Intermittent chaos and multifractal systems

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On the basis of statistical thermodynamics we represent the free energy and the partition function of intermittent multifractal systems. It is pointed out that even in the case of the chaotic system in which the value of minimum free energy cannot be determined because of thermal fluctuations, we can obtain the partition function by using the effective free energy defined in the dominant region of power law indices. We analyze the one-scale Cantor set in the fluctuation mode to show a typical example of this system.

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I. INTRODUCTION

Intermittent phenomena have been investigated to make clear critical phenomena of nonlinear systems [1–5]. Intermittency is characterized by the singular power law behavior of the moment with respect to the event probabilities. Since the index of the moment plays the role of the critical exponent, intermittent phenomena are considered to manifest the characteristic features of critical phenomena of nonlinear systems.

Intermittent phenomena are well accounted for by the fractal geometry based on the properties of self-similarity with the constant scaling index. As for nonlinear chaotic systems, however, the multifractal theory is used instead of the fractal geometry because of fluctuations of scaling indices α. Hitherto, several authors [6–9] have explored intermittent phenomena of nonlinear systems, taking account of the multifractal structure of these systems.

In the multifractal analysis [6] the qth moment of the event probabilities $Z(q)$ is assumed to be given by $Z(q) = \lambda^{f(\alpha)}$, where $f(\alpha) = q\alpha - D(\alpha)$ and $D(\alpha)$ is the fractal dimension, according to the power law behavior of $Z(q)$ in terms of the size of subpieces λ. This formula is obtained on the assumption that the dominant value of $Z(q)$ must be determined by the minimum value of $f(\alpha)$. However, in nonlinear chaotic systems generally we cannot determine the minimum value of $f(\alpha)$ definitely.

In the formulas of statistical thermodynamics $Z(q)$ and $f(\alpha)$ play the role of the partition function and the free energy, respectively. Because of fluctuations of the free energy in nonlinear open systems, its lowest value cannot be determined uniquely. This is the same condition that in the field theory [10] the energy density of the quantum vacuum fluctuates according to the vacuum states.

The aim of this paper is to propose a method to obtain a useful formula of the qth moment of multifractal systems in analogy with statistical thermodynamics. In this formula the fluctuation effect of the free energy is taken into consideration, and hence we can get the available partition function. The partition function, i.e., the qth moment, is given by the effective values of $\lambda^{f(\alpha)}$ in the dominant region of α.

In the next section, we will present a statistical-thermodynamical formalism of multifractal systems, in which the partition function and the free energy are defined in a convenient way. In Sec. III, we show the way to calculate the qth moment of chaotic multifractal systems in contrast with the method based on the assumption of the minimum free energy condition. Section III is our conclusion.

II. STATISTICAL-THERMODYNAMICAL FORMALISM OF MULTIFRACTAL SYSTEMS

In multifractal systems the event probability p_i in the region of λ_i labeled by $i (i = 1, 2, \ldots, n)$ is given by

$$p_i \sim \lambda_i^{\alpha}, \tag{2.1}$$

like the order parameter near the critical point with the critical parameter α. It is not necessary that α should be constant in these systems. In terms of p_i we define the qth moment as

$$Z(q) = \sum_{i=1}^{n} p_i^q. \tag{2.2}$$

In the limit of $n \to \infty$, we rewrite the summation of Eq. (2.2) into integration with respect to α as follows:

$$Z_n(q) = \int d\alpha \eta(\alpha) \rho(\lambda_n) \lambda_n^{q\alpha}, \tag{2.3}$$

where $\rho(\lambda_n) \sim \lambda_n^{-D(\alpha)}$ is the density of the number of subpieces in the fractal dimensionality $D(\alpha)$, and the weight function $\eta(\alpha)$ is normalized as $\int d\alpha \eta(\alpha) = 1$.

In the case where $\eta(\alpha)$ is the δ function at $\alpha = \alpha_c$, Eq. (2.3) yields

$$Z_n(q) = \lambda_n^{q\alpha_c - D(\alpha_c)} = \lambda_n^{f(q)}, \tag{2.4}$$

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where we put $f(q) = q\alpha_c - D(\alpha_c)$. If we set $\lambda_n = \exp(-n)$, we obtain from Eq. (2.4)

$$Z_n(q) = Z(q)^n = \exp[-f(q)\,n]. \tag{2.5}$$

This is the same partition function that is assumed in Ref. [9] by using Renyi entropy [11]. It is reasonable to identify $f(q)$ with the free energy per site. The partition function of this system is given by $Z(q) = \exp(-f)$ in the unit of $kT = 1$, where T is temperature and k is the Boltzmann constant.

The convolution relation of $Z(q)$ [12] yields the configuration sum in terms of n as follows:

$$Z_{\langle n\rangle}(q) = \sum_n P(n, \langle n\rangle)\, Z(q)^n, \tag{2.6}$$

where $Z_{\langle n\rangle}(q)$ is the partition function of the whole system and $P(n, \langle n\rangle)$ is the normalized weight function. In the thermodynamic limit of the infinite average value of n, i.e., $\langle n\rangle \to \infty$, the δ function $P(n, \langle n\rangle) = \delta(n/\langle n\rangle - 1)$ provides

$$Z_{\langle n\rangle}(q) = Z(q)^{\langle n\rangle} = \exp(-f(q)\langle n\rangle). \tag{2.7}$$

Therefore, we get the free energy of the whole system such as

$$F(q, \langle n\rangle) = -\ln Z_{\langle n\rangle}(q) = \langle n\rangle f(q). \tag{2.8}$$

The proportional relation between $F(q, \langle n\rangle)$ and $f(q)$ represents self-similarity of multifractal systems at $\alpha = \alpha_c$.

III. THE qTH MOMENT OF CHAOTIC MULTIFRACTAL SYSTEMS

Various soluble strange sets have been investigated [6, 13] to comprehend dynamical structure of multifractal systems under the minimum free energy condition at $\alpha = \alpha_c$ in the limit of $\lambda_n \to 0$ as

$$\left.\frac{df(\alpha)}{d\alpha}\right|_{\alpha=\alpha_c} = 0. \tag{3.1}$$

It turns out from Eqs. (3.1) and (2.3) that we have

$$Z_n(q) = \lambda_n^{f(\alpha_c)}. \tag{3.2}$$

This is the same partition function as Eq. (2.4), which is obtained by using the δ function for weight density $\eta(\alpha)$ at $\alpha = \alpha_c$ such as $\eta(\alpha) = \delta(\alpha - \alpha_c)$.

Since $f(\alpha) = q\alpha - D(\alpha)$, at $\alpha = \alpha_c$ Eq. (3.1) leads to

$$D(\alpha) = q\alpha - C(q), \tag{3.3}$$

where $f(\alpha(q)) = C(q) = (q-1)D_q$ because of $f(\alpha) = 0$, i.e., $Z_n(q) = 1$ at $q = 1$. Equations (3.2) and (2.2) provide

$$\frac{dZ_n(q)}{dq} = \sum_i p_i^q \ln p_i = \left\{\frac{df(\alpha_c(q))}{dq}\right\} \lambda_n^{f(\alpha_c(q))} \ln\lambda_n. \tag{3.4}$$

Therefore, we get

$$\alpha_c(q) = \frac{\sum_i p_i^q \ln p_i}{\lambda_n^{f(\alpha_c(q))}\ln\lambda_n}, \tag{3.5}$$

because we have

$$\alpha_c(q) = \frac{d}{dq}\{(q-1)D_q\} = \frac{df(\alpha_c(q))}{dq}. \tag{3.6}$$

As for the case where $\eta(\alpha)$ is dominant in the region of $\Delta\alpha = \alpha_2 - \alpha_1$, where the minimum free energy condition is not valid because of free energy fluctuations, we set [14] in comparison with the δ function for weight density $\eta(\alpha)$ as follows:

$$\eta(\alpha) = \frac{f'(\alpha)}{f(\alpha_2) - f(\alpha_1)}, \tag{3.7}$$

where $f'(\alpha) = df(\alpha)/d\alpha$ and $\eta(\alpha)$ is normalized as $\int_{\alpha_1}^{\alpha_2} d\alpha\,\eta(\alpha) = 1$. It follows from Eqs. (3.7) and (2.3) that the partition function is given by

$$Z_n(q) = \frac{\lambda_n^{f(\alpha_2)} - \lambda_n^{f(\alpha_1)}}{\ln\lambda_n\,[f(\alpha_2) - f(\alpha_1)]}. \tag{3.8}$$

If we put $\alpha_2 = \alpha_c + (\Delta\alpha/2)$ and $\alpha_1 = \alpha_c - (\Delta\alpha/2)$, where $0 < \Delta\alpha < 1$, Eq. (3.8) yields

$$Z_n(q) \sim \lambda_n^{f(\alpha_c)} \left\{\frac{\lambda_n^{\Delta\alpha f'(\alpha_c)/2} - \lambda_n^{-\Delta\alpha f'(\alpha_c)/2}}{\Delta\alpha f'(\alpha_c)\ln\lambda_n}\right\}$$
$$\sim \lambda_n^{f(\alpha_c)}\left\{1 + \frac{1}{24}\left(\Delta\alpha f'(\alpha_c)\ln\lambda_n\right)^2\right\}. \tag{3.9}$$

Equation (3.9) leads to $Z_n \sim \lambda_n^{f(\alpha_c)}$ for $\Delta\alpha \to 0$ as in Eq. (3.2).

In order to comprehend the validity of the weight density (3.7), we analyze the one-scale Cantor set in the fluctuation mode as a simple example of multifractal systems. In the process of dividing the interval $[0,1]$ into small segments, we get this Cantor set. The size of each segment λ_i $(i = 1, 2, \ldots, n)$ is assumed to fluctuate in the region of $\Lambda_1 < \lambda_1 < \Lambda_2$ so as to hold the weight density $\eta(\alpha)$ as in Eq. (3.7) with parameters $\alpha_1 = \ln p_1/\ln\Lambda_1$ and $\alpha_2 = \ln p_2/\ln\Lambda_2$. The fluctuation effect is assumed to decrease in accordance with the progress of construction of this Cantor set.

In this example two values of free energy $f(\alpha)$ with respect to two boundary indices α_1 and α_2 are given by

$$f(\alpha_1) = (q-1)D_q(\alpha_1) = q\alpha_1 - D(\alpha_1),$$
$$f(\alpha_2) = (q-1)D_q(\alpha_2) = q\alpha_2 - D(\alpha_2). \tag{3.10}$$

It turns out from Eq. (3.10) that at $q = 0$, $D_q(\alpha_1)$, and $D_q(\alpha_2)$ are written by the fractal dimension $D(\alpha_1)$ and $D(\alpha_2)$, respectively, like

$$D_0(\alpha_1) = D(\alpha_1) \quad\text{and}\quad D_0(\alpha_2) = D(\alpha_2). \tag{3.11}$$

In this Cantor set we have different values $D_0(\alpha_1)$ and $D_0(\alpha_2)$ at $q = 0$ in contrast with the ordinary example

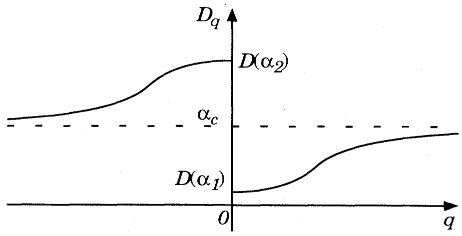

FIG. 1. Plot of $D_q = f/(q-1)$ vs q. D_q approaches α_c in accordance with increasing the absolute value of q.

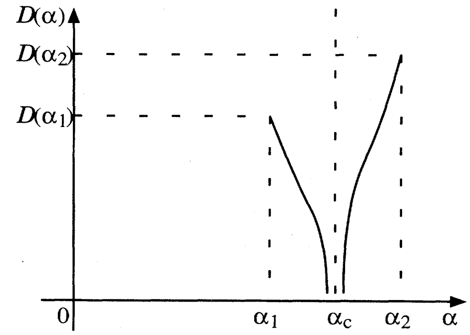

FIG. 2. Plot of the fractal dimension $D(\alpha)$ vs α. Solid lines show that D_q approaches to α_c in accordance with decreasing the fractal dimension $D(\alpha)$.

in which the minimum free energy condition is assumed.

In the case of $p_1 + p_2 = 1$, this Cantor set is considered a two-scale Cantor set with two rescaling parameters Λ_1 and Λ_2. However, in our example, since the fluctuation effect decreases according to advancing construction of segments, we obtain $D_0(\alpha_1) \neq D_0(\alpha_2)$ and $D_\infty = D_{-\infty} = \alpha_c$ as shown in Fig. 1, which is remarkably different from the ordinary two-scale Cantor set. We present the characteristic feature of the fractal dimension $D(\alpha)$ of our example in Fig. 2.

IV. CONCLUSIONS

We have shown a method to make correspondence between the qth moment of chaotic multifractal systems and the partition function of statistical-thermodynamical systems. In this method $f(\alpha) = q\alpha - D(\alpha)$, where α is the power law index and $D(\alpha)$ is the multifractal dimension, plays the role of the free energy. In contrast with the ordinary partition function, which is given by the assumption that the system has the unique minimum free energy value, we have obtained the partition function in the dominant α density region in the case where the minimum free energy fluctuates. To show a typical example we have presented the characteristic feature of the one-scale Cantor set in the fluctuation mode.

Our formulas should be useful especially for nonlinear chaotic multifractal systems, which dominate in the limited α density region $\Delta\alpha$. If $\Delta\alpha < 1$, we can comprehend that these systems behave like fractal systems even if the minimum free energy condition is not valid.

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[14] This is reasonable extension of the δ function, since the mean value theorem gives $\eta(\alpha) = 1/\Delta\alpha = 1/(\alpha_2 - \alpha_1)$ which becomes the δ function in the limit of $\Delta\alpha \to 0$.