Kinetics of multidimensional fragmentation

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We present two classes of exact solutions to a geometric model which describes the kinetics of fragmentation of d-dimensional hypercuboid-shaped objects. The first class of exact solutions is described by a fragmentation rate $a(x_1,...,x_d)=1$ and daughter distribution function $b(x_1,...,x_d|x'_1,...,x'_d)=(\alpha_1+2)\cdots(\alpha_d+2)$ $x_1^{\alpha_1}\cdots x_d^{\alpha_d}/x_1'^{(\alpha_1+1)}\cdots x_d'^{(\alpha_d+1)}$. The second class of exact solutions is described by a fragmentation rate $a(x_1,\ldots,x_d)=x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}/2^d$ and daughter distribution function $b(x_1,\ldots,x_d|x'_1,\ldots,x'_d)=2^d\delta(x_1-x'_1/2)\cdots\delta(x_d-x'_d/2)$. Each class of exact solutions is analyzed in detail for the presence of scaling solutions and the occurrence of shattering transitions; the results of these analyses are also presented.

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I. INTRODUCTION

Fragmentation occurs in numerous important physical, chemical, and geological processes. These include droplet breakup [1] and fiber length reduction [2]; depolymerization through shear action [3,4], chemical attack [5], and exposure to nuclear, ultraviolet, and ultrasonic radiation [6,7]; and rock crushing and grinding (communition) [2]. Theoretical predictions of the evolution with the time of the probability distribution functions of the fragmenting objects during such processes is of great interest and importance. There are essentially two approaches in use for determining the evolution in time of the object probability distributions as a function of the initial conditions and the fragmentation rates. The first approach relies upon statistical and combinatorial arguments [8–10]. The second approach has been through the analysis of the kinetic equation modeling the fragmentation process [11–13].

In the kinetic equation approach the fragmentation process can be described by the evolution in time of the probability distribution function $c(x,t)$, where x is the size of the fragments and t is the time, through a kinetic equation. This theoretical approach is of a mean field nature since fluctuations are ignored. Fragments are assumed to be distributed homogeneously at all times throughout the system, i.e., there is perfect mixing and the shape of the fragments is ignored. Consequently, the size of the fragments is the only dynamical variable that characterizes a fragment in the kinetic equation approach. A number of authors have expended much effort in finding exact solutions to the kinetic equation, in order to study specific practical problems and to provide a greater understanding of the behavior of physical, chemical, and geological systems in which fragmentation occurs [5,14–21]. Although the basic kinetic equations are linear, and in principle soluble, the number of exact solutions is few, mainly because of the nonlocal structure of the kinetic equations.

Of considerable importance are the scaling solutions. These are essentially the solutions in the long-time ($t\rightarrow\infty$), small-size ($x\rightarrow 0$) limit where the probability distribution function $c(x,t)$ evolves to a simpler universal form. This form is universal in the sense that it does not depend on the initial conditions. Most experimental systems evolve to the point where this behavior is reached. Scaling theories based on a linear kinetic equation have been derived for a large class of models which undergo fragmentation [18,22–24].

The time evolution of the fragmentation process depends qualitatively on the behavior of the probability of the breakup for the fragments. For breakup rates increasing sufficiently quickly with decreasing size (or mass), a cascading breakup occurs in which a finite part of the total size (or mass) is transferred to fragments of zero or infinitesimal size (or mass). This so-called "shattering" [17,25] or "disintegration" [26] phenomenon is accompanied by a violation of the usual dynamical scaling as well as violation of size (or mass) conservation. This shattering regime is also characterized by the presence of large fluctuations and the absence of self-averaging.

The general form of the one-dimensional multiple fragmentation equation is given by

$$\frac{\partial c(x,t)}{\partial t}=-a(x)c(x,t)+\int_x^\infty dx'a(x')b(x|x')c(x',t), \tag{1}$$

where $a(x)$ is the rate of fragmentation of objects of size x, the daughter distribution function $b(x|x')$ is the average number of objects of size x produced when an object of size x' breaks up, and $c(x,t)$ is the probability distribution function of objects of size x at time t. To ensure that the size (or mass) of the fragmenting objects is conserved per fragmentation event, we insist that

$$x=\int_0^x dx'x'b(x'|x) \tag{2}$$

holds. The average number of objects produced when an object of size x fragments is given by

$$\langle N(x)\rangle=\int_0^x dx' b(x'|x). \tag{3}$$

On physical grounds we must have $\langle N(x)\rangle \geq 2$, which, together with (2) and (3), places constraints on the possible choices for $b(x|x')$ available to us.

In the special case of binary fragmentation where two objects are produced per fragmentation event, (1)–(3) can be rewritten in terms of the single symmetric function $F(x,x')=F(x',x)$ as follows. First, write

$$a(x)=\int_0^x dx' F(x-x',x'). \tag{4}$$

Then to ensure that there are precisely two objects produced per fragmentation event, choose

$$b(x|x')=\frac{2F(x,x'-x)}{a(x')}, \tag{5}$$

in which case (1) becomes

$$\frac{\partial c(x,t)}{\partial t}=-c(x,t)\int_0^x dx' F(x-x',x') +2\int_x^\infty dx' F(x,x'-x)c(x',t), \tag{6}$$

where $F(x,x')$ describes the rate at which objects of size $(x+x')$ fragments into objects of size x and x'. As mentioned above, the kinetics of such one-dimensional fragmentation processes is now well understood with numerous explicit exact solutions, scaling solutions, and quantitative descriptions of shattering transitions known to us.

In realistic fragmentation processes objects have both size and shape, and it is clear that the geometry of these fragmenting objects will influence the fragmentation process. For example, an object may be selected for fragmentation at a rate which is dependent on its area or volume, but the manner in which the fragmentation of the object is implemented will, in general, depend on its precise dimensions. If two objects have the same area but one is needle shaped and the other is square shaped, they may be equally likely to fragment as a consequence of their possessing the same area, but the needle-shaped object is much more likely to fragment across its longer side, whereas the square-shaped object is equally likely to fragment across either side. Until recently, all these properties were represented by a single parameter, namely the size (or mass) of the fragmenting object.

Recently, various authors [27–29] introduced and investigated simple kinetic models describing the fragmentation of two-dimensional, and more generally d-dimensional objects. These authors present several simple classes of explicit exact solutions for two-dimensional models. Krapivsky and Ben-Naim [28] also discuss the presence of scaling and multiscaling in their models of fragmentation for d-dimensional objects. In Ref. [29], the shattering transition in a two-variable fragmentation model is investigated.

In this paper we present two classes of exact solutions to a geometric model which describes the kinetics of fragmentation of d-dimensional hypercuboid-shaped objects, for general d. Each class of exact solutions is analyzed in detail for the presence of scaling solutions, and the occurrence of shattering transitions.

II. FRAGMENTATION IN d DIMENSIONS

The general form of the d-dimensional multiple fragmentation equation is given by [27]

$$\begin{aligned}\frac{\partial c(x_1,\ldots,x_d,t)}{\partial t}=&-a(x_1,\ldots,x_d)c(x_1,\ldots,x_d,t)\\&+\int_{x_1}^\infty dx_1'\cdots\int_{x_d}^\infty dx_d' a(x_1',\ldots,x_d')\\&\times b(x_1,\ldots,x_d|x_1',\ldots,x_d')c(x_1',\ldots,x_d' t)\end{aligned} \tag{7}$$

where $c(x_1,\ldots,x_d,t)$ is the probability distribution function of a d-dimensional hypercuboid-shaped object of hypervolume $x_1\ldots x_d$ characterized by $(x_1,\ldots,x_d)$ at time t, $a(x_1,\ldots,x_d)$ is the rate at which an object characterized by $(x_1,\ldots,x_d)$ fragments, and the daughter distribution function $b(x_1,\ldots,x_d|x_1',\ldots,x_d')$ is the rate at which an object characterized by $(x_1,\ldots,x_d)$ is produced from an object characterized by $(x_1',\ldots,x_d')$.

Consequently, the average number of objects produced per fragmentation event is

$$\langle N(x_1,\ldots,x_d)\rangle=\int_0^{x_1}dx_1'\cdots\int_0^{x_d}dx_d' b(x_1',\ldots,x_d'|x_1,\ldots,x_d), \tag{8}$$

and hypervolume conservation per fragmentation event requires

$$x_1\ldots x_d=\int_0^{x_1}dx_1'\cdots\int_0^{x_d}dx_d' x_1'\cdots x_d' b(x_1',\ldots,x_d'|x_1,\ldots,x_d). \tag{9}$$

Equations (7)–(9) form a complete set of equations, which define the fragmentation process given $a(x_1,\ldots,x_d)$, $b(x_1,\ldots,x_d|x_1',\ldots,x_d')$, and suitable initial conditions. Of course, the functions $a(x_1,\ldots,x_d)$ and $b(x_1,\ldots,x_d|x_1',\ldots,x_d')$ must be chosen to make the equations physically meaningful, and furthermore $b(x_1,\ldots,x_d|x_1',\ldots,x_d')$ must be chosen to ensure that (8) and (9) hold, as well as the obvious physical constraint $\langle N(x_1,\ldots,x_d)\rangle\geq 2$.

As a special case, consider fragmentation processes such that a given d-dimensional hypercuboid-shaped object fragments into 2^d pieces per fragmentation event. In this case, as in Sec. I above, we may rewrite (7)–(9) in terms of a single function $F(x_1,x_1';\ldots;x_d,x_d')$ as follows. First, choose

$$a(x_1,\ldots,x_d)=\int_0^{x_1}dx_1'\cdots\int_0^{x_d}dx_d' \times F(x_1-x_1',x_1';\ldots;x_d-x_d',x_d'), \tag{10}$$

where $F(x_1,x'_1;\dots;x_d,x'_d)$ is the rate of fragmentation of an object characterized by $[(x_1+x'_1),\dots,(x_d+x'_d)]$ into 2^d smaller objects characterized by $(x_1,\dots,x_d),\dots,(x'_1,\dots,x'_d)$. Now choose

$$b(x_1,\dots,x_d|x'_1,\dots,x'_d)=\frac{2^dF(x_1,x'_1-x_1;\dots;x_d,x'_d-x_d)}{a(x'_1,\dots,x'_d)}, \tag{11}$$

where, of course,

$$F(x_1,x'_1;\dots;x_k,x'_k;\dots;x_d,x'_d)=F(x_1,x'_1;\dots;x'_k,x_k;\dots;x_d,x'_d), \tag{12}$$

i.e., $F(x_1,x'_1;\dots;x_k,x'_k;\dots;x_d,x'_d)$ is symmetric in all pairs of arguments (x_k,x'_k) for $k=1,\dots,d$.

It is easily shown that $\langle N(x_1,\dots,x_d)\rangle=2^d$, as required, and that hypervolume conservation per fragmentation event requires

$$x_1\dots x_d=\frac{2^d\int_0^{x_1}dx'_1\cdots\int_0^{x_d}dx'_dx'_1\cdots x'_dF(x_1-x'_1,x'_1;\dots;x_d-x'_d,x'_d)}{\int_0^{x_1}dx'_1\cdots\int_0^{x_d}dx'_dF(x_1-x'_1,x'_1;\dots;x_d-x'_d,x'_d)}. \tag{13}$$

The fragmentation equation (7) now becomes

$$\begin{aligned}\frac{\partial c(x_1,\dots,x_d,t)}{\partial t}=&-c(x_1,\dots,x_d,t)\int_0^{x_1}dx'_1\cdots\int_0^{x_d}dx'_dF(x_1-x'_1,x'_1;\dots;x_d-x'_d,x'_d)\\&+2^d\int_{x_1}^{\infty}dx'_1\cdots\int_{x_d}^{\infty}dx'_dF(x'_1-x_1,x_1;\dots;x'_d-x_d,x_d)c(x'_1,\dots,x'_d,t),\end{aligned} \tag{14}$$

where $F(x_1,x'_1;\dots;x_d,x'_d)$ is defined by (10)–(13).

Before we begin solving the d-dimensional multiple fragmentation equation, we state briefly that all latin indices i, k,..., etc., will range over the values 1,...,d, unless otherwise stated.

Model 1

This model is described by the fragmentation rate and daughter distribution function, given, respectively, by

$$a(x_1,\dots,x_d)=x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}, \tag{15}$$

$$b(x_1,\dots,x_d|x'_1,\dots,x'_d)=\frac{(\alpha_1+2)\cdots(\alpha_d+2)x_1^{\alpha_1}\dots x_d^{\alpha_d}}{x_1'^{\alpha_1+1}\dots x_d'^{\alpha_d+1}}. \tag{16}$$

This model is the natural d-dimensional generalization of the one-dimensional model of McGrady and Ziff [17].

Insisting that the hypervolume is conserved per fragmentation event places the following restrictions on the indices $\alpha_1,\dots,\alpha_d$:

$$\alpha_i>-2,\quad \forall i=1,\dots,d. \tag{17}$$

It is easily shown that the number of fragments produced per fragmentation event is

$$\langle N(x_1,\dots,x_d)\rangle=\begin{cases}\dfrac{(\alpha_1+2)\cdots(\alpha_d+2)}{(\alpha_1+1)\cdots(\alpha_d+1)} & \text{if all } \alpha_1>-1\\ \infty & \text{if some or all } \alpha_1\leqslant-1.\end{cases} \tag{18}$$

On physical grounds, when $\langle N(x_1,\dots,x_d)\rangle$ is finite $\forall\alpha_i>-1$, $\langle N(x_1,\dots,x_d)\rangle$ must satisfy $\langle N(x_1,\dots,x_d)\rangle\geqslant 2$. This further constrains the α_i by

$$\frac{(\alpha_1+2)\cdots(\alpha_d+2)}{(\alpha_1+1)\cdots(\alpha_d+1)}\geqslant 2. \tag{19}$$

The d-dimensional multiple fragmentation equation (7) now becomes

$$\frac{\partial c(x_1,\dots,x_d,t)}{\partial t}=-x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}c(x_1,\dots,x_d,t)+(\alpha_1+2)\cdots(\alpha_d+2)x_1^{\alpha_1}\cdots x_d^{\alpha_d}\int_{x_1}^{\infty}\frac{dx'_1}{x_1'^{\alpha_1-\beta_1}}\cdots\int_{x_d}^{\infty}\frac{dx'_d}{x_d'^{\alpha_d-\beta_d}}c(x'_1,\dots,x'_d,t). \tag{20}$$

Here we will consider the special case when $\beta_i=-1$, and $\forall i=1,\ldots,d$, in which case (20) becomes

$$\frac{\partial c(x_1,\ldots,x_d,t)}{\partial t}=-c(x_1,\ldots,x_d,t)+(\alpha_1+2)\cdots(\alpha_d+2)x_1^{\alpha_1}\cdots x_d^{\alpha_d}\int_{x_1}^{\infty}\frac{dx_1'}{x_1'^{\alpha_1+1}}\cdots\int_{x_d}^{\infty}\frac{dx_d'}{x_d'^{\alpha_d+1}}c(x_1',\ldots,x_d',t). \tag{21}$$

We now need to solve (21) subject to appropriate initial conditions, which we will assume to be

$$c(x_1,\ldots,x_d,0)=f(x_1,\ldots,x_d)\neq 0. \tag{22}$$

Define the Laplace transform of $c(x_1,\ldots,x_d,t)$ with respect to t, $\phi(x_1,\ldots,x_d,s)$, by

$$\phi(x_1,\ldots,x_d,s)=\int_0^{\infty}dt\; e^{-st}c(x_1,\ldots,x_d,t), \tag{23}$$

in which case we may recover $c(x_1,\ldots,x_d,t)$ from the inverse Laplace transform

$$c(x_1,\ldots,x_d,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}ds e^{st}\phi(x_1,\ldots,x_d,s), \tag{24}$$

where $\mathrm{Re}(s)>\gamma$ to ensure convergence.

Taking the Laplace transform of (21) with respect to t yields the following integral equation for $\phi(x_1,\ldots,x_d,s)$:

$$\phi(x_1,\ldots,x_d,s)=\frac{f(x_1,\ldots,x_d)}{(s+1)}+\frac{(\alpha_1+2)\cdots(\alpha_d+2)x_1^{\alpha_1}\cdots x_d^{\alpha_d}}{(s+1)}\int_{x_1}^{\infty}\frac{dx_1'}{x_1'^{\alpha_1+1}}\cdots\int_{x_d}^{\infty}\frac{dx_d'}{x_d'^{\alpha_d+1}}\phi(x_1',\ldots,x_d',s). \tag{25}$$

Following the approach introduced in Ref. [30] one easily finds

$$\begin{aligned}\phi(x_1,\ldots,x_d,s)=&\frac{f(x_1,\ldots,x_d)}{(s+1)}+\frac{(\alpha_1+2)\cdots(\alpha_d+2)x_1^{\alpha_1}\cdots x_d^{\alpha_d}}{(s+1)^2}\\&\times\int_{x_1}^{\infty}\frac{dx_1'}{x_1'^{\alpha_1+1)}}\cdots\int_{x_d}^{\infty}\frac{dx_d'}{x_d'^{\alpha_d+1}}f(x_1',\ldots,x_d')\sum_{r=0}^{\infty}\frac{1}{(r!)^d}\left[\frac{(\alpha_1+2)\cdots(\alpha_d+2)}{(s+1)}\ln\left(\frac{x_1'}{x_1}\right)\cdots\ln\left(\frac{x_d'}{x_d}\right)\right]^r.\end{aligned} \tag{26}$$

Performing a simple contour integration yields the following expression for the probability distribution function $c(x_1,\ldots,x_d,t)$:

$$\begin{aligned}c(x_1,\ldots,x_d,t)=e^{-t}\Bigg(&f(x_1,\ldots,x_d)+(\alpha_1+2)\cdots(\alpha_d+2)x_1^{\alpha_1}\cdots x_d^{\alpha_d}\int_{x_1}^{\infty}\frac{dx_1'}{x_1'^{\alpha_1+1}}\cdots\int_{x_d}^{\infty}\frac{dx_d'}{x_d'^{\alpha_d+1}}f(x_1',\ldots,x_d')\\&\times\sum_{r=0}^{\infty}\frac{t^{r+1}}{(r+1)!(r!)^d}\left[(\alpha_1+2)\cdots(\alpha_d+2)\ln\left(\frac{x_1'}{x_1}\right)\cdots\ln\left(\frac{x_d'}{x_d}\right)\right]^r\Bigg).\end{aligned} \tag{27}$$

For monodisperse initial conditions

$$f(x_1,\ldots,x_d)=\delta(x_1-l_1)\cdots\delta(x_d-l_d), \tag{28}$$

(27) becomes

$$\begin{aligned}c(x_1,\ldots,x_d,t)=e^{-t}\Bigg(&\delta(x_1-l_1)\cdots\delta(x_d-l_d)+(\alpha_1+2)\cdots(\alpha_d+2)\frac{x_1^{\alpha_1}\ldots x_d^{\alpha_d}}{l_1^{\alpha_1+1}\ldots l_d^{\alpha_d+1}}\\&\times\sum_{r=0}^{\infty}\frac{t^{r+1}}{(r+1)!(r!)^d}\left[(\alpha_1+2)\cdots(\alpha_d+2)\ln\left(\frac{l_1}{x_1}\right)\cdots\ln\left(\frac{l_d}{x_d}\right)\right]^r\Bigg).\end{aligned} \tag{29}$$

For $d=1$ and $\alpha_1=\alpha=0$, (29) reduces to the result of Ziff and McGrady [16]. For $d=2$ and $\alpha_i=0$ (29), reduces to the solution presented by Rodgers and Hassan [27]. When $d=1$ and $\alpha_1=\alpha$ is completely general, (29) is equivalent to the exact solution for the model investigated by McGrady and Ziff [17], with $\beta=-1$.

Model 2

In this model we investigate a fragmentation rate and daughter distribution function given, respectively, by

$$a(x_1,\ldots,x_d)=\frac{1}{2^d}x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}, \tag{30}$$

$$b(x_1,\ldots,x_d|x_1',\ldots,x_d')=2^d\delta\left(x_1-\frac{x_1'}{2}\right)\cdots\delta\left(x_d-\frac{x_d'}{2}\right). \tag{31}$$

It is easily shown that hypervolume conservation per single fragmentation event holds, and that the average number of objects per fragmentation event, $\langle N(x_1,\ldots,x_d)\rangle$, is 2^d.

This particular choice for the fragmentation rate $a(x_1,\ldots,x_d)$ and the daughter distribution function $b(x_1,\ldots,x_d|x_1',\ldots,x_d'$ can be implemented by $F(x_1,x_1';\ldots;x_d,x_d'$ with

$$F(x_1,x_1';\ldots;x_d,x_d')=(x_1+x_1')^{\beta_1+1}\cdots(x_d+x_d')^{\beta_d+1}\times\delta(x_1-x_1')\cdots\delta(x_d-x_d'). \tag{32}$$

In this form the kinetics of the model become a little more transparent. The fragmentation rate $F(x_1,x_1';\ldots;x_d,x_d')$ describes a fragmentation process in which a d-dimensional hypercuboid-shaped object splits into 2^d fragments of equal hypervolume.

This choice for $a(x_1,\ldots,x_d)$ and $b(x_1,\ldots,x_d|x_1',\ldots,x_d')$, or equivalently, $F(x_1,x_1';\ldots;x_d,x_d')$, reduces the d-dimensional multiple fragmentation equation (7) to the following form:

$$\frac{\partial c(x_1,\ldots,x_d,t)}{\partial t}=-\frac{x_1^{\beta_1+1}\ldots x_d^{\beta_d+1}}{2^d}c(x_1,\ldots,x_d,t)+2^{2d+\beta}x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}c(2x_1,\ldots,2x_d,t) \tag{33}$$

where $\beta=\beta_1+\cdots+\beta_d$.

Solving (33) subject to the initial conditions

$$c(x_1,\ldots,x_d,0)=f(x_1,\ldots,x_d)\neq 0 \tag{34}$$

via the approach outlined in Ref. [30], one finds

$$c(x_1,\ldots,x_d,t)=e^{-x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}t/2^d}\left(f(x_1,\ldots,x_d)+\sum_{r=1}^{\infty}2^{r(r+1)\beta/2+r(r+5)d/2}f(2^rx_1,\ldots,2^rx_d)\sum_{k=0}^{r}\frac{e^{x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}(1-2^{k(\beta+d)})t/2^d}}{\prod_{m\in I_k^r}(2^{m(\beta+d)}-2^{k(\beta+d)})}\right) \tag{35}$$

with $\beta=\beta_1+\cdots+\beta_d\neq -d$ and $I_k^r=\{0,1,2,\ldots,k-1,k+1,\ldots,r\}$.

Using the fact that

$$\lim_{\alpha\to 0}\sum_{k=0}^{r}\frac{e^{x_1^{\alpha_1}\cdots x_d^{\alpha_d}(1-2^{k\alpha})t/2^d}}{\Pi_{m\in I_k^r}(2^{m\alpha}-2^{k\alpha})}=\left(\frac{x_1^{\alpha_1}\ldots x_d^{\alpha_d}t}{2^d}\right)^r\frac{1}{r!}, \tag{36}$$

we can obtain $c(x_1,\ldots,x_d,t)$ for $\beta=\beta_1+\cdots+\beta_d=-d$ without any extra effort.

Explicitly,

$$c(x_1,\ldots,x_d,t)=e^{-x_1^{\beta_1+1}\ldots x_d^{\beta_d+1}t/2^d}\sum_{r=0}^{\infty}\frac{(2^dt)^r}{r!}(x_1^{\beta_1+1}\cdots x_d^{\beta_d+1})^rf(2^rx_1,\ldots,2^rx_d), \tag{37}$$

where, of course, $\beta=\beta_1+\cdots+\beta_d=-d$.

For monodisperse initial conditions

$$f(x_1,\ldots,x_d)=\delta(x_1-l_1)\cdots\delta(x_d-l_d), \tag{38}$$

we find

$$c(x_1,\ldots,x_d,t)=\begin{cases} e^{-l_1^{\beta_1+1}\ldots l_d^{\beta_d+1}t/2^d}\left(\delta(x_1-l_1)\cdots\delta(x_d-l_d)+\sum_{r=1}^{\infty}2^{r(r+1)\beta/2+r(r+3)d/2}\delta\left(x_1-\frac{l_1}{2^r}\right)\cdots\delta\left(x_d-\frac{l_d}{2^r}\right)\right. \\ \left.\times\sum_{k=0}^{r}\frac{e^{l_1^{\beta_1+1}\ldots l_d^{\beta_d+1}(1-2)^{(k-r)(\beta+d)}t/2^d}}{\prod_{m\in I_k^r}(2^{m(\beta+d)-2^{k(\beta+d)}})}\right), \quad \beta\neq -d, \\ e^{-l_1^{\beta_1+1}\cdots l_d^{\beta_d+1}t/2^d}\sum_{r=0}^{\infty}\frac{t^r}{r!}(l_1^{\beta_1+1}\cdots l_d^{\beta_d+1})^r\delta\left(x_1-\frac{l_1}{2^r}\right)\cdots\delta\left(x_d-\frac{l_d}{2^r}\right), \quad \beta=-d. \end{cases} \tag{39}$$

For $d=1$, these results reduce to those presented in Ref. [30]. For $d=1$ and $\beta_1=\beta=-1$, one recovers the exact solution of Bak and Bak [19].

III. SCALING THEORY

We now introduce the d-tuple Mellin transform of the probability distribution function $c(x_1,\ldots,x_d,t)$ defined by

$$M(s_1,\ldots,s_d,t)=\int_0^\infty dx_1\cdots\int_0^\infty dx_d x_1^{s_1-1}\cdots x_d^{s_d-1}c(x_1,\ldots,x_d,t). \tag{40}$$

The functions $M(s_1,\ldots,s_d,t)$ for fixed $s_1,\ldots,s_d$ are known as the moments of the probability distribution function $c(x_1,\ldots,x_d,t)$.

Combining (7) and (40) gives

$$\frac{\partial M(s_1,\ldots,s_d,t)}{\partial t}=-\int_0^\infty dx_1\cdots\int_0^\infty dx_d a(x_1,\ldots,x_d)$$
$$\times c(x_1,\ldots,x_d,t)\left(x_1^{s_1-1}\cdots x_d^{s_d-1}-\int_0^{x_1}dx_1'\cdots\int_0^{x_d}dx_d' x_1'^{s_1-1}\cdots x_d'^{s_d-1}b(x_1',\ldots,x_d'|x_1,\ldots,x_d)\right). \tag{41}$$

We now proceed to investigate the two classes of exact solutions to the d-dimensional multiple fragmentation equation, presented in Sec. II, for the presence of scaling.

Model 1

In this case (41) becomes

$$\frac{\partial M(s_1,\ldots,s_d,t)}{\partial t}=-\left(1-\frac{(\alpha_1+2)\cdots(\alpha_d+2)}{(s_1+\alpha_1)\cdots(s_d+\alpha_d)}\right)\times M(s_1+\beta_1+1,\ldots,s_d+\beta_d+1,t) \tag{42}$$

provided $s_i+\alpha_i>0,\forall i$. In the special case when $\beta_i=-1$, $\forall i$, (42) becomes

$$\frac{\partial M(s_1,\ldots,s_d,t)}{\partial t}=-\left(1-\frac{(\alpha_1+2)\cdots(\alpha_d+2)}{(s_1+\alpha_1)\cdots(s_d+\alpha_d)}\right)\times M(s_1,\ldots,s_d,t) \tag{43}$$

provided $s_i+\alpha_i>0,\forall i$. Then it follows that

$$M(s_1,\ldots,s_d,t)=M(s_1,\ldots,s_d,0)\times e^{-\{1-[(\alpha_1+2)\cdots(\alpha_d+2)]/[(s_1+\alpha_1)\cdots(s_d+\alpha_d)]\}t}, \tag{44}$$

provided $s_i+\alpha_i>0,\forall i$.

To obtain the total number of objects, $N(t)$, in our fragmenting system, we must take $s_i=1\forall i$ in (44). We find that

$$N(t)=M(1,\ldots,1,t)=M(1,\ldots,1,0)e^{-\{1-[(\alpha_1+2)\cdots(\alpha_d+2)]/[(\alpha_1+1)\cdots(\alpha_d+1)]\}t}, \tag{45}$$

which is only valid for $\alpha_i>-1,\forall i$. When some or all of the $\alpha_i\leqslant -1$, then (43) and (44) are not valid, since it can easily be shown that the number of particles in the system in this case becomes infinite, even though the total hypervolume $V(t)$ of the system is conserved. If we set $s_i=2\forall i$ in (44), we find that the total hypervolume $V(t)$ of our system is constant, i.e.,

$$V(t)=M(2,\ldots,2,t)=M(2,\ldots,2,0), \tag{46}$$

provided $\alpha_i>-2\forall i$, which indeed is the case in general, as shown by (17).

An interesting feature of (43), for $s_i+\alpha_i>0\forall i$, is that it implies the existence of an infinite number of conservation laws, apart from the usual total hypervolume conservation law one usually encounters. The moments $M(s_1,\ldots,s_d,t)$ with $s_1,\ldots,s_d$, satisfying

$$\frac{(\alpha_1+2)\cdots(\alpha_d+2)}{(s_1+\alpha_1)\cdots(s_d+\alpha_d)}=1, \tag{47}$$

are all time independent. Of course, $s_i\neq 1\forall i$ in this case, otherwise (47) would contradict (19) even if $\alpha_i>-1\forall i$. Besides, the total number of objects, $N(t)$, in a fragmenting system cannot possibly be conserved. Thus, in addition to the conservation of the total hypervolume $V(t)$, there are an infinite number of hidden conserved integrals for those s_i that lie on the hypersurface defined by (47). According to a suggestion by Krapivsky and Ben-Naim [28], it is precisely these integrals which are responsible for the absence of scaling solutions for d-dimensional fragmentation processes. Indeed, the scaling solution

$$c(x_1,\ldots,x_d,t)\sim t^w\phi(t^z x_1,\ldots,t^z x_d) \tag{48}$$

implies an infinite number of scaling relations

$$w=zs, \tag{49}$$

where $s=s_1+\cdots+s_d$, which together with (47) cannot all be satisfied by the scaling exponents w and z. This rules out the possibility of scaling solutions in this model.

Model 2

For this case (41) becomes

$$\frac{\partial M(s_1,\ldots,s_d,t)}{\partial t}=\left(\frac{1}{2^{(s-d)}}-\frac{1}{2^d}\right)M(s_1+\beta_1+1,\ldots,s_d+\beta_d+1,t), \tag{50}$$

where $s=s_1+\cdots+s_d$. Again, as in model 1, we observe that (50) implies the existence of an infinite number of conservation laws. The moments $M(s_1,\ldots,s_d,t)$ with the s_i satisfying

$$s=2d, \tag{51}$$

where $s=s_1+\cdots+s_d$, are all time independent. Thus, in addition to the conservation of the total hypervolume $V(t)=M(2,\ldots,2,t)$, there are an infinite number of hidden conserved integrals for those s_i that lie on the hypersurface defined by (51). Due to the existence of an infinite number of hidden conserved integrals, we do not expect there to be any scaling solutions to this model. However, this does not appear to be the case, and we demonstrate this explicitly by finding the scaling solution to this model.

In this case assume a scaling solution of the form

$$c(x_1,\ldots,x_d,t)\sim v(t)^{-2d}\phi[x_1/v(t),\ldots,x_d/v(t)], \tag{52}$$

as $t\to\infty$, where $v(t)$ is the hypervolume of a typical time-dependent cluster. This scaling form is fully equivalent to the one given by (48); however, it is more convenient for our calculations [18]. The exponent $-2d$ ensures conservation of total hypervolume of the complete system of fragmenting objects.

A short calculation will show that

$$M(s_1,\ldots,s_d,t)\sim v(t)^{s-2d}m(s_1,\ldots,s_d), \tag{53}$$

where the scaling moments $m(s_1,\ldots,s_d)$ are defined by

$$m(s_1,\ldots,s_d)=\int_0^\infty d\xi_1\cdots\int_0^\infty d\xi_d\xi_1^{s_1-1}\cdots\xi_d^{s_d-1}\phi(\xi_1,\ldots,\xi_d). \tag{54}$$

If the moments are to be conserved, then they must be constant. This occurs when

$$s=2d, \tag{55}$$

which is in complete agreement with (51), as it should be.

Substituting (52) into (33) yields

$$-\frac{1}{v(t)^{\beta+d+1}}\frac{dv(t)}{dt}=\omega=\frac{\left[-\frac{1}{2^d}\phi(\xi_1,\ldots,\xi_d)+2^{2d+\beta}\phi(2\xi_1,\ldots,2\xi_d)\right]\xi_1^{\beta_1+1}\cdots\xi_d^{\beta_d+1}}{\left[\xi_1\frac{\partial\phi(\xi_1,\ldots,\xi_d)}{\partial\xi_i}+2d\phi(\xi_1,\ldots,\xi_d)\right]} \tag{56}$$

where the separation constant ω is positive since $v(t)$ must be a decreasing function of time in a fragmenting system, and, $\xi_i=x_i/v(t)\forall i$. In the previous equation we have implicitly assumed an implied summation over repeated indices.

Then

$$v(t)\sim\begin{cases} t^{-1/(\beta+d)}, & \beta>-d, \quad t\to\infty \\ e^{-\omega t}, & \beta=-d, \quad t\to\infty \\ (t_c-t)^{1/|\beta+d|}, & \beta<-d, \quad t<t_c. \end{cases} \tag{57}$$

These expressions are only valid provided scaling holds. For $\beta<-d$ a singularity is encountered within a finite time t_c, and scaling becomes invalid. In this instance, we anticipate shattering, which will be discussed below. When $\beta=-d$, as in the one-dimensional case [24,26,30], a scaling form does not exist which is consistent with the boundary conditions $\phi(\xi_1,\ldots,\xi_d)\to$const as $\xi_k\to 0$ and $\phi(\xi_1,\ldots,\xi_d)\to 0$ as $\xi_k\to\infty$ for $\forall k$. Consequently, we need only look at the case when $\beta>-d$, for which scaling is clearly valid.

When $\beta > -d$, we assume that $\phi(\xi_1,\ldots,\xi_d)$ vanishes at $\xi_k = 0$ and $\xi_k = \infty \forall k$. It can be shown from (56) that

$$\phi(\xi_1,\ldots,\xi_d) \sim \frac{e^{-\xi_1^{\beta_1+1}\cdots\xi_d^{\beta_d+1}/2^d\omega(\beta+d)}}{(\xi_1^2+\cdots+\xi_d^2)^d} \tag{58}$$

as $\xi_k \to \infty \forall k$. Therefore, we will assume that our scaling function $\phi(\xi_1,\ldots,\xi_d)$ is of the following form:

$$\phi(\xi_1,\ldots,\xi_d) = \frac{e^{-\xi_1^{\beta_1+1}\cdots\xi_d^{\beta_d+1}/2^d\omega(\beta+d)}}{(\xi_1^2+\cdots+\xi_d^2)^d} f(\xi_1,\ldots,\xi_d), \tag{59}$$

where we insist that $f(\xi_1,\ldots,\xi_d) = 1$ at $\xi_k = \infty \forall k$.

Substituting (59) into (56) and performing a lengthy calculation yields a solution for $f(\xi_1,\ldots,\xi_k)$ of the form

$$f(\xi_1,\ldots,\xi_d) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{n(\beta+d)}}{\prod_{m=1}^{n}(2^{m(\beta+d)}-1)} \times e^{-\xi_1^{\beta_1+1}\cdots\xi_d^{\beta_d+1}(2^{n(\beta+d)}-1)/2^d\omega(\beta+d)}. \tag{60}$$

Thus, a rather unusual situation occurs in this d-dimensional model. A scaling solution exists, in spite of the fact that we have an infinite number of hidden conserved integrals, of the form (52) as $t \to \infty$ with $v(t)$ given by (57), and $\phi(\xi_1,\ldots,\xi_d)$ given by (59) and (60). These results are consistent with the exact solution of model 2 presented in Sec. II above. When $d = 1$, these results reduce to those in Ref. [30], and are of a similar form to those of Cheng and Redner [24].

IV. SHATTERING TRANSITIONS

Formally, the total hypervolume $V(t)$ of the multidimensional fragmenting system is defined by

$$V(t) = \int_0^\infty dx_1 \cdots \int_0^\infty dx_d x_1 \cdots x_d c(x_1,\ldots,x_d,t), \tag{61}$$

so that, with the aid of (7) and (9), it can easily be shown that

$$\frac{dV(t)}{dt} = 0, \tag{62}$$

indicating that $V(t)$ is conserved. However, when the fragmentation rate increases sufficiently fast as the hypervolume of the fragments decreases to zero, a cascading of the fragmentation occurs such that hypervolume is lost to fragments of zero or infinitesimal hypervolume. This cascading process, which has been named "shattering" [17,25] or "disintegration" [26], is somewhat similar to gelation in coagulating systems, where mass is lost to an infinite gel molecule [31,32]. Gelation and shattering are both signaled by the condition $dV(t)/dt < 0$. When shattering is suspected, a more subtle analysis to that used to derive (62) is required.

To analyze the shattering transition define a cutoff hypervolume $V_\varepsilon(t)$ with $0 < \varepsilon \ll 1$, by

$$V_\varepsilon(t) = \int_\varepsilon^\infty dx_1 \cdots \int_\varepsilon^\infty dx_d x_1 \cdots x_d c(x_1,\ldots,x_d,t), \tag{63}$$

with

$$V(t) = \lim_{\varepsilon \to 0^+} V_\varepsilon(t). \tag{64}$$

It is easily shown that the cutoff hypervolume loss is given by

$$\frac{dV_\varepsilon(t)}{dt} = -\int_\varepsilon^\infty dx_1 \cdots \int_\varepsilon^\infty dx_d a(x_1,\ldots,x_d) c(x_1,\ldots,x_d,t) \times \int_0^\varepsilon dx_1' \cdots \int_0^\varepsilon dx_d' x_1' \cdots x_d' b(x_1',\ldots,x_d'|x_1,\ldots,x_d), \tag{65}$$

where, of course,

$$\frac{dV(t)}{dt} = \lim_{\varepsilon \to 0^+} \frac{dV_\varepsilon(t)}{dt}. \tag{66}$$

Model 1

In this model, for $\beta_i = -1 \forall i$, shattering does not occur for any values of the indices α_i. This is easily demonstrated from the definition of the total hypervolume $V(t)$ of the system via (61) and the exact solution for this particular model (27). One finds that the total hypervolume $V(t)$ of our system is both finite and time independent, a sure indicator that shattering is indeed absent.

Model 2

As can be easily demonstrated by substituting the exact solution for this particular model (37) into the definition of the total hypervolume $V(t)$ of system (61), shattering does not occur when the sum of the homogeneity indices, β, satisfies $\beta = -d$. Again, one finds that the total hypervolume $V(t)$ of the system is both finite and time independent. It therefore remains to investigate the case when the sum of the homogeneity indices, β, satisfies $\beta < -d$.

To analyze the shattering regime, $\beta < -d$, we need to know the behavior of $c(x_1,\ldots,x_d,t)$ as $t \to \infty$ and $x_k \to 0$, such that tx_k remains fixed $\forall k$. A short calculation will show that the asymptotic form of $c(x_1,\ldots,x_d,t)$ in the shattering regime is given by

$$c(x_1,\ldots,x_d,t)\sim T(t)x_1^{\lambda_1}\cdots x_d^{\lambda_d}, \tag{67}$$

where $T(t)\neq 0$. The exponents $\lambda_1,\ldots,\lambda_d$ are restricted by

$$\lambda=|\beta+d|-2d, \tag{68}$$

where $\lambda=\lambda_1+\cdots+\lambda_d$. In this analysis it does not matter what each individual value of the λ_k is; all that matters is what their sum λ is. Therefore, we need not concern ourselves with explicitly determining the λ_k.

For $\beta<-d$, (65) becomes

$$\frac{dV_\varepsilon(t)}{dt}=-\frac{1}{2^d}\int_\varepsilon^{2\varepsilon}dx_1\cdots\int_\varepsilon^{2\varepsilon}dx_0 x_1^{\beta_1+2}\cdots x_d^{\beta_d+2}\times c(x_1,\ldots,x_d,t) \tag{69}$$

for this particular model. Substituting (67) into (69) yields

$$\frac{dV_\varepsilon(t)}{dt}=-\frac{1}{2^d}T(t)(\ln 2)^d, \tag{70}$$

$\forall \beta_i+\lambda_i=-3$, with $1\leq i\leq d$, and

$$\frac{dV_\varepsilon(t)}{dt}=-\frac{1}{2^d}T(t)(\ln 2)^k\left(\frac{2^{(\beta_{k+1}+\lambda_{k+1}+3)}-1}{\beta_{k+1}+\lambda_{k+1}+3}\right)\cdots\left(\frac{2^{(\beta_d+\lambda_d+3)}-1}{\beta_d+\lambda_d+3}\right) \tag{71}$$

if $\beta_i+\lambda_i=-3$ for $i=1,\ldots,k$ and $\beta_i+\lambda_i\neq-3$ for $i=k+1,\ldots,d$, where $0\leq k\leq d-1$. Hence we see that

$$\frac{dV(t)}{dt}=\lim_{\varepsilon\to 0^+}\frac{dV_\varepsilon(t)}{dt}\neq 0, \tag{72}$$

which proves that shattering does indeed take place for $\beta<-d$.

V. VOLUME DISTRIBUTIONS

In this section we briefly consider the total hypervolume probability distribution function $C(V,t)$ defined by

$$C(V,t)=\int_0^\infty dx_1\cdots\int_0^\infty dx_d\,\delta(x_1\cdots x_d-V)c(x_1,\ldots,x_d,t), \tag{73}$$

which is usually very useful for providing a partial description of a fragmenting system.

Model 1

For $\beta_i=-1$, $\alpha_i=0$, $\forall i$, and monodisperse initial conditions, the relevant exact solution to this model is

$$c(x_1,\ldots,x_d,t)=e^{-t}\left\{\delta(x_1-l_1)\cdots\delta(x_d-l_d)+\frac{1}{l_1\ldots l_d}\sum_{r=0}^{\infty}\frac{(2^d t)^{(r+1)}}{(r+1)!(r!)^d}\times\left[\ln\left(\frac{l_1}{x_1}\right)\cdots\ln\left(\frac{l_d}{x_d}\right)\right]^r\right\}. \tag{74}$$

Substituting this into (73) yields the following expression for the probability distribution function for the total hypervolume:

$$C(V,t)=e^{-t}\left\{\delta(V-l_1\cdots l_d)+\frac{2^d t}{l_1\cdots l_d}\sum_{r=0}^{\infty}\frac{(2^d t)^r}{(r+1)![d(r+1)-1]!}\times\left[\ln\left(\frac{l_1\cdots l_d}{V}\right)\right]^{[d(r+1)-1]}\right\}. \tag{75}$$

This expression reduces to the result of Ziff and McGrady [16] when $d=1$, and to the result of Rodgers and Hassan [27] for $d=2$. Expanding (75) for small t gives

$$C(V,t)\sim e^{-t}\left\{\delta(V-l_1\cdots l_d)+\frac{2^d t}{l_1\cdots l_d}\frac{1}{(d-1)!}\left[\ln\left(\frac{l_1\cdots l_d}{V}\right)\right]^{(d-1)}\right\}. \tag{76}$$

As d increases by 1, the power of the logarithmic divergence in the second term in (76) also increases by a factor of 1. Analogous to the one-dimensional case, the d-dimensional case for $\beta_i=-1\,\forall i$ forms the borderline case for the shattering transition.

Define the normalized nth moments of the total hypervolume V by

$$\langle V^n\rangle=\frac{\int_0^\infty dV\,V^n C(V,t)}{\int_0^\infty dV C(V,t)}. \tag{77}$$

Then it follows that

$$\langle V^n\rangle^{1/n}\sim e^{-2^d[1-1/(n+1)^d]t/n} \tag{78}$$

which indicates that $C(V,t)$ does not exhibit a scaling form.

Model 2

In this model, for monodisperse initial conditions, the relevant exact solution is given by (39). Substituting (39) into (73) yields

$$C(V,t)=\begin{cases} e^{-l_1^{\beta_1+1}\cdots l_d^{\beta_d+1}t/2^d}\Bigg(\delta(V-l_1\cdots l_d)+\sum_{r=1}^{\infty}2^{r(r+1)\beta/2+r(r+3)d/2} \\ \times\delta\left(V-\frac{l_1\cdots l_d}{2^{rd}}\right)\sum_{k=0}^{r}\frac{e^{l_1^{\beta_1+1}\cdots l_d^{\beta_d+1}(1-2^{(k-r)(\beta+d)})t/2^d}}{\prod_{m\in I_k^r}(2^{m(\beta+d)}-2^{k(\beta+d)})}\Bigg), & \beta\neq -d, \\ e^{-l_1^{\beta_1+1}\cdots l_d^{\beta_d+1}t/2^d}\times\sum_{r=0}^{\infty}\frac{t^r}{r!}(l_d^{\beta_1+1}\cdots l_d^{\beta_d+1})^r\delta\left(V-\frac{l_1\cdots l_d}{2^{rd}}\right), & \beta=-d. \end{cases} \tag{79}$$

These results may be compared with the results when $d=1$ for $c(x_1,\ldots,x_d,t)$ given in (39) or in [30]. The similarity is quite remarkable, indicating that the scaling and shattering behavior of $C(V,t)$ given by (79) will closely match that observed for $c(x_1,\ldots,x_d,t)$ given by (39), when $d=1$.

VI. CONCLUSIONS

In reality, fragmenting objects will have both size and shape, i.e., a geometry. Intrigued by the possibility that the geometry of the fragmenting objects may influence the fragmentation process, we have investigated two distinct d-dimensional fragmentation models for $d\geqslant 1$. Two classes of exact solutions to these geometric models, which describe the kinetics of fragmentation of d-dimensional hypercuboid-shaped objects, are presented. The first class is described by a fragmentation rate $a(x_1,\ldots,x_d)=1$ and a daughter distribution function $b(x_1,\ldots,x_d|x'_1,\ldots,x'_d=(\alpha_1+2)\cdots(\alpha_d+2)$ $x_1^{\alpha_1}\ldots x_d^{\alpha_d}/x_1'^{(\alpha_1+1)}\ldots x_d'^{(\alpha_d+1)}$. For $d>1$, this particular class of exact solutions does not exhibit scaling, and does not permit the occurrence of a shattering transition. This model is a generalization to d dimensions of the model investigated by McGrady and Ziff [17], with $\beta=-1$. The second class of exact solutions is described by a fragmentation rate $a(x_1,\ldots,x_d)=x_1^{\beta_1+1}\cdots x_d^{\beta_d+1}/2^d$ and a daughter distribution function $b(x_1,\ldots,x_d|x'_1,\ldots,x'_d)=2^d\delta(x_1-x'_1/2)\cdots\delta(x_d-x'_d/2)$. This particular class of exact solutions describes a type of fragmentation process in which d-dimensional hypercuboid-shaped objects always break up into 2^d pieces of equal hypervolume at various rates which depend upon the geometry of the fragmenting objects and the homogeneity indices $\beta_1,\ldots,\beta_d$. This type of fragmentation has been observed and studied when polymers degrade under tension (stretching) [33], or in the presence of a destructive force field such as ultrasound [6]. Defining β to be the sum of all the homogeneity indices $\beta_1,\ldots,\beta_d$, it is shown that this particular class of exact solutions exhibits scaling for $\beta>-d$, and this scaling form is explicitly determined. For $\beta=-d$, we show that a scaling solution to this model does not exist, and that the shattering transition is not permitted. When $\beta<-d$, we show that a scattering transition occurs.

An interesting scenario occurs in our investigation into the scaling behavior of the second class of exact solutions. When $\beta>-d$, we have shown that a scaling solution to our model exists, and we explicitly find this scaling solution. This is very surprising, since it has been suggested [28] that scaling solutions are not supposed to exist when we have an infinite number of hidden conserved integrals present. We suggest that the existence of an infinite number of hidden conserved integrals is not always indicative of the absence of scaling solutions. There must exist other important criteria which conclusively indicate the absence of scaling solutions in a particular d-dimensional fragmentation process. We propose to investigate the nature of these criteria in subsequent work.

An investigation into the occurrence of a shattering transition in the second class of exact solutions presented in Sec. II is quite intriguing. When $\beta<-d$ it is shown that shattering occurs. It is interesting to see that whether shattering occurs or not is determined exclusively by the fact that $\beta<-d$, and not on the values of the individual $\beta_1,\ldots,\beta_d$, which add up to give β. Of course, the intensity of the shattering transition will depend on how negative $\beta+d$ is. In this case, there is also the possibility that competing effects between the positive and negative homogeneity indices β_i will act to moderate the intensity of this shattering transition.

The total hypervolume probability distribution function $C(V,t)$ is very useful for providing a partial description of a fragmenting system. Consider $C(V,t)$ for the first class of exact solutions for small t. In this case, the two-dimensional model differs from the one-dimensional model by the presence of a logarithmic divergence term. The three-dimensional model differs from the one-dimensional model by the presence of a logarithm squared divergence term, and so on. An analysis of the normalized nth moments of the total hypervolume V indicate that $C(V,t)$ for model 1 does not exhibit scaling. For the second class of exact solutions presented in Sec. II, it is shown that the similarity between $C(V,t)$ and $c(x_1,\ldots,x_d,t)$, with $d=1$, is quite remarkable. This is a strong indicator that the scaling and shattering behavior of $C(V,t)$ will be very similar to that observed for $c(x_1,\ldots,x_d,t)$, with $d=1$.

We have found that the introduction of more than one dimension (or parameter) to characterize the geometry of the fragmenting object can have a significant effect on the kinetics of the fragmentation process. As a prospect for future research, one could investigate problems with sources and sinks in d dimensions, given that one now has a useful method available for determining exact solutions to geometric models which describe the kinetics of fragmentation of d-dimensional hypercuboid-shaped objects.

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