Generalized Lyapunov exponents for products of correlated random matrices

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We give the exact expressions for the generalized Lyapunov exponents of products of random matrices extracted with a Markovian rule. In analogy to the uncorrelated case, these expressions are obtained via a replica trick method, and exponents are given by the largest eigenvalue in modulus of appropriate matrices. As an application we study the distribution of the electronic dc conductance in the random dimer model, which is of interest because it possesses an extended state. We find that in the vicinity of this state the distribution is lognormal and characterized by one single parameter, which is the localization length.

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Products of random matrices play an important role in many areas of physics [1], such as, for example, magnetic disordered systems, chaotic dynamical systems, and electronic, vibrational, or optical disordered systems; generally speaking, in all systems which are described by coupled linear equations with random coefficients. For instance, the dc conductance of a disordered electronic system can be obtained, by means of the Landauer formula, from the knowledge of the maximum characteristic Lyapunov exponent of the product of appropriate random matrices. The maximum Lyapunov exponent of a similar product of random transfer matrices is related to the thermodynamic properties of one-dimensional random Ising models. In addition, a first approximation of the behavior of chaotic systems can also be obtained in terms of products of convenient random matrices.

At the same time, the dynamics of these systems can be sensibly affected by the presence of correlation in the disorder [2]. Development of computing techniques appropriate for these cases is therefore important for the understanding of the consequences of disorder correlation on the system properties. In this Brief Report we derive the expressions for the generalized (maximum) Lyapunov exponents of a product of random matrices when finite range correlations are introduced in the disorder through a Markovian process.

The generalized Lyapunov exponents describe the fluctuations of the Lyapunov exponent due to finite system size, i.e., when only a finite number of matrices is involved, and also those of some non-self-averaging quantities in infinite systems, such as dc electronic conductance, as we shall see below. Averages over realizations of disorder are obtained via a replica trick, generalizing the method which is used in the uncorrelated case [3,1]. In order to see how this works in practice, we use our results for computing the electronic dc conductance g of the random dimer model [4], which is a one-dimensional model with correlated disorder that is particularly interesting because it is a one-dimensional random system with an extended state. It turns out that, for energies approaching that of the extended state, g possesses a lognormal probability distribution described by a single parameter, the localization length.

The (maximum) Lyapunov exponent of an infinite product of random matrices is a nonrandom quantity, describing the rate of exponential growth of the largest product's eigenvalue when the number of matrices involved approaches infinity. It is defined as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \left\langle \ln \left| \frac{\mathbf{A}_N \mathbf{A}_{N-1} \cdots \mathbf{A}_2 \mathbf{A}_1 \mathbf{u}_0}{\mathbf{u}_0} \right| \right\rangle, \tag{1}$$

where A_i are random $d \times d$ matrices, $\mathbf{u}_0$ is a generic vector of R^d, and $\langle\ \rangle$ is the average over the possible realizations of disorder. It can be shown under very general assumptions [5] that λ is a self-averaging quantity, so that the brackets in the above expression can be dropped in almost any realization. On the other hand, when only a finite number of matrices is considered the corresponding exponential growth rate λ_N can display very large sample to sample fluctuations. These fluctuations can be characterized through the generalized Lyapunov exponents, which in turn are related to the exponential growth rates of the moments of the matrix product:

$$L(q) = \lim_{N\to\infty} \frac{1}{N} \ln \left\langle \left| \frac{\mathbf{A}_N \mathbf{A}_{N-1} \cdots \mathbf{A}_2 \mathbf{A}_1 \mathbf{u}_0}{\mathbf{u}_0} \right|^q \right\rangle. \tag{2}$$

It can be shown that $\lambda = \lim_{q\to 0} \partial L(q)/\partial q$, and Eq. (2) implies that for a large enough N $\langle |\mathbf{u}_N/\mathbf{u}_0|^q \rangle \approx \exp[NL(q)]$ (with $\mathbf{u}_N = \mathbf{A}_N \mathbf{A}_{N-1} \cdots \mathbf{A}_2 \mathbf{A}_1$). This quantity constitutes the continuation of the characteristic function $\langle e^{ikx} \rangle$ to imaginary values of k with $x = |\mathbf{u}_N/\mathbf{u}_0|$. Owing to the central limit theorem, finite N fluctuations of λ_N for large N are generally well approximated by a Gaussian distribution,

$$P(\lambda_N) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(\lambda_N - N\lambda)^2}{2N\sigma^2} \right\}. \tag{3}$$

However, for large values of the argument $(\lambda_N - N\lambda)$, such a distribution can fail in describing the tail of the actual distribution and cases are known in which this

discrepancy affects also the core of the distribution [1]. Therefore in general λ and σ are not sufficient to fully describe the probability distribution of λ_N, and only represent the first and second order terms in the Taylor expansion of $L(q)$:

$$L(q) = \lambda q + \frac{1}{2}\sigma^2 q^2 + \cdots . \tag{4}$$

We are interested here in finding expressions for computing $L(q)$ in the case of Markovian correlated random matrices. Apart from exceptional cases, the product in the Eq. (2) is dominated for large N by its largest eigenvalue in modulo. Thus $L(q)$ can be determined by computing $\lim_{N\to\infty} N^{-1}\ln\langle|\ \mathrm{Tr}\ \mathbf{X}_N\ |^q\rangle$, with $\mathbf{X}_N = \mathbf{A}_N\mathbf{A}_{N-1}\cdots\mathbf{A}_2\mathbf{A}_1$ [3]. Moreover, we can use the identity (holding for any matrix $\mathbf{M}$ and for positive integer q) $(\mathrm{Tr}\ \mathbf{M})^q = \mathrm{Tr}(\ \mathbf{M}^{\otimes q})$, where $\otimes$ indicates the direct product, and

$$\mathbf{M}^{\otimes q} = \underbrace{\mathbf{M}\otimes\mathbf{M}\otimes\ldots\otimes\mathbf{M}}_{q\ \text{times}}.$$

The point is now to evaluate the average $\langle|\mathrm{Tr}(\ {\mathbf{X}_N}^{\otimes q})|\rangle$.

Let each matrix be identified by an index ϵ_i and let ϵ_i be chosen from a discrete set of L values. By defining for shortness $\mathbf{B} = \mathbf{A}^{\otimes q}$, in the case of a Markovian correlated process the average of ${\mathbf{X}_N}^{\otimes q}$ is obtained as

$$\langle\mathbf{X}_N^{\otimes q}\rangle = \sum_{\{\epsilon_i\}}\mathbf{B}(\epsilon_N)T(\epsilon_N,\epsilon_{N-1})\mathbf{B}(\epsilon_{N-1})T(\epsilon_{N-1},\epsilon_{N-2})$$

$$\times\cdots\mathbf{B}(\epsilon_2)T(\epsilon_2,\epsilon_1)\mathbf{B}(\epsilon_1)P_1(\epsilon_1) \tag{5}$$

where $T(\epsilon_i,\epsilon_j)$ is the stochastic matrix element, giving the conditional probability of getting $\mathbf{A}(\epsilon_i)$ after $\mathbf{A}(\epsilon_j)$. The sum is extended over all the possible sequences $\{\epsilon_i\}$, and P_1 is the probability for the sequence to start with ϵ_1. Let us now introduce the matrix $\mathbf{Y}$ whose elements $Y(l,\epsilon;l',\epsilon')$ are given by

$$Y(l,\epsilon;l',\epsilon') = B_{ll'}(\epsilon)T(\epsilon,\epsilon'). \tag{6}$$

This is an $Ld^q \times Ld^q$ matrix in which we preserve the indices of the parent matrices for more clarity in computations. For increasing N, the average Eq. (5) becomes less and less dependent on the initial matrix $\mathbf{B}(\epsilon_1)P_1(\epsilon_1)$. For this reason, we can choose $P_1(\epsilon_1) \equiv \sum_{\epsilon_0} T(\epsilon_1,\epsilon_0)$, and rewrite the above equation as

$$\langle(X_N^{\otimes q})_{lm}\rangle = \sum_{\epsilon_N}\sum_{\epsilon_0} Y^{\,N}(l,\epsilon_N,;m,\epsilon_0)$$

so that, at least for even q,

$$\langle|\mathrm{Tr}[\ {\mathbf{X}_N}^{\otimes q}]|\rangle = \langle\mathrm{Tr}[\ {\mathbf{X}_N}^{\otimes q}]\rangle$$

$$= \mathrm{Tr}\langle\ {\mathbf{X}_N}^{\otimes q}\rangle$$

$$= \sum_l\langle(X_N^{\otimes q})_{ll}\rangle$$

$$= \sum_l\sum_{\epsilon_N}\sum_{\epsilon_0} Y^{\,N}(l,\epsilon_N,;l,\epsilon_0).$$

For large N, the last expression is dominated by the largest eigenvalue $y(q)$ in modulus of $\mathbf{Y}$, that is,

$$\langle|\mathrm{Tr}[\ {\mathbf{X}_N}^{\otimes q}]|\rangle \simeq a|y(q)|^N,$$

where a is a constant independent of N. Therefore we get finally

$$\lim_{N\to\infty}\frac{1}{N}\ln\langle|\mathrm{Tr}\ \mathbf{X}_N\ |^q\rangle = \lim_{N\to\infty}\frac{1}{N}\ln\mathrm{Tr}\ \mathbf{Y}^N = \ln|y(q)| \tag{7}$$

and the generalized Lyapunov exponents are given by

$$L(q) = \ln|y(q)|. \tag{8}$$

Equation (7), obtained for q even, is valid for every integer q whenever $Y_{ij} > 0$, $\forall i,j$. In the case of independent matrices, the stochastic matrix $T(\epsilon_i,\epsilon_j)$ reduces to a matrix with identical columns, and it is easy to verify that the usual expression [3,1] for uncorrelated disorder is recovered.

The expressions of the generalized Lyapunov exponents derived above can be used, for instance, to study the dc conductance statistics of electronic systems with correlated disorder. As an example we consider here the random dimer model [4], which has been introduced for explaining the exceptionally high electronic transport properties of some conjugated polymer chains [6], and which is particularly interesting because it is a one-dimensional disordered system with an extended state.

The stationary equation for one electron in one dimension in the base of the atomic orbitals can be written in the matrix form

$$\mathbf{u}_i = \mathbf{A}_i\mathbf{u}_{i-1}$$

where

$$\mathbf{u}_{i-1} = \begin{pmatrix}\psi_i\\ \psi_{i-1}\end{pmatrix} \quad\text{and}\quad \mathbf{A}_i = \begin{pmatrix}E-\epsilon_i & -1\\ 1 & 0\end{pmatrix}. \tag{9}$$

Here E is the system energy, ϵ_n the site energy, and ψ_n plays the role of the wave function amplitude at site n; moreover $i \geq 1$ and $\psi_0 = 0$. The typical conductance s at $T = 0$ for a system with Fermi energy E and length $N \gg 1$ is given by the Landauer formula [7]

$$s = \frac{2q_e^2}{h}\ e^{-2N\lambda}, \tag{10}$$

where q_e is the charge of the carriers, h the Planck constant, and λ the Lyapunov exponent of the matrix product appropriate to the electronic system. It is well known that the typical conductance of a disordered electronic system $s = \exp(-\langle\ln g\rangle)$ is a physical quantity that can be considered statistically meaningful, whereas on the other hand the conductance g itself can exhibit very large fluctuations, leading eventually to a distribution function with infinite variance [8]. The self-averaging properties of s are a direct consequence of those of λ, Eq. (1), whereas the probability distribution of g can be characterized through its moments $\langle g\ ^q\rangle$ that, in analogy with Eq. (10), are also related to the generalized Lyapunov exponents Eq. (2). Thus calculation of conductance statis-

tics essentially consists in computing $L(q)$.

In the random dimer model the site energy ϵ_i, Eq. (9), assume randomly two values, ϵ_a and ϵ_b, but one of them, say ϵ_b, can appear only in pairs. When ϵ_b is in the energy band of the pure ϵ_a system (i.e., $\mid \epsilon_a - \epsilon_b \mid \leq 2$), such a random model possesses one extended state at the energy $E_0 = \epsilon_b$, and exhibits quantum diffusion, which on the contrary does not take place in one-dimensional systems with uncorrelated disorder [5,9]. This is therefore a very remarkable example of how Markovian correlations can modify drastically the properties of disordered systems.

It can be verified by direct inspection that for the random dimer model the transition matrix of the Markovian process is

$$\mathbf{T} = \begin{pmatrix} p & 0 & p \\ 1-p & 0 & 1-p \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix element $T_{\alpha\beta}$ yields the probability of having a site of type α after one of type β. Here p is the probability of having ϵ_a, which can appear only after an integer number of ϵ_b pairs, or after another ϵ_a. On the whole, the relative frequency of sites with energy ϵ_a and ϵ_b is given by $f_a = p/(2-p)$ and $f_b = 2(1-p)/(2-p)$, respectively, and according to Eq. (6) we have

$$\mathbf{Y} = \begin{pmatrix} p\mathbf{A}_a^{\otimes q} & \mathbf{0} & p\mathbf{A}_a^{\otimes q} \\ (1-p)\mathbf{A}_b^{\otimes q} & \mathbf{0} & (1-p)\mathbf{A}_b^{\otimes q} \\ \mathbf{0} & \mathbf{A}_b^{\otimes q} & \mathbf{0} \end{pmatrix}.$$

We have computed $L(q)$ through Eq. (8) numerically, by selecting $y(q)$ through iterated application of $\mathbf{Y}$ on an arbitrary initial vector $\mathbf{v}$. It can be seen that the eigenvalues of $\mathbf{Y}$ are the same as those of the matrix

$$\begin{pmatrix} p\mathbf{A}_a^{\otimes q} & \mathbf{A}_b^{\otimes q} \\ (1-p)\mathbf{A}_b^{\otimes q} & \mathbf{0} \end{pmatrix},$$

the remaining eigenvalues being zero. This reduces considerably the amount of required numerical resources.

Figure 1 shows the behavior of $L(q)/q$ up to $q = 8$ at some different energies for the case $\epsilon_a = -0.5$ and $\epsilon_b = 0.5$, and transition probability $p = 0.5$. A linear behavior of $L(q)/q$ indicates a Gaussian distribution of λ_N, Eqs. (3) and (4), whence a lognormal distribution of the conductance. It is seen that this is indeed the case when the energy approaches the "mobility edge" $E_0 = 0.5$, while far from this energy higher terms in expression (4) become more and more important. It can be useful to recall that $L(q)$ is in general a convex function of q. In the energy range where the linear behavior is observed, the parameters of the distribution can be derived from two values of $L(q)$. In fact in this case λ and σ in Eq. (4) are the only nonzero cumulants of the distribution. Thus the two quantities

$$\alpha = L(2) - \frac{L(4)}{4}, \quad \beta = \frac{L(4)}{4} - \frac{L(2)}{2} \tag{11}$$

must be, respectively, equal to λ and σ^2. Computed values for the energy range around E_0 are reported in Fig. 2. Interestingly, in this region the values of λ and σ^2 so obtained are about equal. This indicates that the distribution $P(g_N)$ of the conductance of a chain of N sites depends only upon one parameter. By recalling that the Lyapunov exponent λ is a measure of the inverse of the localization length [5], such a distribution can be written as

$$P(g_N) = \frac{1}{\sqrt{2\pi\xi}} \exp\left\{-\frac{(\ln g_N - N/\xi)^2}{2N/\xi}\right\}. \tag{12}$$

It is interesting to recall that the same peculiar behavior has been observed in several disordered systems under many different circumstances [10].

We have also computed the Lyapunov exponent via a Monte Carlo method, and compared the values obtained in this way with those given by α, Eq. (11).

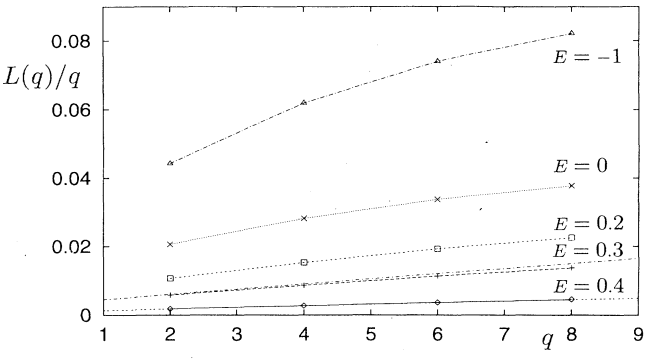

FIG. 1. Behavior of $L(q)/q$ for the random dimer model at some different energies. For energies approaching the extended state, $E_0 = 0.5$, the behavior becomes more and more linear, implying a lognormal distribution of the conductance probability. The two lines spanning the whole E range are the lines $L(q)/q = [1 + (1/2)q]\lambda$, where λ is the Lyapunov exponent obtained by Monte Carlo simulations for $E = 0.3$ and $E = 0.4$.

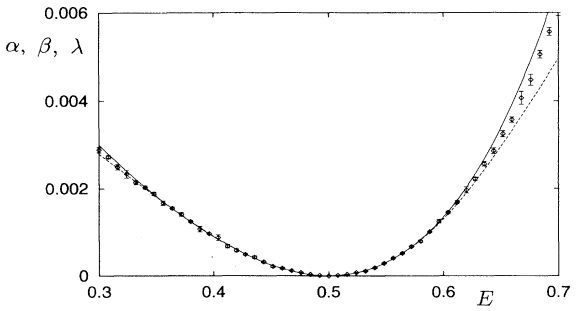

FIG. 2. Quantities α (continuous line) and β (dashed line) from Eq. (11) in the energy range around $E_0 = 0.5$; α approximates better and better the Lyapunov exponent λ (diamonds) as the energy approaches E_0, implying a lognormal distribution of conductance in this region. λ and its statistical error at each energy have been computed by Monte Carlo simulations of 10 chains made of 10^6 sites.

Such values are reported in Fig. 2, together with the corresponding statistical error. It can be seen that the agreement between the two determinations of λ is excellent in the range of energies around the extended state. Monte Carlo values of λ have also been used for fitting $L(q)/q$ in the linear cases. In fact, λ being the only parameter of the probability distribution, $L(q)/q$ takes the form $L(q)/q = [1 + (1/2)q]\lambda$. This quantity is shown in Fig. 1 for $E = 0.3$ and $E = 0.4$ (lines spanning the whole x range). Again the agreement is found to be very good as E approaches E_0.

In conclusion, we derived exact expressions for computing the generalized Lyapunov exponents $L(q)$ for the product of Markovian correlated matrices. These expressions take into account all the possible realizations of the disorder and can be easily employed in numerical computations. As an example, we investigated the fluctuations of dc conductance statistics of the random dimer model, and found that in the vicinity of the extended state the conductance has a lognormal distribution, characterized by a single parameter, i.e., the localization length. The expressions for the moments given here are of a general form, and can be used in those problems where products of large numbers of random matrices with Markovian correlations are involved.

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[1] See, e.g., A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices in Statistical Physics* (Springer, Heidelberg, 1993).

[2] A. Crisanti, G. Paladin, and A. Vulpiani, Phys. Rev. A **39**, 6491 (1989).

[3] J.B. Pendry, J. Phys. C **15**, 4821 (1982).

[4] David H. Dunlap, H.-L. Wu, and Philip W. Phillips, Phys. Rev. Lett. **65**, 88 (1990); other mathematically equivalent models have also been investigated: see, e.g., S.N. Evangelou and D.E. Katsanos, Phys. Lett. A **164**, 456 (1992).

[5] K. Ishii, Suppl. Prog. Theor. Phys. **53**, 77 (1973), and references therein.

[6] P.A. Schulz, D.S. Gãlvao, and M.J. Caldas, Phys. Rev. B **44**, 6079 (1991).

[7] See, e.g., Bernard Souillard, in *Chance and Matter*, edited by J. Souletie, J. Vannimenus, and R. Stora (Elsevier, Amsterdam, 1987), pp. 305-380.

[8] Elihu Abrahams and Michel J. Stephen, J. Phys. C **13**, L377 (1980).

[9] D.J. Thouless, Phys. Rep. **13**, 93 (1974).

[10] See, e.g., J. Flores, P.A. Mello, and G. Monsiváis, Phys. Rev. B **35**, 2144 (1987); Boris Shapiro, Philos. Mag. B **56**, 1031 (1987); Hiroaki Yamada, Masaki Goda, and Yoji Aizawa, J. Phys. Soc. Jpn. **60**, 3501 (1991); Bernard Kramer, Arisato Kawabata, and Michael Schreiber, Philos. Mag. B **65**, 595 (1992), and references therein.