma}(\omega)=\alpha_{\rho\sigma}(-\omega)\ , \tag{2.5}$$

which reflects the fact that $\vec{r}$ is a Hermitian operator. Thus the real and imaginary parts of $\alpha_{\rho\sigma}(\omega)$ are even and odd functions of ω, respectively. Second, no element of the matrix $\alpha_{\rho\sigma}(\omega)$ has poles in the upper half-plane (UHP) (see Appendix A). Furthermore, for the three diagonal elements $\alpha_{\rho\rho}(\omega)$ $[\equiv\alpha^s_{\rho\rho}(\omega)]$ (with $\rho=1,2,3$), we have

$$\mathrm{Im}\alpha_{\rho\rho}(\omega)>0 \quad \text{for } \omega>0\ , \tag{2.6}$$

thereby $-i\omega\alpha_{\rho\rho}(\omega)$ ($\rho=1,2,3$) are real positive functions (see Appendix B).

The factor $\coth(\hbar\omega/2kT)$ in (2.1) is a monotonically increasing function of temperature T, so is $\langle r_\rho^2\rangle$ as deduced from (2.1) and (2.6), i.e.,

$$\frac{\partial}{\partial T}\langle r_\rho^2\rangle>0 \quad \text{with } \rho=1,2,3\ , \tag{2.7}$$

similar to the one-dimensional case [7].

Without loss of generality, we assume for the rest of the calculations in this paper that the magnetic field is along the z axis. Then, from (2.2), the only nonzero elements of $\alpha^s_{\rho\sigma}(\omega)$ are

$$\alpha^s_{xx}(\omega)=\alpha^s_{yy}(\omega)=\frac{\lambda}{\lambda^2-(e/c)^2\vec{B}^2\omega^2} \tag{2.8}$$

and

$$\alpha^s_{zz}(\omega)=\frac{1}{\lambda}=\frac{1}{-m\omega^2+K-i\omega\tilde{\mu}(\omega)}\ . \tag{2.9}$$

Correspondingly, the only nonzero equal-time position autocorrelation functions here are the mean square displacements (also called dispersions) $\langle x^2\rangle$, $\langle y^2\rangle$, and $\langle z^2\rangle$.

We note here that $\alpha^s_{zz}(\omega)$ is the same as that for a one-dimensional problem without the magnetic field, and that it can be obtained formally by setting $\vec{B}$ equal to zero in the expression (2.8) for α^s_{xx} or α^s_{yy}. Hence, $\alpha^s_{zz}(\omega)$ and $\langle z^2\rangle$ are independent of the magnetic field, which is expected because a magnetic field does not affect the motion of particles along the field line itself.

The dispersions $\langle x^2\rangle$ or $\langle y^2\rangle$ may also be expressed in a series form by means of the theorem of residues from the theory of functions of a complex variable. First, noting that the integrand in (2.1) is an even function of ω because of the reality condition (2.5) on $\alpha_{11}(\omega)$, (2.1) can be rewritten as

$$\langle x^2\rangle=\frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}d\omega\alpha_{11}(\omega)\coth\left[\frac{\hbar\omega}{2kT}\right]\ . \tag{2.10}$$

We may now close the contour in the UHP, where only the factor $\coth(\hbar\omega/2kT)$ in the integrand in (2.10) contributes simple poles at $\omega=i\nu_n$ ($n=1,2,\ldots$). Here $\nu_n=2\pi kTn/\hbar$ are the usual Matsubara frequencies [10]. The summation over the residues yields

$$\langle x^2\rangle=\frac{kT}{m}\left\{\frac{1}{\omega_0^2}+2\sum_{n=1}^{\infty}\frac{\hat{\lambda}(\nu_n)}{\hat{\lambda}^2(\nu_n)+(\nu_n\omega_c)^2}\right\}\ , \tag{2.11}$$

where $\omega_0=(K/m)^{1/2}$ is the bare-oscillator frequency and $\omega_c\equiv eB/mc$ is the cyclotron frequency, and where

$$\hat{\lambda}(\nu_n)\equiv\lambda(i\nu_n)/m=\nu_n^2+\omega_0^2+\nu_n\hat{\gamma}(\nu_n) \tag{2.12}$$

with

$$\hat{\gamma}(\nu_n)\equiv\tilde{\mu}(i\nu_n)/m\ . \tag{2.13}$$

Since $\tilde{\mu}(iz)>0$ for $z>0$ [9], it follows from (2.12) and (2.13) that $\hat{\lambda}(\nu_n)>0$ ($n=1,2,\ldots$). Therefore $\langle x^2\rangle$ decreases monotonically with increasing strength of the magnetic field

$$\frac{\partial}{\partial B}\langle x^2\rangle<0\ . \tag{2.14}$$

We conclude this section by emphasizing that Eqs. (2.7) and (2.14) hold for any strength of the magnetic field and any type of heat baths restricted only by general physical principles. Equation (2.14) is also closely related to the fact that the dissipative system of a charged quantum oscillator in an external magnetic field is still generally diamagnetic (see Appendix C).

III. DISSIPATIVE EFFECT ON THE LOCALIZATION OF A MAGNETO-OSCILLATOR

To evaluate the effect of dissipation on the localization of a charged oscillator in a magnetic field, we restrict ourselves to the case of Ohmic heat bath for simplicity. Then $\tilde{\mu}(\omega)=m\gamma$ is frequency independent, where γ is the so-called friction constant. It follows, by using (2.4) and (2.8), that

$$\mathrm{Im}\alpha^s_{xx}(\omega)=\frac{\gamma\omega}{2m}\left[\frac{1}{(\omega^2-\omega_0^2+\omega_c\omega)^2+\gamma^2\omega^2}+\frac{1}{(\omega^2-\omega_0^2-\omega_c\omega)^2+\gamma^2\omega^2}\right]=\frac{\omega}{2m}\mathrm{Im}\left\{\frac{1}{\zeta}\left[\frac{1}{\omega^2+\omega_2^2}-\frac{1}{\omega^2+\omega_1^2}\right]\right\}\ , \tag{3.1}$$

where

$$\omega_{2,1}=\left[\frac{\gamma}{2}\pm\left[\frac{b-a}{2}\right]^{1/2}\right]-i\left[\frac{\omega_c}{2}\pm\left[\frac{b+a}{2}\right]^{1/2}\right], \tag{3.2a}$$

$$\zeta=\left[\frac{b+a}{2}\right]^{1/2}+i\left[\frac{b-a}{2}\right]^{1/2}, \tag{3.2b}$$

with

$$a=\left[\frac{\omega_c}{2}\right]^2+\omega_0^2-\left[\frac{\gamma}{2}\right]^2 \tag{3.3}$$

and

$$b=\left[a^2+\left[\frac{\gamma\omega_c}{2}\right]^2\right]^{1/2}. \tag{3.4}$$

Substituting (3.1) into (2.1) and carrying out the integration, we obtain

$$\langle x^2\rangle=\frac{kT}{m\omega_0^2}+\frac{1}{2\pi m}\mathrm{Im}\left\{\frac{1}{\zeta}[\psi(1+\varpi_1)-\psi(1+\varpi_2)]\right\}, \tag{3.5}$$

where $\varpi_{1,2}=\hbar\omega_{1,2}/2\pi kT$, and $\psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$ [11]. In the high-temperature region $kT\gg\hbar\omega_{1,2}$, by expanding the $\psi(z)$ functions involved about 1, (3.5) reduces to

$$\langle x^2\rangle=\frac{kT}{m\omega_0^2}+O(T^{-1}), \tag{3.6}$$

in accord with the classical equipartition law. While for low temperatures $kT\ll\hbar\omega_{1,2}$, we may insert the asymptotic expansion of $\psi(z)$ in (3.5) and find

$$\langle x^2\rangle=\frac{\hbar}{2\pi mb}\left\{2\left[\frac{b+a}{2}\right]^{1/2}\tan^{-1}\left[\frac{2}{\gamma}\left[\frac{b+a}{2}\right]^{1/2}\right]+\left[\frac{b-a}{2}\right]^{1/2}\ln\left[\frac{\gamma/2+\sqrt{(b-a)/2}}{\gamma/2-\sqrt{(b-a)/2}}\right]\right\}+\frac{\pi\gamma(kT)^2}{3m\hbar\omega_0^4}+O((kT)^4), \tag{3.7}$$

which has the T^2 power-law correction characteristic of the Ohmic heat bath. We note in passing that this leading-order correction term is independent of the magnetic field.

Setting $\omega_c=0$ in (3.7), we have at $T=0$ K

$$\langle x^2\rangle=\begin{cases}\dfrac{\hbar}{\pi m\sqrt{\omega_0^2-\gamma^2/4}}\tan^{-1}\left[\dfrac{2}{\gamma}\sqrt{\omega_0^2-\gamma^2/4}\right] & \text{if } \omega_0>\dfrac{\gamma}{2} \quad (3.8)\\[2ex] \dfrac{\hbar}{2\pi m\sqrt{\gamma^2/4-\omega_0^2}}\ln\left[\dfrac{\gamma/2+\sqrt{\gamma^2/4-\omega_0^2}}{\gamma/2-\sqrt{\gamma^2/4-\omega_0^2}}\right] & \text{if } \omega_0<\dfrac{\gamma}{2}, \quad (3.9)\end{cases}$$

in agreement with known zero-temperature one-dimensional results in the absence of a magnetic field [6,7].

In order to examine the effect of dissipation on localization at zero temperature, we evaluate the partial derivative of $\langle x^2\rangle$ with respect to the friction constant γ. From (3.7), it is straightforward to check that

$$\frac{\partial}{\partial\gamma}\langle x^2\rangle=\frac{\hbar\omega_c}{4\pi mb^2}\left[\frac{b[\gamma^2/4+(b+a)/2]-(b+a)(\omega_c^2/4+\omega_0^2+\gamma^2/4)}{b\sqrt{(b+a)/2}}\tan^{-1}\left[\frac{2}{\gamma}\left[\frac{b+a}{2}\right]^{1/2}\right]+\frac{b[\gamma^2/4-(b-a)/2]-(b-a)(\omega_c^2/4+\omega_0^2+\gamma^2/4)}{2b\sqrt{(b-a)/2}}\ln\left[\frac{\gamma/2+\sqrt{(b-a)/2}}{\gamma/2-\sqrt{(b-a)/2}}\right]+\gamma\right]. \tag{3.10}$$

This form is somewhat complicated for the purpose of ascertaining whether it is negative definite or not. Thus we first consider its value at zero dissipation by setting γ to zero in (3.10):

$$\left.\frac{\partial}{\partial\gamma}\langle x^2\rangle\right|_{\gamma=0}=\frac{\hbar}{2\pi m(\omega_c^2/4+\omega_0^2)}\left[\frac{\omega_c}{4\sqrt{\omega_c^2/4+\omega_0^2}}\ln\left[\frac{\sqrt{\omega_c^2/4+\omega_0^2}+\omega_c/2}{\sqrt{\omega_c^2/4+\omega_0^2}-\omega_c/2}\right]-1\right], \tag{3.11}$$

which can be easily shown to be negative (with a smaller absolute value than the case with $\omega_c=0$) if $\omega_c<3.018\omega_0$, but positive if $\omega_c>3.018\omega_0$. Therefore, for magnetic fields less than the critical value $B_c=mc\omega_c/e=3.018mc\omega_0/e$, the Ohmic dissipation still results in enhanced localization, but to a less extent than the case without a magnetic field. On the other hand, for a magnetic field surpassing that critical value, the dissipation instead reduces the localization of the oscillator. This result is quite intriguing. It might be understood qualitatively by noting that both the Lorentz force and the frictional force depend on the velocity of the particle, with the latter tending to slow down and hence localize the particle whereas the former tending to delocalize it [see Eq. (3.11)]. It is these opposite tendencies of the dissipation and the magnetic field that give rise to this interesting phenomenon.

In general, the critical value of ω_c is a function of γ for nonzero friction constant, viz., $\omega_c = f(\gamma)$, which is the solution to the equation obtained by setting the right-hand side of (3.10) to zero. From the discussion following (3.11), we immediately have $f(0)=3.018\omega_0$. Some other properties of this function can be obtained by analyzing (3.10) in detail.

For both large ω_c and γ (i.e., $\omega_c \gg \omega_0$ and $\gamma \gg \omega_0$), we have, from (3.10),

$$\frac{\partial}{\partial\gamma}\langle x^2\rangle = \frac{2\hbar}{\pi m}\left[\frac{\omega_c^2-\gamma^2}{(\omega_c^2+\gamma^2)^2}\ln\left[\frac{\sqrt{\omega_c^2+\gamma^2}}{\omega_0}\right] - \frac{\omega_c^2-\gamma^2}{(\omega_c^2+\gamma^2)^2} - \frac{2\gamma\omega_c}{(\omega_c^2+\gamma^2)^2}\tan^{-1}\left[\frac{\omega_c}{\gamma}\right]\right], \tag{3.12}$$

which implies that the leading term of the asymptotic expansion of $f(\gamma)$ is γ, i.e., $f(\gamma)\approx\gamma+\cdots$ for $\gamma \gg \omega_0$. Furthermore, taking the derivative of $f(\gamma)$ with respect to γ on both sides of (3.10), we find

$$f'(0) = \frac{-\left.\dfrac{\partial^2}{\partial\gamma^2}\langle x^2\rangle\right|_{\gamma=0}}{\left.\dfrac{\partial^2}{\partial\gamma\,\partial\omega_c}\langle x^2\rangle\right|_{\gamma=0}} \cong 1.420\ . \tag{3.13}$$

Finally, we turn to the case of strong dissipation (i.e., $\gamma \gg \omega_0$ and $\gamma \gg \omega_c$) and obtain

$$\langle x^2\rangle \sim \frac{2\hbar}{\pi m\gamma}\left[\ln\left[\frac{\gamma}{\omega_0}\right] - \frac{\omega_c^2-2\omega_0^2}{\gamma^2}\ln\left[\frac{\gamma}{\omega_0}\right]\cdots\right]. \tag{3.14}$$

The second leading term in this asymptotic expansion decreases with increasing ω_c. Hence we can see that strong dissipation leads to strong localization and that the magnetic field only slightly enhances this effect.

IV. CONCLUSIONS

We have considered the problem of calculating the symmetrized equal-time position correlation functions for a charged quantum oscillator linearly coupled to a heat bath, and in the presence of a constant homogeneous magnetic field. We have started off by examining some general properties of the generalized susceptibility tensor of the dynamical system involved, which in turn have enabled us to reach two general conclusions about the equal-time position autocorrelation functions (also called dispersions) of the magnetic system in an arbitrary heat bath. In addition to the transversal dispersions of a charged quantum particle, the free energy of such a system can also be shown to decrease monotonically with increasing intensity of the magnetic field, hence indicating the diamagnetism of the system even in the presence of a physical heat bath. The generality of these theorems stems from the fact that, because of the neutrality of the heat-bath oscillators implied in the underlying independent-oscillators model [9], the magnetic field enters into the GLE only through the Lorentz-force term so that the external field and the dissipation do not affect each other. It may be of interest to note in passing a similar theorem on the magnetoconductivity of metals that states under rather general assumptions that if an external magnetic field has no bearing on scattering mechanisms, then the electric conductivity of metals is a monotonically nonincreasing function of the magnitude of the magnetic field [12].

We have calculated explicitly the equal-time position autocorrelation functions, in the presence of a magnetic field $\vec{B}$, for a charged quantum harmonic oscillator in the Ohmic heat bath. The motion along $\vec{B}$ is unaffected by it, as expected, but the motion perpendicular to it displays, at zero temperature, an interesting phenomenon due to interplay between the dissipation and the magnetic field $\vec{B}$. For weak dissipation, the effect of a magnetic field opposes that of the dissipation. For a $\vec{B}$ field less than a certain critical value, the dissipation effect still dominates over the magnetic-field effect, resulting in a localization weakened by $\vec{B}$ for motion normal to it. However, for a magnetic field larger than this critical value, weak dissipation is simply overwhelmed by the magnetic field, causing an overall reduction in the transversal localization of the particle. Hence the overall shape of the orbit of the oscillator looks somewhat like an oblate ellipsoid with the magnetic field along its symmetry axis. Only in the strong dissipation regime does the magnetic field reinforce the effect of dissipation, leading to stronger localization in the direction orthogonal to the field, and thus the corresponding orbital shape of the oscillator would look more like a football, a symmetric ellipsoid elongated along the direction of the magnetic field.

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APPENDIX A

Since $1/\lambda \equiv \alpha^{(0)}(\omega)$ is simply the generalized susceptibility for a one-dimensional oscillator, $-iz/\lambda(z) = -iz\alpha^{(0)}(z)$ is a positive real function for $K>0$ [13], and thus its real part everywhere in the UHP is positive [9],

$$\text{Re}[-iz/\lambda(z)]>0 \quad \text{for } \text{Im}z>0\ . \tag{A1}$$

Let us now suppose that

$$\lambda(z) = \pm m\omega_c z \tag{A2}$$

for some z in the UHP. Then we would get

$$-iz/\lambda(z) = \mp i/m\omega_c\ , \tag{A3}$$

which contradicts (A1). Therefore (A2) has no roots in the UHP. It follows that $\alpha_{\rho\sigma}(\omega)$, given by (2.2), has no poles in the UHP.

APPENDIX B

To prove (2.6), we need first to establish the following two useful identities of $\alpha_{\rho\sigma}(\omega)$:

$$\alpha_{\nu\mu}(\omega)-\alpha^*_{\mu\nu}(\omega)=2i\alpha_{\sigma\mu}(\omega)\alpha^*_{\sigma\nu}(\omega)\omega\,\mathrm{Re}\tilde{\mu}(\omega) \qquad \text{(B1)}$$

and

$$\alpha_{\nu\mu}(\omega)-\alpha^*_{\mu\nu}(\omega)=2i\alpha_{\nu\sigma}(\omega)\alpha^*_{\mu\sigma}(\omega)\omega\,\mathrm{Re}\tilde{\mu}(\omega)\ . \qquad \text{(B2)}$$

For this, let us introduce the inverse matrix, denoted by $D_{\rho\sigma}(\omega)$, of $\alpha_{\rho\sigma}(\omega)$ [8]:

$$D_{\rho\sigma}(\omega)=\lambda\delta_{\rho\sigma}+i\frac{e}{c}\omega\varepsilon_{\rho\sigma\eta}B_\eta\ , \qquad \text{(B3)}$$

which is related to $\alpha_{\rho\sigma}(\omega)$ by definition through the equations

$$D_{\rho\eta}(\omega)\alpha_{\eta\sigma}(\omega)=\delta_{\rho\sigma} \qquad \text{(B4)}$$

and

$$\alpha_{\rho\eta}(\omega)D_{\eta\sigma}(\omega)=\delta_{\rho\sigma}\ , \qquad \text{(B5)}$$

where the Kronecker delta function $\delta_{\rho\sigma}$ is unity for $\rho=\sigma$, and zero otherwise.

From (B3) and (2.4), we have

$$D^*_{\rho\sigma}(\omega)-D_{\sigma\rho}(\omega)=2i\delta_{\rho\sigma}\omega\,\mathrm{Re}\tilde{\mu}(\omega)\ . \qquad \text{(B6)}$$

Multiplying (B6) by $\alpha_{\rho\mu}(\omega)\alpha^*_{\sigma\nu}(\omega)$ and using (B4), we obtain (B1) and, similarly, (B2) by multiplying (B6) by $\alpha_{\nu\sigma}(\omega)\alpha^*_{\mu\rho}(\omega)$ with the aid of (B5).

Now we turn to the proof of (2.6) itself by calculating the work done by an external, c-number force $\vec{f}$ (aside from the magnetic field) in a complete cycle on an otherwise isolated system [9]

$$\begin{aligned} W&=\int_{-\infty}^{\infty}dt f_\rho(t)\langle v_\rho(t)\rangle \\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\tilde{f}_\rho(\omega)\langle\tilde{v}_\rho(-\omega)\rangle\ , \end{aligned} \qquad \text{(B7)}$$

where the second equality is obtained by using the Parseval theorem [14] and where $v_\rho(t)$ is the velocity operator of the particle, and $\vec{f}(t)$ is assumed to be arbitrary except for the requirement that it vanish at both the distant past and the distant future. Here tilde denotes the Fourier transform as usual, e.g.,

$$\tilde{v}_\rho(\omega)=\int_{-\infty}^{\infty}dt e^{i\omega t}v_\rho(t)\ . \qquad \text{(B8)}$$

From (B8) and $v_\rho(t)\equiv\dot{r}_\rho(t)$, one can easily see that

$$\tilde{v}_\rho(\omega)=-i\omega\tilde{r}_\rho(\omega)\ , \qquad \text{(B9)}$$

where the displacement $\tilde{r}_\rho(\omega)$ is related to the external force $\tilde{f}_\sigma(\omega)$ and the fluctuation force $\tilde{F}_\sigma(\omega)$ via the solution of the GLE [8]:

$$\tilde{r}_\rho(\omega)=\alpha_{\rho\sigma}(\omega)[\tilde{f}_\sigma(\omega)+\tilde{F}_\sigma(\omega)]\ . \qquad \text{(B10)}$$

Putting (B9) and (B10) in (B7) and averaging out the random force $\tilde{F}$ gives

$$W=\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega\omega\alpha^*_{\mu\nu}(\omega)\tilde{f}_\mu(\omega)\tilde{f}^*_\nu(\omega)\ , \qquad \text{(B11)}$$

where we have used the reality condition on $\tilde{v}_\rho(\omega)$: $\tilde{v}_\rho(-\omega)=\tilde{v}^*_\rho(\omega)$. Forming complex conjugate of (B11) and interchanging the dummy indices μ and ν, one then finds

$$W=-\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega\omega\alpha_{\nu\mu}(\omega)\tilde{f}_\mu(\omega)\tilde{f}^*_\nu(\omega)\ . \qquad \text{(B12)}$$

Assembling (B11), (B12), (B1), (B9), and (B10), one finally obtains

$$\begin{aligned} W&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\omega\frac{1}{2i}[\alpha_{\nu\mu}(\omega)-\alpha^*_{\mu\nu}(\omega)]f_\mu(\omega)\tilde{f}^*_\nu(\omega) \\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\omega^2\mathrm{Re}[\tilde{\mu}(\omega)]\sum_\sigma|\alpha_{\sigma\mu}(\omega)\tilde{f}_\mu(\omega)|^2 \\ &=\frac{1}{\pi}\int_0^{\infty}d\omega\,\mathrm{Re}[\tilde{\mu}(\omega)]\sum_\sigma|\langle\tilde{v}_\sigma(\omega)\rangle|^2\ , \end{aligned} \qquad \text{(B13)}$$

which is positive as demanded by the second law of thermodynamics.

Equation (B11) may also be written as

$$\begin{aligned} W=\frac{1}{\pi}\int_0^{\infty}d\omega\omega\{&\mathrm{Im}\alpha_{\mu\nu}(\omega)\mathrm{Re}[\tilde{f}_\mu(\omega)\tilde{f}^*_\nu(\omega)] \\ &-\mathrm{Re}\alpha_{\mu\nu}(\omega)\mathrm{Im}[\tilde{f}_\mu(\omega)\tilde{f}^*_\nu(\omega)]\}\ , \end{aligned} \qquad \text{(B14)}$$

where we have used the fact that, due to the reality conditions on $\alpha_{\mu\nu}(\omega)$ and $\tilde{f}_\mu(\omega)$, $\mathrm{Re}\alpha_{\mu\nu}(\omega)$ and $\mathrm{Im}\alpha_{\mu\nu}(\omega)$, as well as $\mathrm{Re}\tilde{f}_\mu(\omega)$ and $\mathrm{Im}\tilde{f}_\mu(\omega)$, are even and odd functions of ω, respectively. Since $\tilde{f}_\mu(\omega)$ are arbitrary other than the boundary conditions $\lim_{\omega\to\pm\infty}\tilde{f}_\mu(\omega)=0$, $\tilde{f}_\mu(\omega)$ $(\mu=1,2,3)$ may well be chosen all real (and thus even functions of ω). Then the integrand in (B14), according to (B13), must be positive for all ω,

$$\mathrm{Im}[\alpha_{\mu\nu}(\omega)\tilde{f}_\mu(\omega)\tilde{f}_\nu(\omega)]=\mathrm{Im}[\alpha^s_{\mu\nu}(\omega)\tilde{f}_\mu(\omega)\tilde{f}_\nu(\omega)]>0 \quad \text{for } \omega>0\ . \qquad \text{(B15)}$$

Hence, $\mathrm{Im}\alpha^s_{\mu\nu}(\omega)$ must be a positive definite matrix for all $\omega>0$, and (2.6) readily follows as a corollary.

APPENDIX C

The free energy of a charged quantum oscillator linearly coupled to a neutral heat bath, and in a magnetic field, defined as the free energy of the composite system of the oscillator interacting with the heat bath minus that of the bath itself, assumes the form [8]

$$F_O(T,B)=\frac{1}{\pi}\int_0^{\infty}d\omega f(\omega,T)\mathrm{Im}\left\{\frac{d}{d\omega}\ln[\det\alpha(\omega+i0^+)]\right\}\ , \qquad \text{(C1)}$$

where $f(\omega,T)$ is the free energy (including zero-point energy) of a free oscillator of frequency ω:

$$f(\omega,T=kT\ln[2\sinh(\hbar\omega/2kT)]\ , \qquad \text{(C2)}$$

and where

$$\det\alpha(\omega)=\left\{\lambda\left[\lambda^2-\left[\omega\frac{e}{c}\right]^2\vec{B}^2\right]\right\}^{-1} \qquad \text{(C3)}$$

is the determinant of the matrix $\alpha_{\rho\sigma}(\omega)$ given in (2.2) [8].

Since the heat bath is neutral, the magnetic moment M of the charged oscillator is related to the free energy $F_O(T,B)$ through the equation [15]

$$M=-\frac{\partial F_O}{\partial B} . \tag{C4}$$

Substituting (C1)–(C3) in (C4) and integrating by parts once yields

$$\begin{aligned} M&=B\frac{\hbar e^2}{\pi c^2}\int_0^\infty d\omega\omega^2\coth\left[\frac{\hbar\omega}{2kT}\right]\mathrm{Im}[\lambda^2-(e/c)^2B^2\omega^2]^{-1} \\ &=B\left[\frac{e}{c}\right]^2\frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}d\omega\omega^2\coth\left[\frac{\hbar\omega}{2kT}\right] \\ &\quad\times[\lambda^2-(e/c)^2B^2\omega^2]^{-1} , \end{aligned} \tag{C5}$$

where we have used in the last line the reality condition on the quantity in the brackets. Before we move on, it would be of interest to check the classical limit of (C5). Expanding $\coth(\hbar\omega/2kT)$ for small $\hbar$ and exploiting the analyticity of the integrand in the UHP (see Appendix A), we get

$$M=B\left[\frac{e}{c}\right]^2\frac{kT}{\pi i}\int_{-\infty}^{\infty}d\omega\omega[\lambda^2-(e/c)^2B^2\omega^2]^{-1}=0 , \tag{C6}$$

which is expected on account of the quantum nature of magnetism (the Bohr–van Leeuwen theorem) [16].

The integration in (C5) may be performed by closing the contour in the UHP and by using the partial fractional expansion of $\coth(z)$ [17]

$$\coth(z)=\sum_{n=-\infty}^{\infty}\frac{1}{z+in\pi} . \tag{C7}$$

The resulting serial expression for M is

$$M=-2kTB\left[\frac{e}{mc}\right]^2\sum_{n=1}^{\infty}\frac{\nu_n^2}{\hat{\lambda}^2(\nu_n)+(\nu_n\omega_c)^2}<0 , \tag{C8}$$

where $\nu_n=2\pi kTn/\hbar$ are again the Matsubara frequencies. Hence, the magnetic moment due to the orbital motion of a charged oscillator is still diamagnetic, unaltered by the presence of an arbitrary heat bath. The same holds for a charged Brownian particle as one takes the limit $\omega_0^2\to 0$ in (C8).

For an Ohmic heat bath at zero temperature, the magnetic moment of a charged oscillator can be calculated explicitly [8]:

$$\begin{aligned} M=-\frac{\hbar e^2B}{2\pi m^2c^2B}\Bigg\{ &\frac{\frac{\gamma^2}{4}+\left[\frac{b+a}{2}\right]}{\left[\frac{b+a}{2}\right]^{1/2}}\tan^{-1}\left[\frac{2}{\gamma}\left[\frac{b+a}{2}\right]^{1/2}\right] \\ &-\frac{\frac{\gamma^2}{4}-\left[\frac{b-a}{2}\right]}{2\left[\frac{b-a}{2}\right]^{1/2}}\ln\left[\frac{\frac{\gamma}{2}+\left[\frac{b-a}{2}\right]^{1/2}}{\frac{\gamma}{2}-\left[\frac{b-a}{2}\right]^{1/2}}\right]\Bigg\} , \end{aligned} \tag{C9}$$

where the quantity within the braces is positive [see Eqs. (4.8) and (4.9) of Ref. [8]]. For a charged Brownian particle, this reduces in the limit $\omega_0^2\to 0$ to

$$M=-\frac{\hbar e}{\pi mc}\tan^{-1}\left[\frac{\omega_c}{\gamma}\right] . \tag{C10}$$

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