Linear stability analysis for propagating fracture

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To study the stability of mode I (opening-mode) fracture, we consider a two-dimensional system in which a crack moves along the centerline of a very wide, infinitely long strip. We compute the first-order response of the crack to a spatially periodic, perturbing shear stress. We assume isotropic linear elasticity in the strip and a cohesive-zone model of the crack tip. The behavior of this system is strongly sensitive to the dynamics within the cohesive zone; stability cannot be deduced simply from properties of the far-field stress-intensity factors. When the mode I and mode II (sliding-mode) fracture energies are equal, the crack is marginally stable at zero speed and is unstable against deflection at all nonzero speeds. However, when the cohesive stress has a shear component that strongly resists bending into mode II, there is a nonvanishing critical velocity for the onset of instability.

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I. INTRODUCTION AND SUMMARY OF RESULTS

The need for an analytic understanding of the dynamic stability of propagating fracture has become especially urgent in the past several years. This renewed interest in an old subject is largely the result of remarkable experiments by Fineberg *et al.* [1,2] and also has been stimulated by numerical experiments by Abraham *et al.* [3] in which similar phenomena are observed. Our purpose in this paper is to describe a mathematical approach to the study of fracture stability and to use this approach to show that crack propagation in a broad class of conventional cohesive-zone models is strongly unstable. In developing and interpreting this result, we point out some important fundamental difficulties in modern theories of fracture dynamics.

The analysis presented here is quite complicated. It is useful, therefore, to start with a summary of our strategy and main results.

Both the laboratory and numerical experiments appear to indicate that propagating mode I (opening-mode) [4] cracks in a variety of materials encounter some kind of oscillatory instability at velocities appreciably below the Rayleigh speed. Motions at speeds higher than the onset velocity for the instability are dissipative; the oscillation radiates elastic energy and, as seen in the experiments, induces deformations near the fracture surface. It is not clear whether the onset velocity is an absolute upper bound for the propagation speed or is simply a speed above which further acceleration becomes much more difficult. The experiments also seem to indicate that the oscillation involves the direction of propagation and not just the speed. That is, the crack moves along an irregular trajectory through the solid.

A basic idea of why such an instability might occur was suggested many years ago by Yoffe [5]. (See Freund's book [6] for a more modern treatment.) The idea is conceptually quite simple. Consider an idealized crack in the form of a simple straight slit in a two-dimensional linear elastic material and suppose, without asking for a dynamic explanation, that the tip of this crack is moving forward at a constant speed v. The singular stress field near the tip can be obtained by transforming the equations of elasticity into the moving frame. The result, in effect, is a Lorentz transformation of the field that deforms the pattern of longitudinal and shear stresses. Yoffe observed that, at some critical v less than the Rayleigh speed v_R, the maximum in the component of the stress that pulls the fracture surfaces apart from each other shifts away from the forward direction. In effect, the Lorentz-Fitzgerald contraction enhances the stress tangential to the direction of crack motion. At speeds higher than the critical v, this maximum occurs at an angle of about 60° on either side of the forward direction; it is tempting to speculate that the crack might undergo a directional instability under these conditions.

The shortcoming of the Yoffe analysis is that it contains no consideration of the forces required to keep the crack moving or any prediction of how the crack actually responds to those forces. A real crack cannot support the singular stresses at the tip that are produced by a simple slit. There must be some mechanism by which these stresses are regularized, for example, by blunting of the tip or by deformation of the material in its neighborhood. Yoffe's singular stress cannot be and indeed is not the actual stress acting at the physical tip of the crack; therefore, it is not clear how or whether the Yoffe stress can determine the dynamic response of the system. It is that area of uncertainty that we address in this paper.

In order to construct a fully dynamic theory of fracture, we need a fully dynamic model, that is, a model for which the equations of motion are complete and mathematically well posed. So far as we know, the simplest such possibility is the cohesive-zone model of Barenblatt [7] and Dugdale [8]. (For a substantially different approach to this problem, see recent work by Marder [9].) We consider a two-dimensional isotropic material in either plane stress or plane strain and assume that linear elasticity is valid everywhere up to the fracture surface, including at the tip. The cohesive force in the neighborhood of the tip provides a fracture energy and a mechanism for regularizing the stress singularity. We emphasize that the use of such a model removes the need to speculate about whether some far-field condition such as the vanishing of $K_{\rm II}$, the mode II (sliding-mode) [4] stress-intensity factor, might determine the direction in which the crack extends. As we shall see, the elementary condition that the stresses at the crack tip contain no nonphysical singularities is sufficient to determine both the geometry of the cohesive zone and the direction of motion. Equivalently, the dynamics of these models is determined by the basic principle that the system must move in such a way that the stresses remain nonsingular at all times.

In principle, it is absolutely essential to include some dissipative mechanism in a model of this kind. The point here is simple; it emerged clearly in our recent study of one-dimensional fracture stability [10]. In the absence of dissipation, subsonic steady-state fracture can occur only at exactly the Griffith threshold where all of the stored elastic energy is converted into fracture energy at the crack tip. At this threshold, all speeds between zero and the Rayleigh speed are mathematically possible. A linear stability theory, however, is an analysis of the first-order response of the system to some change in the driving force; but such a calculation makes no sense if only one special driving force is allowed.

Nevertheless, we choose to neglect dissipation in all of the analysis in this paper. Our special justification is that we have confirmed, at least tentatively, that dissipation does not make a qualitative change in the particular results to be presented here and omitting it allows us to avoid adding yet one more complication to an already very complicated presentation. We propose to devote a later paper in this series to a study of dissipative effects in dynamic fracture. More generally, dissipation may not play so important a role for present purposes because we are computing only the change in the trajectory as a response to forces that tend to deflect the crack and these bending forces do not change the energy balance in a linear approximation.

We consider a crack moving along the centerline of an infinite elastic strip occupying the region $(-\infty < x < +\infty, -W < y < +W)$ in the x,y plane. Far ahead of the crack, the strip is uniformly strained by an amount $\varepsilon_{xx} = \varepsilon_{T\infty}$ (the strain "tangential" to the crack axis), $\varepsilon_{yy} = \varepsilon_{N\infty}$ (the "normal" strain), and, for the moment, $\varepsilon_{xy} = 0$. From the beginning of the analysis, we assume that the half-width W is very much larger than any other length scale in the problem, thus we carry out most of our calculations in the limit $W \to \infty$ and impose outgoing-wave boundary conditions at large $|y|$. However, there are several places where we need to reintroduce the length W. For example, the fully relaxed width of the crack must scale like W and the stress-intensity factor that characterizes the forces transmitted to the crack tip is proportional to $\sqrt{W}$. Our technique for determining these W dependences is based on an earlier paper in this series [11].

Our first important result has to do with the stresses at the crack tip. These stresses are entirely nonsingular because of the cohesive force. In fact, the normal stress Σ_{yy} must be exactly equal to the yield stress at the tip. The tangential stress Σ_{xx} is also finite everywhere including at the tip. A simple steady-state calculation, described in Sec. III, indicates that Σ_{xx} exceeds Σ_{yy} everywhere along the x axis by an amount that is very large, of order the yield stress. This stress difference is proportional to v^2 for small crack speeds v and diverges as v approaches the Rayleigh speed v_R. Thus, for any moving crack, the actual tractions on the fracture surface, as opposed to the Yoffe stresses in the asymptotic region of the singular field, always favor motion perpendicular to the original direction of propagation. We believe that this result, by itself, implies intrinsic instability of this class of fracture models. Some parts of this result have appeared in the fracture literature in the past 30 years [12] and important elements of it were included in a recent paper by one of the present authors [13]. So far as we know, however, its generality and importance have not been emphasized before now, certainly not in connection with a fully dynamic stability analysis of the kind to be described here.

A conventional investigation of the dynamic stability of fracture would be extremely difficult. In such an analysis, we should start by linearizing the equations of motion about a steady-state solution and then look to see whether the eigenstates of the resulting linear operator grow or decay exponentially as functions of time. To prove stability, we would have to determine that no member of the complete set of eigenfunctions is a growing mode. That program seems harder than is necessary for present purposes. In fact, we do not know how to implement it for fracture dynamics and it might not provide useful answers even if it could be carried out rigorously and completely. (For a discussion of the limitations of eigenvalue analyses in stability problems, see [14].)

Instead, we have adopted a limited version of linear stability theory that seems to give us the information that we need. Specifically, we compute the steady-state response of our system to a small (i.e., first-order) external force that produces a spatially oscillating shear stress along the x axis:

$$\Sigma_{xy}^{(\rm ext)}(x,0) = \hat{\varepsilon}_m \, e^{imx}. \tag{1.1}$$

Here the dimensionless stress Σ_{xy} is measured in units 2μ, where μ is the elastic modulus, and $\hat{\varepsilon}_m$ is the amplitude of the perturbation whose wave number is m. In Sec. II we define Σ_{xy} in the entire x,y plane so that, in principle, it can be the result of tractions applied at the edges of the strip or of material irregularities near the centerline. The goal of the calculation is to compute the perturbed centerline $y = Y_{\rm cen}(x)$ of the resulting fracture to first order in $\hat{\varepsilon}_m$, that is,

$$Y_{\rm cen}(x) = \hat{Y}_m \, e^{imx} \equiv \hat{\chi}_Y(m,v)\, \hat{\varepsilon}_m \, e^{imx}. \tag{1.2}$$

Here $\hat{\chi}_Y$ is a complex steady-state response coefficient that depends on the wave number m and the average crack speed v. If $\hat{\chi}_Y$ diverges at some v and some real value of m, then we would conclude that the system undergoes a change in dynamic stability at that wave number and speed. More generally, poles of $\hat{\chi}_Y$ in the complex m plane are equivalent to stability eigenvalues. According to (1.2), poles in the lower half m plane correspond to stable modes and changes in stability occur when poles cross the real m axis.

The wavy crack described by (1.2) is similar to that considered by Gao [15], whose work draws on that of Cotterell and Rice [16]. It is also reminiscent of the interesting quasistatic fracture patterns observed by Yuse and Sano [17,18], but is, in fact, quite different because here we are considering very fast, freely propagating fracture. The technique of looking at the first-order response to small perturbations also has been used recently by Rice *et al.* [19,20], who have studied the in-plane stability of a three-dimensional crack. Our work, like that of Gao, focuses entirely on out-of-plane deformations.

Our strategy for computing $\hat{\chi}_Y(m,v)$ is to use steady-state techniques developed in an earlier paper by two of the present authors [21] to calculate terms up to first order in $\hat{\varepsilon}_m$. The crucial ingredient is the condition that all stresses be nonsingular at the crack tip. We start this calculation by transforming the equations of elasticity into a frame of reference moving in the negative x direction at a speed such that the tip of the crack is always at $x' = 0$, that is, $x = x' + x_{\rm tip}(t)$, where

$$\dot{x}_{\rm tip}(t) = -v - \hat{v}_m e^{-imvt}. \tag{1.3}$$

This transformation into a nonuniformly moving frame is essential because it allows us to deal nonperturbatively with the various mathematical singularities that occur at the crack tip. Note that the first-order part of the velocity $\dot{x}_{\rm tip}$ oscillates with a frequency mv and has an as yet unknown complex amplitude $\hat{v}_m$, which, like $\hat{Y}_m$, is proportional to $\hat{\varepsilon}_m$:

$$\hat{v}_m \equiv \hat{\chi}_v(m,v)\, \hat{\varepsilon}_m. \tag{1.4}$$

The next step is to write down formal solutions of the equations of elasticity separately in the two regions of the x,y plane above and below $Y_{\rm cen}(x)$ and then to evaluate the unknown coefficients that occur in those solutions by imposing boundary conditions on the centerline. Ahead of the crack tip, the centerline is purely fictitious and the boundary conditions are simply statements that the stresses and displacements must be continuous there. Behind the tip, on the other hand, these boundary conditions are statements about tractions on the fracture surfaces. Here we must be careful to recognize that the cohesive force, which is exactly normal to the fracture surface for pure mode I fracture, may acquire a shear component when the crack bends away from its initial direction, that is, when it becomes partially mode II. All of these basic elements of the theory are described in Sec. II.

The above combination of elasticity and boundary conditions produces a set of Wiener-Hopf equations that can be solved for the unknown stresses ahead of the tip and the unknown crack-opening displacements behind it. At zeroth order in $\hat{\varepsilon}_m$, the Wiener-Hopf equation is a restatement of the conventional, steady-state, cohesive-zone model. For purposes of completeness, and also to introduce some mathematical devices that are useful later in the paper, we write out the zeroth-order analysis in detail in Secs. III and IV. As a by-product of this analysis, we obtain the information about the stresses on the fracture surface that we described earlier in this Introduction.

Sections V–VII are devoted to the first-order calculations in $\hat{\varepsilon}_m$. This problem separates for reasons of symmetry into two decoupled equations, one of which involves only the tangential displacement along the fracture surface, the shear stress ahead of the crack tip, and the amplitude $\hat{Y}_m$. The second contains only the normal stresses and displacements and the amplitude $\hat{v}_m$. Both equations contain terms proportional to the zeroth-order displacement. Because these equations are decoupled, each can be solved by Wiener-Hopf techniques. Then the unknown amplitudes $\hat{Y}_m$ and $\hat{v}_m$ can be determined uniquely by requiring that the first-order shear and normal stresses be nonsingular at the crack tip. We expect that the most important of these results is the first, i.e., the one that pertains to the amplitude $\hat{Y}_m$, which describes the deformations perpendicular to the direction of propagation that would be affected by some kind of Yoffe instability. Therefore, we shall consider only $\hat{\chi}_Y$, and not $\hat{\chi}_v$, in this paper.

Our final result for the response coefficient $\hat{\chi}_Y$ has the form

$$\hat{\chi}_Y^{-1}(m,v) = -im \left[\Delta\varepsilon_\infty + K_{\rm I}(-im)^{1/2}\, \mathcal{D}(m\ell, v)\right], \tag{1.5}$$

where $\Delta\varepsilon_\infty = \varepsilon_{N\infty} - \varepsilon_{T\infty}$, $K_{\rm I}$ is the static, mode I stress-intensity factor, and $\mathcal{D}(m\ell, v)$ is a function of v and the dimensionless product $m\ell$, ℓ being the length of the cohesive zone at the tip of the crack.

Section VIII is devoted to the interpretation of this formula. We start by showing that (1.5) can be made to reproduce the stability theory of Cotterell and Rice [16] by taking the limit $m\ell \to 0$, which is equivalent to ignoring all dynamic effects on length scales comparable to ℓ and, consequently, using $K_{\rm II} = 0$ as the determining condition for crack extension. Closer inspection of $\mathcal{D}(m\ell, v)$ for nonzero $m\ell$, however, reveals that the crack that actually is being described in the far-field calculation is strongly unstable. A correct analysis requires careful attention to the details of the cohesive forces. In particular, it turns out that a strong cohesive-shear force is required in order to produce stability even at small speeds v in this model.

Section VIII is written in such a way as to be readable without a detailed understanding of the derivation of the equations or even the definitions of terms other than those discussed in this Introduction. Sections II–V contain the basic theory that is essential for understanding the strategy being employed here. Sections VI and VII are devoted to details of the mathematical methods used in evaluating the formulas derived in the previous sections.

II. DEFINITIONS AND BASIC EQUATIONS

For completeness, and in order to establish our notation, we start by writing the equations of motion for an isotropic elastic material. We use the elastic potentials $\Phi(x,y)$ and $\Psi(x,y)$. In terms of these functions, the displacements $u_x(x,y)$ and $u_y(x,y)$ are

$$u_x = \frac{\partial\Phi}{\partial x} + \frac{\partial\Psi}{\partial y}, \qquad u_y = \frac{\partial\Phi}{\partial y} - \frac{\partial\Psi}{\partial x} \tag{2.1}$$

and the stresses (measured in units 2μ) are

$$\Sigma_{xx} \equiv \frac{\sigma_{xx}}{2\mu} = \left(\frac{\kappa}{2}\right)\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\kappa}{2}-1\right)\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Psi}{\partial x\partial y}, \tag{2.2a}$$

$$\Sigma_{yy} \equiv \frac{\sigma_{yy}}{2\mu} = \left(\frac{\kappa}{2}-1\right)\frac{\partial^2\Phi}{\partial x^2} + \left(\frac{\kappa}{2}\right)\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x\partial y}, \tag{2.2b}$$

$$\Sigma_{xy} \equiv \frac{\sigma_{xy}}{2\mu} = \frac{\partial^2\Phi}{\partial x\partial y} + \frac{1}{2}\frac{\partial^2\Psi}{\partial y^2} - \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}. \tag{2.2c}$$

The parameter κ is the square of the ratio of the longitudinal to transverse sound speeds, which is a function of the Poisson ratio ν,

$$\frac{\kappa}{2} = \begin{cases} \dfrac{1-\nu}{1-2\nu}, & \text{plane strain} \\ \dfrac{1}{1-\nu}, & \text{plane stress.} \end{cases} \tag{2.3}$$

The functions Φ and Ψ can be written in the form

$$\Phi(x,y,t) = \frac{1}{2}\left(\varepsilon_{T\infty}x^2 + \varepsilon_{N\infty}y^2\right) + \frac{\hat{\varepsilon}_m\, y}{im}\, e^{imx} + \phi(x,y,t), \tag{2.4a}$$

$$\Psi(x,y,t) = \psi(x,y,t), \tag{2.4b}$$

where ϕ and ψ satisfy the wave equations

$$\ddot{\phi} = \kappa\nabla^2\phi, \qquad \ddot{\psi} = \nabla^2\psi. \tag{2.5}$$

We have scaled the time t so that the transverse sound speed is unity. Accordingly, the crack speed v and all related quantities with dimensions of velocity are expressed here in units of the transverse sound speed.

The first two terms on the right-hand side of (2.4a) produce, respectively, the uniform tensile stress that drives the crack and the oscillating shear stress (1.1) that perturbs the rectilinear motion. The latter already is written in an approximate form that is legal because we consider only first-order deviations of the crack away from the x axis and also because we do not need to be specific about the mechanism that causes the perturbing shear stress. If this stress is produced by tractions at the edges of the strip, for example, a correct form for this term would be the harmonic function $(\hat{\varepsilon}_m/im^2)\,\sinh(my)\,\exp(imx)$; but the exponential growth at large y is not necessary and, in any case, is irrelevant for our purposes. The fields ϕ and ψ are the changes in Φ and Ψ caused by the presence of the crack. So long as we do not immediately look for static solutions by setting $v = 0$, we may use the decoupled wave equations (2.5). Our results will be correct in the limit $v \to 0$.

The next step is to transform the wave equations (2.5) into a frame of reference moving with the tip of the crack and to look for steady-state solutions in this frame. We write ϕ and ψ in the forms $\phi(x - x_{\rm tip}, y, t)$ and $\psi(x - x_{\rm tip}, y, t)$ so that the tip is always at $x' = x - x_{\rm tip} = 0$. Then, for simplicity, we redefine $x' \to x$. The transformed equations are

$$\ddot{\phi} - 2\dot{x}_{\rm tip}\frac{\partial\dot{\phi}}{\partial x} + \dot{x}_{\rm tip}^2\frac{\partial^2\phi}{\partial x^2} - \ddot{x}_{\rm tip}\frac{\partial\phi}{\partial x} = \kappa\frac{\partial^2\phi}{\partial x^2} + \kappa\frac{\partial^2\phi}{\partial y^2}, \tag{2.6a}$$

$$\ddot{\psi} - 2\dot{x}_{\rm tip}\frac{\partial\dot{\psi}}{\partial x} + \dot{x}_{\rm tip}^2\frac{\partial^2\psi}{\partial x^2} - \ddot{x}_{\rm tip}\frac{\partial\psi}{\partial x} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}. \tag{2.6b}$$

In principle, one might also transform to the frame that moves with the tip in the y direction. We have done this and find that it is not necessary for present purposes.

At this point it is convenient to separate the problem into parts that are zeroth and first order in the perturbation $\hat{\varepsilon}_m$. We do this by using (1.3) and by noting that the only explicit time dependence remaining in ϕ and ψ must be a first-order oscillation of frequency mv. Therefore we write

$$\phi(x,y,t) = \phi_0(x,y) + \phi_1(x,y)\,e^{-imvt}, \tag{2.7a}$$

$$\psi(x,y,t) = \psi_0(x,y) + \psi_1(x,y)\,e^{-imvt}. \tag{2.7b}$$

At zeroth order, ϕ_0 and ψ_0 satisfy

$$\beta_l^2\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial^2\phi_0}{\partial y^2} = 0, \qquad \beta_t^2\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial y^2} = 0, \tag{2.8}$$

where

$$\beta_l^2 \equiv 1 - \frac{v^2}{\kappa}, \qquad \beta_t^2 \equiv 1 - v^2. \tag{2.9}$$

The first-order equations are

$$\beta_l^2 \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} + \frac{2imv^2}{\kappa}\frac{\partial\phi_1}{\partial x} + \frac{m^2v^2}{\kappa}\phi_1 = \frac{2v\hat{v}_m}{\kappa}\frac{\partial^2\phi_0}{\partial x^2} - \frac{imv\hat{v}_m}{\kappa}\frac{\partial\phi_0}{\partial x}, \quad (2.10a)$$

$$\beta_t^2 \frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} + 2imv^2\frac{\partial\psi_1}{\partial x} + m^2v^2\psi_1 = 2v\hat{v}_m\frac{\partial^2\psi_0}{\partial x^2} - imv\hat{v}_m\frac{\partial\psi_0}{\partial x}. \quad (2.10b)$$

The general solutions of (2.8) are

$$\phi_0^{[\pm]}(x,y) = \int \frac{dk}{2\pi}\, \hat{\phi}_0^{[\pm]}(k)\, e^{\mp\beta_l|k|y+ikx}, \quad (2.11a)$$

$$\psi_0^{[\pm]}(x,y) = \int \frac{dk}{2\pi}\, \hat{\psi}_0^{[\pm]}(k)\, e^{\mp\beta_t|k|y+ikx}, \quad (2.11b)$$

where the superscripts in square brackets $[\pm]$ mean that the fields pertain to regions above or below the centerline, $y > Y_{\rm cen}(x)$ or $y < Y_{\rm cen}(x)$. Symbols with carets such as $\hat{\varepsilon}_m$ or $\hat{\phi}_0^{[\pm]}(k)$, as always throughout this paper, denote Fourier amplitudes. Note that we include only decaying exponentials in (2.11), which means that we are immediately going to the limit in which the width of the strip W is very large. We shall return to this point shortly. Using (2.11) to evaluate the inhomogeneous terms on the right-hand sides of (2.10), we find first-order solutions of the form

$$\phi_1^{[\pm]}(x,y) = \int \frac{dk}{2\pi}\left[\hat{\phi}_1^{[\pm]}(k)\, e^{\mp q_l y} + \frac{\hat{v}_m k}{mv}\hat{\phi}_0^{[\pm]}(k)\, e^{\mp\beta_l|k|y}\right] e^{ikx}, \quad (2.12a)$$

$$\psi_1^{[\pm]}(x,y) = \int \frac{dk}{2\pi}\left[\hat{\psi}_1^{[\pm]}(k)\, e^{\mp q_t y} + \frac{\hat{v}_m k}{mv}\hat{\psi}_0^{[\pm]}(k)\, e^{\mp\beta_t|k|y}\right] e^{ikx}, \quad (2.12b)$$

where

$$q_l^2 = k^2 - \frac{v^2}{\kappa}(k-m)^2, \qquad q_t^2 = k^2 - v^2(k-m)^2. \quad (2.13)$$

Again, we include only decaying exponentials, so that we choose the real parts of q_l and q_t to be non-negative. For this oscillatory part of the field, we also must specify outgoing-wave conditions; that is, we must choose the imaginary parts of q_l and q_t to be nonpositive.

Starting in Sec. III, we shall use the above expressions to evaluate the displacements and stresses along the centerline $Y_{\rm cen}(x)$. To do this, we first must refer these vector and tensor quantities to coordinate axes, say, x' and y', that are normal and tangential to the local orientation of the crack. To first order in $Y_{\rm cen}$, the normal displacement is

$$u_{y'}^{[\pm]}(x,Y_{\rm cen}) \cong u_y^{[\pm]}(x,0) + \left.\frac{\partial u_y^{[\pm]}}{\partial y}\right|_{y=0} Y_{\rm cen}(x) - u_x^{[\pm]}(x,0)\frac{dY_{\rm cen}}{dx}$$
$$= \left[\frac{\partial\Phi^{[\pm]}}{\partial y} - \frac{\partial\Psi^{[\pm]}}{\partial x}\right]_{y=0} + \left[\left(\frac{\partial^2\Phi^{[\pm]}}{\partial y^2} - \frac{\partial^2\Psi^{[\pm]}}{\partial x\partial y}\right) - im\left(\frac{\partial\Phi^{[\pm]}}{\partial x} + \frac{\partial\Psi^{[\pm]}}{\partial y}\right)\right]_{y=0} \hat{Y}_m\, e^{imx-imvt} \quad (2.14)$$

and the tangential displacement is

$$u_{x'}^{[\pm]}(x,Y_{\rm cen}) \cong u_x^{[\pm]}(x,0) + \left.\frac{\partial u_x^{[\pm]}}{\partial y}\right|_{y=0} Y_{\rm cen}(x) + u_y^{[\pm]}(x,0)\frac{dY_{\rm cen}}{dx}$$
$$= \left[\frac{\partial\Phi^{[\pm]}}{\partial x} + \frac{\partial\Psi^{[\pm]}}{\partial y}\right]_{y=0} + \left[\left(\frac{\partial^2\Phi^{[\pm]}}{\partial x\partial y} + \frac{\partial^2\Psi^{[\pm]}}{\partial y^2}\right) + im\left(\frac{\partial\Phi^{[\pm]}}{\partial y} - \frac{\partial\Psi^{[\pm]}}{\partial x}\right)\right]_{y=0} \hat{Y}_m\, e^{imx-imvt}. \quad (2.15)$$

We need similar expressions for the normal stress

$$\Sigma_{y'y'}^{[\pm]}(x,Y_{\rm cen}) \cong \Sigma_{yy}^{[\pm]}(x,Y_{\rm cen}) - 2\Sigma_{xy}^{[\pm]}(x,0)\frac{dY_{\rm cen}}{dx}$$
$$\cong \left[\left(\frac{\kappa}{2}-1\right)\frac{\partial^2\Phi^{[\pm]}}{\partial x^2} + \left(\frac{\kappa}{2}\right)\frac{\partial^2\Phi^{[\pm]}}{\partial y^2} - \frac{\partial^2\Psi^{[\pm]}}{\partial x\partial y}\right]_{y=0}$$
$$+ \left[\left(\frac{\kappa}{2}-1\right)\frac{\partial^3\Phi^{[\pm]}}{\partial x^2\partial y} + \left(\frac{\kappa}{2}\right)\frac{\partial^3\Phi^{[\pm]}}{\partial y^3} - \frac{\partial^3\Psi^{[\pm]}}{\partial x\partial y^2}\right]_{y=0} (\hat{Y}_m\, e^{imx-imvt}) \quad (2.16)$$

and for the shear stress

$$\begin{aligned}\Sigma^{[\pm]}_{x'y'}(x,Y_{\rm cen}) &\cong \Sigma^{[\pm]}_{xy}(x,Y_{\rm cen}) + [\Sigma^{[\pm]}_{yy}(x,0)-\Sigma^{[\pm]}_{xx}(x,0)]\frac{dY_{\rm cen}}{dx}\\ &\cong \left[\frac{\partial^2\Phi^{[\pm]}}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2\Psi^{[\pm]}}{\partial y^2}-\frac{1}{2}\frac{\partial^2\Psi^{[\pm]}}{\partial x^2}\right]_{y=0} + \left[\left(\frac{\partial^3\Phi^{[\pm]}}{\partial x\partial y^2}+\frac{1}{2}\frac{\partial^3\Psi^{[\pm]}}{\partial y^3}-\frac{1}{2}\frac{\partial^3\Psi^{[\pm]}}{\partial x^2\partial y}\right)\right.\\ &\quad \left. +im\left(\frac{\partial^2\Phi^{[\pm]}}{\partial y^2}-\frac{\partial^2\Phi^{[\pm]}}{\partial x^2}-2\frac{\partial^2\Psi^{[\pm]}}{\partial x\partial y}\right)\right]_{y=0}\hat{Y}_m\, e^{imx-imvt}. \end{aligned} \tag{2.17}$$

In the second form of (2.16), we have used the fact that $\Sigma^{[\pm]}_{xy}(x,0)$ is automatically of first order in $\hat{\varepsilon}_m$ and therefore makes only a second-order contribution to $\Sigma^{[\pm]}_{y'y'}(x,Y_{\rm cen})$. The analogous expression for $\Sigma^{[\pm]}_{x'x'}(x,Y_{\rm cen})$ is easily derived, but will not be needed beyond zeroth order for present purposes.

The quantities that have physical significance in our analysis are sums and differences of the $[\pm]$ displacements and stresses defined above. First, there are the normal (mode I) crack-opening displacement

$$U_N(x) \equiv \frac{1}{2}[u^{[+]}_{y'}(x,Y_{\rm cen}) - u^{[-]}_{y'}(x,Y_{\rm cen})] \tag{2.18a}$$

and the shear displacement

$$U_S(x) \equiv \frac{1}{2}[u^{[+]}_{x'}(x,Y_{\rm cen}) - u^{[-]}_{x'}(x,Y_{\rm cen})]. \tag{2.18b}$$

The analog of (2.18a) with a sum instead of a difference in the square brackets plays no role in what follows and, in any case, must vanish to first order if we preserve symmetry about the centerline. Similarly, we have no need here for the average tangential displacement $U_T(x)$, the analog of (2.18b) with a plus sign in the brackets; but this is a nonvanishing quantity and could play some role in extensions of the present calculations.

The relevant stresses are the normal stress

$$\Sigma_N(x) \equiv \frac{1}{2}[\Sigma^{[+]}_{y'y'}(x,Y_{\rm cen}) + \Sigma^{[-]}_{y'y'}(x,Y_{\rm cen})], \tag{2.19a}$$

the shear stress

$$\Sigma_S(x) \equiv \frac{1}{2}[\Sigma^{[+]}_{x'y'}(x,Y_{\rm cen}) + \Sigma^{[-]}_{x'y'}(x,Y_{\rm cen})], \tag{2.19b}$$

and the tangential stress

$$\Sigma_T(x) \equiv \frac{1}{2}[\Sigma^{[+]}_{x'x'}(x,Y_{\rm cen}) + \Sigma^{[-]}_{x'x'}(x,Y_{\rm cen})]. \tag{2.19c}$$

We also note that the analogous expressions with differences instead of sums in the square brackets vanish:

$$\delta\Sigma_A = 0, \tag{2.20}$$

where the subscript A denotes N, S (but not T), and, for example,

$$\delta\Sigma_N(x) \equiv \frac{1}{2}[\Sigma^{[+]}_{y'y'}(x,Y_{\rm cen}) - \Sigma^{[-]}_{y'y'}(x,Y_{\rm cen})]. \tag{2.21}$$

While establishing notation, it is convenient to rewrite the Fourier amplitudes of the potential fields that appear in (2.11) and (2.12) in forms consistent with (2.18) and (2.19). The following notation is useful:

$$\hat{\phi}_N(k) \equiv \frac{1}{2}[\hat{\phi}^{[+]}(k) + \hat{\phi}^{[-]}(k)],$$

$$\hat{\phi}_S(k) \equiv \frac{1}{2}[\hat{\phi}^{[+]}(k) - \hat{\phi}^{[-]}(k)] \tag{2.22a}$$

and

$$\hat{\psi}_N(k) \equiv \frac{1}{2}[\hat{\psi}^{[+]}(k) - \hat{\psi}^{[-]}(k)] \; ,$$

$$\hat{\psi}_S(k) \equiv \frac{1}{2}\left[\hat{\psi}^{[+]}(k) + \hat{\psi}^{[-]}(k)\right]. \tag{2.22b}$$

For simplicity, we have omitted the subscripts $0, 1$ that distinguish between zeroth- and first-order quantities in these equations.

Having defined the displacements and stresses in the previous paragraphs, we now can complete the definition of our model by specifying the cohesive stress. We are allowed considerable latitude in this regard. For reasons that will become apparent later, it is especially important to look at the case in which this stress is produced by a central force acting between opposite points on the fracture surfaces. Let the magnitude of this central force per unit area be $\Sigma_c\{|U(x)|\}$, where $|U(x)| = \sqrt{U_N^2(x)+U_S^2(x)}$. Then its normal and shear components are

$$\Sigma_{cN}(x) = \Sigma_c\{|U(x)|\}\,\frac{U_N(x)}{|U(x)|} \tag{2.23a}$$

and

$$\Sigma_{cS}(x) = \Sigma_c\{|U(x)|\}\,\frac{U_S(x)}{|U(x)|}. \tag{2.23b}$$

For all of the specific calculations to be reported here, we use the simple form

$$\Sigma_c\{|U|\} = \begin{cases}\Sigma_0 & \text{for } 0<|U|<\delta\\ 0 & \text{for } |U|>\delta,\end{cases} \tag{2.24}$$

where δ is the range of the cohesive force and Σ_0 is the yield stress. The fracture energy is $\gamma = \delta\Sigma_0$. Equations (2.23), plus the symmetry condition (2.20) and the defining condition for the location of the crack in the moving frame

$$U_N(x) = U_S(x) = 0, \quad x < 0, \tag{2.25}$$

complete the specification of our model.

III. ZEROTH-ORDER CALCULATIONS: STRESSES ON THE FRACTURE SURFACES

The next step in our analysis is to solve the steady-state problem for the case in which the perturbation $\hat{\varepsilon}_m$ vanishes. Many of the details of this calculation have been presented previously [11,21], but we need to reformulate parts of those analyses for present purposes and, therefore, the explanations that follow will be very nearly self-contained. In this section we derive the zeroth-order Wiener-Hopf equation and demonstrate the important inequality relating to the stresses on the fracture surfaces. In Sec. IV we compute the associated crack-opening displacement.

The zeroth-order part of $U_N(x)$, which we denote by $U_{N0}(x)$, is obtained from the first term on the right-hand side of (2.14) and the zeroth-order solutions (2.11). Using the notation introduced in (2.22), we find

$$\hat{U}_{N0}(k) = -\beta_l |k| \hat{\phi}_{N0}(k) - ik\,\hat{\psi}_{N0}(k). \tag{3.1}$$

Similarly, from the first term in (2.16), the zeroth-order normal stress is

$$\hat{\Sigma}_{N0}(k) = 2\pi\Sigma_{N\infty}\delta(k) + k^2\beta_0^2\hat{\phi}_{N0}(k) + i\beta_t k|k|\,\hat{\psi}_{N0}(k), \tag{3.2}$$

where

$$\Sigma_{N\infty} \equiv \left(\frac{\kappa}{2}\right)\varepsilon_{N\infty} + \left(\frac{\kappa}{2} - 1\right)\varepsilon_{T\infty} \tag{3.3}$$

and

$$\beta_0^2 \equiv 1 - \frac{v^2}{2}. \tag{3.4}$$

We also need the symmetry and continuity conditions

$$\delta\hat{\Sigma}_{S0}(k) = -i\beta_l k|k|\hat{\phi}_{N0}(k) + k^2\beta_0^2\hat{\psi}_{N0}(k) = 0, \tag{3.5}$$

$$\hat{U}_{S0}(k) = ik\hat{\phi}_{S0}(k) - \beta_t|k|\hat{\psi}_{S0}(k) = 0, \tag{3.6}$$

and

$$\delta\hat{\Sigma}_{N0}(k) = k^2\beta_0^2\hat{\phi}_{S0}(k) + i\beta_t k|k|\hat{\psi}_{S0}(k) = 0. \tag{3.7}$$

Equations (3.6) and (3.7) immediately tell us that

$$\hat{\phi}_{S0}(k) = \hat{\psi}_{S0}(k) = 0. \tag{3.8}$$

The elimination of $\hat{\phi}_{N0}(k)$ and $\hat{\psi}_{N0}(k)$ from (3.1), (3.2), and (3.5) produces a relation between $\hat{\Sigma}_{N0}(k)$ and $\hat{U}_{N0}(k)$. The result is

$$\begin{aligned}\hat{\Sigma}_{N0}(k) &= \hat{\Sigma}_{c0}^{(+)}(k) + \hat{\Sigma}_{N0}^{(-)}(k) \\ &= 2\pi\Sigma_{N\infty}\delta(k) - \hat{F}(k)\,\hat{U}_{N0}^{(+)}(k),\end{aligned} \tag{3.9}$$

where

$$\hat{F}(k) = b(v)|k|, \qquad b(v) = \frac{2}{\beta_l v^2}(\beta_l\beta_t - \beta_0^4). \tag{3.10}$$

The function $\hat{\Sigma}_{N0}^{(-)}(k)$ is the Fourier transform of the unknown zeroth-order stress in the unbroken region $x < 0$ and $\hat{\Sigma}_{c0}^{(+)}(k)$ is the Fourier transform of the zeroth-order cohesive stress $\Sigma_c\{U_{N0}(x)\}$ as defined in (2.23). We have introduced the superscripts in parentheses $(\pm)$ to indicate functions that have singularities only in the upper (or lower) half k planes because they are Fourier transforms of functions that are nonzero only on the positive (or negative) x axis. In (3.10), note that $b(v)$ vanishes at the Rayleigh speed $b(v_R) = 0$.

Equation (3.9) is the Wiener-Hopf equation that we must solve to compute the zeroth-order crack-opening displacement $U_{N0}(x)$. Before doing that, however, we look at the analog of (3.9) for the zeroth-order tangential stress $\Sigma_{T0}(x)$, defined in (2.19c). Using (2.2a) and the expressions for the potential fields obtained from (3.1) and (3.5), we find

$$\hat{\Sigma}_{T0}(k) = 2\pi\Sigma_{T\infty}\delta(k) + c(v)|k|\,\hat{U}_{N0}^{(+)}(k), \tag{3.11}$$

where

$$c(v) = \frac{2}{v^2\beta_l}\left\{\beta_l\beta_t + \beta_0^2\left[\left(\frac{\kappa}{2} - 1\right)\beta_l^2 - \left(\frac{\kappa}{2}\right)\right]\right\} \tag{3.12}$$

and

$$\Sigma_{T\infty} \equiv \left(\frac{\kappa}{2}\right)\varepsilon_{T\infty} + \left(\frac{\kappa}{2} - 1\right)\varepsilon_{N\infty}. \tag{3.13}$$

We now can subtract (3.9) from (3.11) and invert the Fourier transform to obtain the difference between the T and N stresses anywhere along the x axis:

$$\Sigma_{T0}(x) - \Sigma_{N0}(x)$$

$$= -\Delta\varepsilon_\infty - A(v)\,b(v)\int\frac{dk}{2\pi}\,e^{-ikx}\,|k|\,\hat{U}_{N0}^{(+)}(k), \tag{3.14}$$

where $\Delta\varepsilon_\infty = \varepsilon_{N\infty} - \varepsilon_{T\infty}$ and

$$A(v) = \frac{(2 - v^2/\kappa)(1 - v^2/2) - 2\beta_l\beta_t}{\beta_l\beta_t - \beta_0^4}. \tag{3.15}$$

This is quite a remarkable result. In our cohesive-zone model, the normal stress $\Sigma_{N0}(x)$ remains finite and nonnegative everywhere along the x axis including at the crack tip, where it is exactly equal to the yield stress Σ_0. Equation (3.9) tells us that

$$-b(v)\int\frac{dk}{2\pi}\,e^{-ikx}\,|k|\,\hat{U}_{N0}^{(+)}(k) = \Sigma_{N0}(x) - \Sigma_{N\infty}. \tag{3.16}$$

We also know that the quantities $\Delta\varepsilon_\infty$ and $\Sigma_{N\infty}$ are small of order $W^{-1/2}$ in comparison to Σ_0. Thus, everywhere in the neighborhood of the crack tip,

$$\Sigma_{T0}(x) - \Sigma_{N0}(x) \approx A(v)\,\Sigma_{N0}(x). \tag{3.17}$$

The function $A(v)$ is positive for all v less than the Rayleigh speed. For small v,

$$A(v) \approx \left(\frac{1+1/\kappa^2}{1-1/\kappa}\right)\frac{v^2}{2} \tag{3.18}$$

and, for v approaching the Rayleigh speed, $A(v)$ diverges like $(v_R - v)^{-1}$. As advertised in the Introduction, we conclude from (3.18) that the tangential stress, which deflects the crack away from the x direction, exceeds the normal stress on the fracture surface throughout the tip region and at all nonzero velocities. It is only for the static crack that the two stresses are equal. We interpret this inequality as strong evidence that the moving crack is unstable against perturbations that bend it away from the x axis — a conclusion that is supported by our dynamic stability analysis.

IV. ZEROTH-ORDER CALCULATIONS: CRACK-OPENING DISPLACEMENT

To complete the analysis at zeroth order in $\hat{\varepsilon}_m$, we return to (3.9) to compute $U_{N0}(x)$, which we shall need later in an explicit form. At this point we must pay attention to our assumption that the width of the strip W is much larger than any other length in the problem. In the Fourier representation, that condition is $kW \gg 1$. The one length scale that does not quite satisfy this condition is the width of the fully open crack, which scales like W for fixed $\varepsilon_{N\infty}$ or, more appropriately, like $W^{1/2}$ for fixed stress-intensity factor.

The issue of the W dependence arises because, in (3.9), we need a modification of the Wiener-Hopf kernel $\hat{F}(k) = b(v)|k|$ that will be valid for very small k. We can compute the exact value of $\hat{F}(0)$ by looking infinitely far behind the tip ($x \to +\infty$) where the crack is fully open everywhere and $U_{N0}(\infty)/W = \varepsilon_{N\infty} + (1 - \frac{2}{\kappa})\varepsilon_{T\infty}$ and $\Sigma_{N0}(\infty) = \Sigma_{N\infty} - \hat{F}(0)U_{N0}(\infty) = 0$. Thus

$$\hat{F}(0) = \frac{\Sigma_{N\infty}}{U_{N0}(\infty)} = \frac{\kappa}{2W}. \tag{4.1}$$

Then, in analogy to Ref. [11], we make the simple interpolation

$$\hat{F}(k) \approx \left[\left(\frac{\kappa}{2W}\right)^2 + b^2(v)k^2\right]^{1/2} \equiv b(v)\,(\alpha^2 + k^2)^{1/2}, \tag{4.2}$$

where

$$\alpha \equiv \frac{\kappa}{2\,b(v)\,W}. \tag{4.3}$$

The Wiener-Hopf factorization is

$$\hat{F}(k) = \hat{F}^{(+)}(k)\,\hat{F}^{(-)}(k), \tag{4.4}$$

with

$$\hat{F}^{(+)}(k) = b(v)\,(\alpha + ik)^{1/2}, \quad \hat{F}^{(-)}(k) = (\alpha - ik)^{1/2}. \tag{4.5}$$

Equation (3.9) now can be recast in the form

$$-ik\,\hat{F}^{(+)}(k)\hat{U}_{N0}^{(+)}(k) - ik\,\hat{\Lambda}_{c0}^{(+)}(k) + \frac{\Sigma_{N\infty}}{\hat{F}^{(-)}(0)} = \frac{ik\,\hat{\Sigma}_{N0}^{(-)}(k)}{\hat{F}^{(-)}(k)} + \frac{\Sigma_{N\infty}}{\hat{F}^{(-)}(0)} + ik\,\hat{\Lambda}_{c0}^{(-)}(k) = 0, \tag{4.6}$$

where the functions

$$\hat{\Lambda}_{c0}^{(\pm)}(k) = \mp\int\frac{dk'}{2\pi i}\,\frac{1}{k'-k\pm i\epsilon}\,\frac{\hat{\Sigma}_{c0}^{(+)}(k')}{\hat{F}^{(-)}(k')} \tag{4.7}$$

occur because we have used the Cauchy formula to decompose the ratio $\hat{\Sigma}_{c0}^{(+)}(k)/\hat{F}^{(-)}(k)$ into a sum of (+) and (−) terms. (Here, and throughout this paper, the symbol ϵ denotes an infinitesimally small positive quantity.) The fact that both the (+) and (−) sides of (4.6) are equal to zero is a result of (4.1) and the relations that precede it. The formal solutions of (3.9) are therefore

$$ik\,\hat{U}_{N0}^{(+)}(k) = \frac{1}{\hat{F}^{(+)}(k)}\left[\frac{\Sigma_{N\infty}}{\hat{F}^{(-)}(0)} - ik\,\hat{\Lambda}_{c0}^{(+)}(k)\right] \tag{4.8}$$

and

$$ik\,\hat{\Sigma}_{N0}^{(-)}(k) = -\hat{F}^{(-)}(k)\left[\frac{\Sigma_{N\infty}}{\hat{F}^{(-)}(0)} + ik\,\hat{\Lambda}_{c0}^{(-)}(k)\right]. \tag{4.9}$$

A quick way to obtain the Barenblatt condition for nonsingular stress is the following. In (4.9), the factor $\hat{F}^{(-)}(k) \sim k^{1/2}$ for large k produces the usual $|x|^{-1/2}$ singularity in the stress at small $|x|$. Both terms inside the square brackets in (4.9) are constants at large k; thus the Barenblatt condition is simply the requirement that these constants cancel each other. Specifically,

$$\Sigma_{N\infty} = \hat{F}^{(-)}(0)\int\frac{dk}{2\pi}\,\frac{\hat{\Sigma}_{c0}^{(+)}(k)}{\hat{F}^{(-)}(k)} = \left[\frac{\kappa}{2\pi\,b(v)\,W}\right]^{1/2}\int_0^\infty\frac{dx}{\sqrt{x}}\,\Sigma_c\{U_{N0}(x)\}. \tag{4.10}$$

If we use the special form for Σ_c given in (2.24), then we can define ℓ, the length of the cohesive zone, to be the position at which

$$U_{N0}(\ell) = \delta \tag{4.11}$$

and the Barenblatt condition (4.10) becomes a condition for ℓ,

$$\Sigma_{N\infty} = \left[\frac{\kappa}{2\pi\, b(v)\, W}\right]^{1/2} 2\sqrt{\ell}\,\Sigma_0. \tag{4.12}$$

Substituting the k-space version of (4.10) into (4.8), we find

$$ik\,\hat{U}_{N0}^{(+)}(k) = \frac{1}{\hat{F}^{(+)}(k)} \int \frac{dk'}{2\pi}\,\frac{k'}{k'-k+i\epsilon}\,\frac{\hat{\Sigma}_{c0}^{(+)}(k')}{\hat{F}^{(-)}(k')}. \tag{4.13}$$

With (2.24), we can write (4.13) in the form

$$ik\,\hat{U}_{N0}^{(+)}(k) = \frac{2\,\Sigma_0}{b(v)}\sqrt{\frac{\ell}{\pi}}\,\frac{1}{(\alpha+ik)^{1/2}}\int_0^1 dw\, e^{-ik\ell(1-w^2)}, \tag{4.14}$$

which is valid in the limit $\ell/W \to 0$ but for arbitrarily large $k\ell$. A useful result obtained by inverting the Fourier transform (4.14) is, for $0 < x < \ell$,

$$\frac{dU_{N0}(x)}{dx} = \frac{\Sigma_0}{\pi b(v)}\ln\left(\frac{1+\sqrt{\frac{x}{\ell}}}{1-\sqrt{\frac{x}{\ell}}}\right). \tag{4.15}$$

Integration over x yields

$$U_{N0}(\ell) = \frac{2\Sigma_0\ell}{\pi b(v)} = \delta. \tag{4.16}$$

Combining (4.16) with (4.12), we find

$$\Sigma_{N\infty} = \left(\frac{\kappa\,\gamma}{W}\right)^{1/2} = \Sigma_G, \tag{4.17}$$

where $\gamma = \Sigma_0\,\delta$ is the fracture energy and Σ_G is the Griffith threshold stress. As anticipated, in the absence of any dissipative mechanism, steady-state solutions exist only at threshold where energy balance is possible.

V. FIRST-ORDER CALCULATIONS: A FORMALLY EXACT EXPRESSION FOR $\hat{\chi}_Y$

The first-order calculation separates naturally into parts that involve only the normal (N) and shear (S) components. It is the S part that determines the centerline $Y_{\rm cen}(x)$ and therefore is of interest here.

From (2.15), (2.11), and (2.12), we compute the first-order shear displacement

$$\begin{aligned}\hat{U}_{S1}^{(+)}(k) &= ik\,\hat{\phi}_{S1}(k) - q_t\,\hat{\psi}_{S1}(k)\\ &\quad -\{i\beta_l k|k-m|\,\hat{\phi}_{N0}(k-m)\\ &\quad -(k-m)[k-v^2(k-m)]\hat{\psi}_{N0}(k-m)\}\hat{Y}_m.\end{aligned} \tag{5.1}$$

Similarly, from (2.16) and (2.20),

$$\begin{aligned}\delta\hat{\Sigma}_{N1}(k) &= q_0^2\,\hat{\phi}_{S1}(k) + ikq_t\,\hat{\psi}_{S1}(k)\\ &\quad -\{\beta_0^2\beta_l|k-m|^3\,\hat{\phi}_{N0}(k-m)\\ &\quad +i\beta_t^2(k-m)^3\,\hat{\psi}_{N0}(k-m)\}\hat{Y}_m = 0,\end{aligned} \tag{5.2}$$

where

$$q_0^2 \equiv k^2 - \frac{v^2}{2}(k-m)^2. \tag{5.3}$$

Finally, from (2.17),

$$\begin{aligned}\hat{\Sigma}_{S1}(k) &= -ikq_l\,\hat{\phi}_{S1}(k) + q_0^2\,\hat{\psi}_{S1}(k)\\ &\quad +\Big\{i(k-m)^2\Big(k+m-\frac{v^2k}{\kappa}\Big)\hat{\phi}_{N0}(k-m)\\ &\quad -\beta_t(k-m)|k-m|\Big[k+m-\frac{v^2}{2}(k-m)\Big]\\ &\quad \times\hat{\psi}_{N0}(k-m)\Big\}\hat{Y}_m + 2\pi\,\hat{E}_m\delta(k-m),\end{aligned} \tag{5.4}$$

where

$$\hat{E}_m \equiv \hat{\varepsilon}_m + im\Delta\varepsilon_\infty\hat{Y}_m \tag{5.5}$$

and $\Delta\varepsilon_\infty = \varepsilon_{N\infty} - \varepsilon_{T\infty}$. Note that $\hat{\phi}_{N1}$, $\hat{\psi}_{N1}$, and $\hat{v}_m$ do not appear in these equations. The superscripts $(\pm)$ remind us, for example, that the relative shear displacement vanishes ahead of the crack, thus $\hat{U}_{S1}^{(+)}(k)$ has singularities only in the upper half k plane.

Eliminating $\hat{\phi}_{S1}(k)$ and $\hat{\psi}_{S1}(k)$ from the above equations and using (3.2) and (3.5) to evaluate $\hat{\phi}_{N0}$ and $\hat{\psi}_{N0}$ in terms of $\hat{U}_{N0}^{(+)}$, we find

$$\begin{aligned}\hat{\Sigma}_{S1}(k) &= \hat{\Sigma}_{cS1}^{(+)}(k) + \hat{\Sigma}_{S1}^{(-)}(k)\\ &= 2\pi\,\hat{E}_m\delta(k-m) - \hat{G}_S(k)\,\hat{U}_{S1}^{(+)}(k)\\ &\quad +\hat{L}_S(k)\,\hat{U}_{N0}^{(+)}(k-m)\,\hat{Y}_m,\end{aligned} \tag{5.6}$$

where $\hat{\Sigma}_{cS1}^{(+)}(k)$ is the Fourier transform of the first-order shear component of the cohesive stress from (2.23b),

$$\Sigma_{cS1}(x) = \Sigma_c\{U_{N0}(x)\}\,\frac{U_{S1}(x)}{U_{N0}(x)}, \tag{5.7}$$

and $\hat{\Sigma}_{S1}^{(-)}(k)$ is the Fourier transform of the shear stress in the unbroken region $x < 0$. The Wiener-Hopf kernel — the analog of $\hat{F}(k)$ in (3.10) — is

$$\hat{G}_S(k) = \frac{2}{q_t v^2(k-m)^2}\Big(k^2q_lq_t - q_0^4\Big) \tag{5.8}$$

and

$$\begin{aligned}\hat{L}_S(k) &= \frac{2\,ikq_l}{v^2}\Big[\Big(\frac{m}{k-m}\Big)^2 - \beta_0^2\Big]\\ &\quad +\frac{2\,ikq_0^2}{q_tv^2}\Big[1-\Big(\frac{m}{k-m}\Big)^2 - v^2\Big(\frac{k-m}{k}\Big)\Big]\\ &\quad +\left(\frac{2\,i}{v^2}\right)|k-m|\left[\frac{\beta_0^2}{\beta_l}\Big(\beta_l^2k+m\Big)\right.\\ &\quad \left.-\beta_t\left(\beta_0^2k+m+\frac{mv^2}{2}\right)\right].\end{aligned} \tag{5.9}$$

Despite appearances, $\hat{G}_S(k)$ and $\hat{L}_S(k)$ are finite at $k =$

m and $v = 0$.

Equation (5.6) is algebraically complicated but is solvable by conventional Wiener-Hopf methods. We start by writing $\hat{G}_S(k)$ in the form $\hat{G}_S^{(+)}(k)\,\hat{G}_S^{(-)}(k)$, where the superscripts ($\pm$) have their usual significance. We then divide both sides of (5.6) by $\hat{G}_S^{(-)}(k)$ and rewrite the result in the form

$$\frac{1}{\hat{G}_S^{(-)}(k)}\left[\hat{\Sigma}_{S1}^{(-)}(k) - \frac{i\hat{E}_m}{k-m+i\epsilon}\right] + \frac{i\hat{E}_m}{k-m-i\epsilon}$$
$$\times\left[\frac{1}{\hat{G}_S^{(-)}(k)} - \frac{1}{\hat{G}_S^{(-)}(m)}\right] - \hat{\Lambda}_m^{(-)}(k)\hat{Y}_m + \hat{\Lambda}_{cS1}^{(-)}(k)$$
$$= -\hat{G}_S^{(+)}(k)\hat{U}_{S1}^{(+)}(k) - \frac{i\hat{E}_m}{k-m-i\epsilon}\left[\frac{1}{\hat{G}_S^{(-)}(m)}\right]$$
$$+\hat{\Lambda}_m^{(+)}(k)\hat{Y}_m - \hat{\Lambda}_{cS1}^{(+)}(k). \tag{5.10}$$

The terms proportional to $\hat{E}_m$ come from writing $\delta(k-m)$ as the difference between poles at $k = m \pm i\epsilon$, $\epsilon \to 0^+$, and then writing the product $\left[(k-m-i\epsilon)\hat{G}_S^{(-)}(k)\right]^{-1}$ as a sum of terms each with singularities only in the upper or lower half k plane. Similarly,

$$\hat{\Lambda}_m^{(\pm)}(k) = \mp\int\frac{dk'}{2\pi i}\,\frac{1}{k'-k\pm i\epsilon}\left[\frac{\hat{L}_S(k')\,\hat{U}_{N0}^{(+)}(k'-m)}{\hat{G}_S^{(-)}(k')}\right] \tag{5.11}$$

and

$$\hat{\Lambda}_{cS1}^{(\pm)}(k) = \mp\int\frac{dk'}{2\pi i}\,\frac{1}{k'-k\pm i\epsilon}\,\frac{\hat{\Sigma}_{cS1}^{(+)}(k')}{\hat{G}_S^{(-)}(k')}. \tag{5.12}$$

All of the terms on the left-hand side of (5.10) are ($-$) terms and those on the right are ($+$); therefore, both sides must be equal to the same entire function of k.

Far ahead of the crack tip, $x \to -\infty$, the shear stress on the center line $Y_{\rm cen}(x)$ is $\hat{E}_m\exp(imx)$; thus

$$\lim_{k\to m}(k-m)\,\hat{\Sigma}_{S1}^{(-)}(k) = i\hat{E}_m. \tag{5.13}$$

It is convenient to eliminate the singularity in (5.10) by multiplying through by $(k-m)$. (In x space, we are taking a derivative.) The resulting equation has no simple poles near the real k axis. Moreover,

$$\lim_{k\to m}(k-m)\,\hat{\Lambda}_m^{(-)}(k) = 0. \tag{5.14}$$

To see this, note that the factor $\hat{U}_{N0}^{(+)}(k'-m)$ in the integrand in (5.11) has a simple pole at $k' = m + i\epsilon$ with a residue proportional to the opening displacement $U_{N0}(\infty)$. The contour of integration passes below both this pole and the one at $k' = k + i\epsilon$, thus $\hat{\Lambda}_m^{(-)}(k)$ has no singularity when $k \to m$. On the other hand, the contour of integration for $\hat{\Lambda}_m^{(+)}(k)$ is pinched between $k' = m + i\epsilon$ and $k' = k - i\epsilon$ as $k \to m$, thus $\hat{\Lambda}_m^{(+)}(k)$ does have a pole here.

Equations (5.13) and (5.14) tell us that, after multiplication by $(k-m)$, the entire left-hand side of (5.10) vanishes as $k \to m$, which means that both sides of this equation vanish for all k. Accordingly, the Wiener-Hopf solutions are

$$i(k-m)\hat{\Sigma}_{S1}^{(-)}(k)$$
$$= -\hat{G}_S^{(-)}(k)\left[\frac{\hat{E}_m}{\hat{G}_S^{(-)}(m)} - i(k-m)\,\hat{\Lambda}_m^{(-)}(k)\hat{Y}_m + i(k-m)\,\hat{\Lambda}_{cS1}^{(-)}(k)\right] \tag{5.15}$$

and

$$i(k-m)\,\hat{U}_{S1}^{(+)}(k)$$
$$= \frac{1}{\hat{G}_S^{(+)}(k)}\left[\frac{\hat{E}_m}{\hat{G}_S^{(-)}(m)} + i(k-m)\,\hat{\Lambda}_m^{(+)}(k)\hat{Y}_m - i(k-m)\,\hat{\Lambda}_{cS1}^{(+)}(k)\right]. \tag{5.16}$$

The next step is the analog of the derivation of the Barenblatt condition in (4.10). That is, just as we implicitly determined ℓ in (4.10) so that the zeroth-order normal stress is nonsingular at the tip, we now choose $\hat{Y}_m$ so that the shear stress is nonsingular. On the right-hand side of (5.15), in the limit of large k, the factor $\hat{G}_S^{(-)}(k)$ is proportional to $k^{1/2}$ (we shall see this explicitly in Sec. VI) and the quantity in square brackets goes to a constant. Therefore, without regularization, the shear stress would diverge like $|x|^{-1/2}$. Accordingly, we regularize the stress by requiring that the large-k limit of the quantity in square brackets be zero, thus fixing the value of $\hat{Y}_m$. The result is

$$\hat{E}_m = \hat{\varepsilon}_m + im\Delta\varepsilon_\infty\hat{Y}_m$$
$$= -\hat{G}_S^{(-)}(m)\int\frac{dk}{2\pi}\,\frac{\hat{L}_S(k)\,\hat{U}_{N0}^{(+)}(k-m)}{\hat{G}_S^{(-)}(k)}\,\hat{Y}_m$$
$$+\hat{G}_S^{(-)}(m)\int\frac{dk}{2\pi}\,\frac{\hat{\Sigma}_{cS1}^{(+)}(k)}{\hat{G}_S^{(-)}(k)}. \tag{5.17}$$

In principle, we can solve (5.17) for $\hat{Y}_m$. To do so in a formal way, we define

$$U_{S1}(x) \equiv u_{S1}(x)\,\hat{Y}_m\,e^{imx}, \qquad \hat{U}_{S1}^{(+)}(k) = \hat{u}_{S1}(k-m)\,\hat{Y}_m \tag{5.18}$$

and then write

$$u_{S1}(x) = A(x)\,U_{N0}(x), \tag{5.19}$$

where $A(x)$ is some smooth function of x that, as we shall confirm below, needs to be defined only in the cohesive zone $0 < x < \ell$. Referring to (2.23) and (2.24), we have

$$\hat{\Sigma}_{cS1}^{(+)}(k) = \Sigma_0 \hat{Y}_m \int_0^\ell dx A(x)\, e^{-i(k-m)x}. \quad (5.20)$$

Our formally exact solution, therefore, is

$$\frac{\hat{\varepsilon}_m}{\hat{Y}_m} = \hat{\chi}_Y^{-1}(m,v) = -im\Delta\varepsilon_\infty - \tilde{\mathcal{D}}(m,v), \quad (5.21)$$

where $\tilde{\mathcal{D}}(m,v) \equiv \tilde{\mathcal{D}}_0(m,v) + \tilde{\mathcal{D}}_1(m,v)$ and

$$\tilde{\mathcal{D}}_0(m,v) = \int \frac{dk}{2\pi} \frac{\hat{G}_S^{(-)}(m)}{\hat{G}_S^{(-)}(k)} \hat{L}_S(k)\, \hat{U}_{N0}^{(+)}(k-m), \quad (5.22a)$$

$$\tilde{\mathcal{D}}_1(m,v) = -\int \frac{dk}{2\pi} \frac{\hat{G}_S^{(-)}(m)}{\hat{G}_S^{(-)}(k)} \Sigma_0 \int_0^\ell dx A(x)\, e^{-i(k-m)x}. \quad (5.22b)$$

The remaining unknown ingredient of (5.22) is the function $A(x)$, which we must compute by solving (5.16) for the shear-displacement function that we need in (5.19). Equation (5.16) is an inhomogeneous linear integral equation that determines $\hat{U}_{S1}^{(+)}(k)$, which appears explicitly on the left-hand side and implicitly via $\hat{\Sigma}_{cS1}^{(+)}(k)$ in $\hat{\Lambda}_{cS1}^{(+)}(k)$. This equation is best rewritten by using the regularization condition (5.17) to eliminate $\hat{E}_m$, so that the quantity in square brackets on the right-hand side explicitly exhibits its correct behavior (k^{-1}) at large k. The result is

$$i(k-m)\,\hat{u}_{S1}^{(+)}(k-m) = \frac{1}{\hat{G}_S^{(+)}(k)} \int \frac{dk'}{2\pi} \frac{k'-m}{k'-k+i\epsilon} \frac{1}{\hat{G}_S^{(-)}(k')} \times \left[\Sigma_0 \int_0^\ell dx A(x)\, e^{-i(k'-m)x} - \hat{L}_S(k')\, \hat{U}_{N0}^{(+)}(k'-m) \right]. \quad (5.23)$$

In Fourier transform, (5.23) becomes

$$\frac{d}{dx}[A(x)\,U_{N0}(x)] + e^{-imx} \int_0^x dx'\,\Gamma_S(x') \int \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{\hat{G}_S^{(-)}(k)}\, i(k-m)\,\Sigma_0 \int_0^\ell dx'' A(x'')\, e^{-i(k-m)x''}$$
$$= e^{-imx} \int_0^x dx'\,\Gamma_S(x') \int \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{\hat{G}_S^{(-)}(k)} i(k-m)\,\hat{L}_S(k)\,\hat{U}_{N0}^{(+)}(k-m), \quad (5.24)$$

where

$$\Gamma_S(x) = \int \frac{dk}{2\pi} \frac{e^{ikx}}{\hat{G}_S^{(+)}(k)}. \quad (5.25)$$

In deriving (5.24), we have used

$$\int \frac{dk}{2\pi i} \frac{e^{ikx}}{\hat{G}_S^{(+)}(k)} \frac{1}{k-k'-i\epsilon} = \begin{cases} \int_0^x dx'\,\Gamma_S(x')\, e^{ik'(x-x')}, & x > 0 \\ 0, & x < 0\,. \end{cases} \quad (5.26)$$

It is clear from the form of (5.24) that $A(x)$ needs to be determined only in $0 < x < \ell$. $A(x)$ is the ratio of the two displacement functions $u_{S1}(x)$ and $U_{N0}(x)$, both of which are proportional to $x^{3/2}$ for small values of x within the cohesive zone. Thus we expect $A(x)$ to be approximately a constant for small x. In fact, we shall argue that essentially all of the physical properties of interest for present purposes emerge from the quantity $A(0)$, which can be thought of as the leading term in a series expansion.

VI. MATHEMATICAL DETAILS

At this point we must pay attention to mathematical details, that is, to evaluating the formulas derived in the preceding section. Readers who wish to go directly to the results should skip directly to Sec. VIII, which starts with a summary of the relevant formulas.

We start by finding explicit expressions for the Wiener-Hopf factors $\hat{G}_S^{(\pm)}(k)$. The kernel $\hat{G}_S(k)$ defined in (5.8) has branch points in the k plane at the zeros of $q_l(k)$ and $q_t(k)$. We define these zeros by the relations

$$q_l^2(k) = \beta_l^2\,(k-k_{l+})\,(k-k_{l-}), \qquad k_{l\pm} = \pm\frac{mv}{\sqrt{\kappa} \pm v} \pm i\epsilon, \quad (6.1)$$

$$q_t^2(k) = \beta_t^2\,(k-k_{t+})\,(k-k_{t-}), \qquad k_{t\pm} = \pm\frac{mv}{1 \pm v} \pm i\epsilon. \quad (6.2)$$

Here the signs of the infinitesimal imaginary parts $\pm i\epsilon$ are determined by the rules for the real and imaginary parts of the q's stated following (2.13).

$\hat{G}_S(k)$ also has zeros associated with the Doppler-shifted Rayleigh modes. To see this, note that the last factor on the right-hand side of (5.8) can be written in the form

$$\left(k^2\, q_l\, q_t - q_0^4\right) = k^4\left[\left(1-\frac{v^2(k-m)^2}{\kappa\, k^2}\right)^{1/2}\left(1-\frac{v^2(k-m)^2}{k^2}\right)^{1/2} - \left(1-\frac{v^2(k-m)^2}{2\,k^2}\right)^2\right]. \tag{6.3}$$

Comparing this expression with the formula for $b(v)$ in (3.10) and remembering that $b(v)$ vanishes when $v = v_R$, we see that (6.3) vanishes when $v^2(k-m)^2/k^2 = v_R^2$. Thus the Rayleigh zeros are at

$$k_{R\pm} = \pm\frac{mv}{v_R \pm v} \pm i\epsilon. \tag{6.4}$$

In this case, the signs of the $\pm i\epsilon$ are determined by the requirement that $\hat{G}_S^{-1}(k)$ be a causal Green's function.

With these definitions, we write $\hat{G}_S(k)$ in the form

$$\hat{G}_S(k) = \frac{(k-k_{R+})\,(k-k_{R-})}{q_t}\,\beta_l b(v) \times\left[\frac{2\,(k^2 q_l q_t - q_0^4)}{\beta_l v^2\, b(v)\,(k-m)^2\,(k-k_{R+})\,(k-k_{R-})}\right]. \tag{6.5}$$

The quantity in square brackets goes to $+1$ in the limit $k\to\infty$. Its only singularities are the branch points at $k = k_{l\pm}, k_{t\pm}$; the zeros at $k_{R\pm}$ have been removed. Thus we can write

$$\hat{G}_S(k) = \frac{(k-k_{R+})\,(k-k_{R-})}{q_t}\,\beta_l b(v) \times\exp[N^{(+)}(k) + N^{(-)}(k)], \tag{6.6}$$

where

$$N^{(\pm)}(k) = \mp\int\frac{dk'}{2\pi i}\,\frac{1}{k'-k\pm i\epsilon}\,\ln\left[\frac{2\left(k'^2 q_l(k')q_t(k') - q_0^4(k')\right)}{\beta_l v^2\, b(v)\,(k'-m)^2\,(k'-k_{R+})\,(k'-k_{R-})}\right], \tag{6.7}$$

and we can close the contours of integration at infinity in the k' plane so that the only contributions to the $N^{(\pm)}(k)$ are from the integrations around the branch cuts between k_{l+} and k_{t+} for $N^{(+)}(k)$ and between k_{t-} and k_{l-} for $N^{(-)}(k)$. A convenient form for the results is

$$N^{(+)}(k) = -\int_{k_{l+}}^{k_{t+}}\frac{dp}{\pi}\,\frac{\varphi(p)}{p-k},$$

$$N^{(-)}(k) = \int_{k_{t-}}^{k_{l-}}\frac{dp}{\pi}\,\frac{\varphi(p)}{p-k}, \tag{6.8}$$

where

$$\tan\varphi(p) = \frac{p^2|q_l(p)\,q_t(p)|}{q_0^4(p)}. \tag{6.9}$$

Then our expressions for the Wiener-Hopf factors are

$$\hat{G}_S^{(+)}(k) = \frac{\beta_l\, b(v)}{\beta_t}\,\frac{i(k-k_{R+})}{[\epsilon + i(k-k_{t+})]^{1/2}}\,\exp[N^{(+)}(k)] \tag{6.10}$$

and

$$\hat{G}_S^{(-)}(k) = \frac{-i(k-k_{R-})}{[\epsilon - i(k-k_{t-})]^{1/2}}\,\exp[N^{(-)}(k)]. \tag{6.11}$$

We have chosen the arbitrary prefactors here in analogy to the similar choices for the $\hat{F}^{(\pm)}(k)$ in (4.5). Note that, as anticipated in the derivation of (5.17), the $\hat{G}_S^{(\pm)}(k)$ both are proportional to $k^{1/2}$ at large k.

We now can use (6.10) and (6.11) for evaluating the contributions to $\tilde{\mathcal{D}}(m,v)$ in (5.22). We begin by looking at $\tilde{\mathcal{D}}_0(m,v)$, as defined in (5.22a). Close inspection of the function $\hat{L}_S(k)$ reveals that it is bounded in the limit of large k, thus the integrand goes like k^{-2} and we can close the contour of integration at infinity in the lower half plane even without the help of the convergence factor $\exp[-ik\ell(1-w^2)]$ in the expression (4.14) for $\hat{U}_{N0}^{(+)}(k)$. Our contour of integration is then reduced to contours around the branch cuts between k_{t-} and k_{l-} and between k_{l-} and m, the latter coming from the factor $|k-m|$ in $\hat{L}_S(k)$. There is also a contribution from the pole at $k = k_{R-}$. The only values of k that occur in the integration are of order m and therefore the contribution from the integral over w in (4.14) is unity up to a correction of order $m\ell$.

The question of whether or not terms of order $m\ell$ may be neglected in this analysis figures prominently in the discussions that follow. In general, we expect that ℓ is a microscopically — or at least mesoscopically — small length scale, much smaller than the wavelengths $2\pi/m$ that ought to characterize deformations of fracture trajectories. For the moment, then, let us neglect such corrections to $\tilde{\mathcal{D}}_0(m,v)$ and write

$$\tilde{\mathcal{D}}_0(m,v) \cong im(-imW)^{1/2}\Sigma_{N\infty}\,\mathcal{D}_0(v), \tag{6.12}$$

where

$$\mathcal{D}_0(v) = -\left(\frac{2}{\kappa\, b(v)}\right)^{1/2}\frac{1}{(-im)^{3/2}} \times\int\frac{dk}{2\pi}\,\frac{\hat{G}_S^{(-)}(m)}{\hat{G}_S^{(-)}(k)}\,\frac{\hat{L}_S(k)}{[\epsilon + i(k-m)]^{3/2}} \tag{6.13}$$

is a real-valued function that depends only on v.

Several comments need to be made with regard to this formula. We have used the Barenblatt condition (4.12) to replace the quantity $\Sigma_0\sqrt{\ell}$ in (4.14) by a term proportional to $\Sigma_{N\infty}\sqrt{W}$, which is, apart from constants, the static mode I stress-intensity factor. We also have used the fact that, once we omit corrections of order $m\ell$, $\tilde{\mathcal{D}}_0(m,v)$ is a homogeneous function of m of order 3/2. The latter remark is obvious for dimensional reasons. $\hat{\chi}_Y$ has the dimensions of a length and W can appear only in the dimensionless group mW. Another important feature is that reality of $Y_{\rm cen}(x)$ implies that $\hat{\chi}_Y(-m,v) = \hat{\chi}_Y^*(m,v)$. This, plus our expectation that $\hat{\chi}_Y$ be analytic in the upper half m plane for stable cracks, dictates the combination $(-imW)^{1/2}$. These features of $\hat{\chi}_Y(m,v)$ emerge naturally from our calculation; they do not have to be imposed as independent constraints.

In general, the function $\mathcal{D}_0(v)$ must be evaluated numerically. We have computed it for several different values of κ and find that it is a positive, increasing function of v for $0 \le v < v_R$. Thus there is no apparent change of stability. It will be useful later to have an exact expression for $\mathcal{D}_0$ at $v=0$. For this purpose, we use

$$\lim_{v\to 0}\hat{L}_S(k) = i\Big(1-\frac{1}{\kappa}\Big)\,[(k-m)|k-m| - (k-2m)|k|]. \tag{6.14}$$

A short calculation then yields

$$\lim_{v\to 0}\mathcal{D}_0(v) = \left(\frac{\kappa-1}{2\kappa^2}\right)^{1/2}. \tag{6.15}$$

VII. MORE DETAILS: THE COHESIVE-SHEAR TERM

The next item of mathematical business is to solve the integral equation (5.24) for $A(x)$ and use the result to evaluate the function $\tilde{\mathcal{D}}_1(m,v)$ defined in (5.22b). This is the term that describes the effect of the cohesive-shear stress.

The linear operator acting on $A(x)$ on the left-hand side of (5.24) has an especially interesting and important property. At $m=v=0$, the function $A(x)=$ const is a null eigenvector of this operator. To see this, look at the terms proportional to $A(x)$ in (5.23) or (5.24), set $m=0$ and $\hat{G}_S^{(\pm)}(k) = \hat{F}^{(\pm)}(k)$, and compare the result to the equation for $\hat{U}_{N0}(k)$ shown in (4.13). It follows that $A(x)$, and therefore $\tilde{\mathcal{D}}_1(m,v)$, must be singular at $m=v=0$. In what follows, we focus our attention on computing $\tilde{\mathcal{D}}_1(m,v)$ in the neighorhood of this singularity.

The leading x dependence in (5.24) is determined by

$$\Gamma_S(x) = \frac{\beta_t}{b(v)\,\beta_l}\int\frac{dk}{2\pi}\,\frac{[\epsilon + i(k-k_{t+})]^{1/2}}{i(k-k_{R+})} \times \exp[-N^{(+)}(k)]\,e^{ikx}. \tag{7.1}$$

We need this function in (5.24) only for positive values of x that are small of order ℓ. To see how it behaves at small x, let m be real and positive (and continue analytically into the complex m plane later) and then write $k = mvz$ so that

$$\Gamma_S(x) = \frac{\beta_t}{b(v)\,\beta_l}\,(imv)^{1/2}\int\frac{dz}{2\pi i}H(z)\,e^{imvxz}. \tag{7.2}$$

Here

$$H(z) = \frac{(z-Z_{t+})^{1/2}}{(z-Z_{R+})}\,\exp[-\tilde{N}^{(+)}(z)], \tag{7.3}$$

where

$$Z_{t+}(v) = \frac{1}{1+v},\quad Z_{l+}(v) = \frac{1}{\sqrt{\kappa}+v},\quad Z_{R+} = \frac{1}{v_R+v}, \tag{7.4}$$

$$\tilde{N}^{(+)}(z) = N^{(+)}(mvz) = -\int_{Z_{l+}}^{Z_{t+}}\frac{dz'}{\pi}\,\frac{\tilde{\varphi}(z')}{z'-z}, \tag{7.5}$$

and $\tilde{\varphi}(z) = \varphi(mvz)$, as defined in (6.8), is a (weakly) v-dependent function of z but not m. Because x is positive, we close the contour of integration in the upper half of the z plane; thus the contour of integration in (7.2) runs counterclockwise around the entire positive real z axis. All of the singularities of $H(z)$ lie within this contour; therefore it can be moved arbitrarily far out into the complex z plane. Then we can make a large-z expansion for $H(z)$ and integrate term by term over z:

$$\int\frac{dz}{2\pi i}\,H(z)\,e^{imvxz} = \int\frac{dz}{2\pi i}\,\frac{1}{\sqrt{z}}\sum_{n=0}^{\infty}\frac{h_n}{z^n}\,e^{imvxz} = \sum_{n=0}^{\infty}(-ih_n)(-imvx)^{n-1/2}\,\frac{1}{\pi}\,\Gamma\Big(\frac{1}{2}-n\Big). \tag{7.6}$$

The Γ functions of negative half-integer orders imply that this is a rapidly convergent series. We need only the leading term, for which $h_0 = 1$:

$$\Gamma_S(x) \approx \frac{\beta_t}{b(v)\,\beta_l}\,\frac{1}{\sqrt{\pi x}}\,\{1 + [\text{terms of order } (imvx)]\}. \tag{7.7}$$

With this information, and for small values of $m\ell$, we can determine the leading contributions to $A(x)$ by looking at the leading terms in (5.24) and equating coefficients of powers of x. When inserted into (5.22), a correction to $A(x)$ of relative order $(imvx)$ becomes a contribution to $\tilde{\mathcal{D}}_1(m,v)$ of relative order $(-imv\ell)$, which we expect to be small. For present purposes, we look only at the leading term, say, $A(x\to 0) \equiv A_0(m\ell, v)$.

With (7.7) inserted on both sides of (5.24), the leading term is of order $x^{1/2}$. Equating the coefficients of $x^{1/2}$, we find

$$A_0(m\ell, v) = \frac{\frac{\beta_t}{\beta_l}\,\mathcal{C}_0(m\ell, v)}{1 - \frac{\beta_t}{\beta_l}\,\mathcal{C}_1(m\ell, v)}, \tag{7.8}$$

where

$$\begin{aligned}\mathcal{C}_0(m\ell, v) &= \frac{\sqrt{\pi\ell}}{\Sigma_0} \int \frac{dk}{2\pi}\, \frac{\hat{L}_S(k)}{\hat{G}_S^{(-)}(k)}\, i(k-m)\hat{U}_{N0}^{(+)}(k-m) \\ &= \frac{2\ell}{b(v)} \int \frac{dk}{2\pi}\, \frac{\hat{L}_S(k)}{\hat{G}_S^{(-)}(k)}\, \frac{1}{[\epsilon + i(k-m)]^{1/2}} \\ &\quad \times \int_0^1 dw\, e^{-i(k-m)(1-w^2)\ell}\end{aligned} \tag{7.9}$$

and

$$\mathcal{C}_1(m\ell, v) = \sqrt{\pi\ell} \int \frac{dk}{2\pi}\, \frac{e^{-i(k-m)\ell}}{\hat{G}_S^{(-)}(k)}. \tag{7.10}$$

In (7.9), the final integral over w is needed for small $m\ell$ only to tell us to close the contour of integration in the lower half k plane. It produces no extra ℓ dependence. In (7.10), we have implicitly kept the factor $\exp[ik(x - x')] \to \exp[+ik0]$ from (5.24) because it tells us how to interpret the result that we get from integrating over x'' in the latter equation. In both cases, we are simply using mathematical devices to make sure that we are properly carrying out integrations over the cohesive zone.

With this approximation for $A(x)$, we have

$$\begin{aligned}\tilde{\mathcal{D}}_1(m, v) &\cong -A_0\, \Sigma_{N\infty} \left[\frac{\pi b(v) W}{2\kappa\ell}\right]^{1/2} \\ &\quad \times \int_0^\ell dx \int \frac{dk}{2\pi}\, \frac{\hat{G}_S^{(-)}(m)}{\hat{G}_S^{(-)}(k)}\, e^{-i(k-m)x} \\ &\approx -A_0\, \Sigma_{N\infty}\, (-imW)^{1/2} \\ &\quad \times \left(\frac{2b(v)}{\kappa}\right)^{1/2} \left[\frac{\hat{G}_S^{(-)}(m)}{(-im)^{1/2}}\right].\end{aligned} \tag{7.11}$$

To obtain the second, approximate form of (7.11), note the similarity between the integral over k here and the integral that defines Γ_S in (5.25). The required analysis is precisely the same as that outlined in Eqs. (7.1)–(7.7), except that we must close the contour of integration in the negative half plane and define quantities such as $Z_{t-}(v)$,$Z_{l-}(v)$ in analogy to the quantities with (+) subscripts in (7.4) and (7.5). The details should be obvious. The result in (7.11) is accurate to the lowest nonvanishing order of $m\ell$. As in (6.12), we have eliminated Σ_0 in favor of $\Sigma_{N\infty}$. Despite appearances, the final factor in square brackets is an m-independent function of v.

If we then combine (7.11) with (7.8) and (7.9), we obtain

$$\tilde{\mathcal{D}}_1(m, v) \cong im(-imW)^{1/2}\Sigma_{N\infty}\, \frac{(3im\ell/2)\, \mathcal{D}_1(v)}{1 - \frac{\beta_t}{\beta_l}\, \mathcal{C}_1(m\ell, v)}, \tag{7.12}$$

where

$$\begin{aligned}\mathcal{D}_1(v) &= \frac{4}{3m^2} \left(\frac{2}{\kappa\, b(v)}\right)^{1/2} \left[\frac{\hat{G}_S^{(-)}(m)}{(-im)^{1/2}}\right] \frac{\beta_t}{\beta_l} \\ &\quad \times \int \frac{dk}{2\pi}\, \frac{\hat{L}_S(k)}{\hat{G}_S^{(-)}(k)}\, \frac{e^{-ik0}}{[\epsilon + i(k-m)]^{1/2}}.\end{aligned} \tag{7.13}$$

Like $\mathcal{D}_0(v)$, $\mathcal{D}_1(v)$ is a real function that depends only on v. In analogy with (6.15), we have normalized $\mathcal{D}_1(v)$ so that

$$\lim_{v\to 0} \mathcal{D}_1(v) = \left(\frac{\kappa - 1}{2\kappa^2}\right)^{1/2}. \tag{7.14}$$

Our final mathematical chore is to evaluate $\mathcal{C}_1(m\ell, v)$. It is easy to see that $\mathcal{C}_1(0, v) = 1$ and thus, as expected, the denominator in (7.12) vanishes at $m = v = 0$. Our problem, therefore, is to evaluate the first leading corrections for nonvanishing $m\ell$. As in the analysis of (7.11), note the similarity between $\mathcal{C}_1$ as defined in (7.10) and Γ_S in (5.25). With the same procedure, we find

$$\begin{aligned}\mathcal{C}_1(m\ell, v) &\approx e^{im\ell}\, \{1 + [\text{terms of order } (-imv\ell)]\} \\ &\approx 1 + im\ell.\end{aligned} \tag{7.15}$$

For small v, the corrections of order $(-imv\ell)$ are unimportant.

VIII. INTERPRETATION AND ANALYSIS

In summary, our mathematical results are the following. Our general expression for the response function $\hat{\chi}_Y(m, v)$ is

$$-\frac{1}{im\hat{\chi}_Y(m, v)} = \Delta\varepsilon_\infty + (-imW)^{1/2}\Sigma_{N\infty} \times \left[\mathcal{D}_0(v) + \frac{\frac{3im\ell}{2}\, \mathcal{D}_1(v)}{1 - \frac{\beta_t}{\beta_l}\, \mathcal{C}_1(m\ell, v)}\right], \tag{8.1}$$

where the functions $\mathcal{D}_0(v)$ and $\mathcal{D}_1(v)$ are given in Eqs. (6.13) and (7.13), respectively, and $C_1(m\ell, v)$ is given to first order in $m\ell$ in (7.15). The quantity $(W)^{1/2}\Sigma_{N\infty}$ is proportional to the mode I stress-intensity factor denoted by $K_{\rm I}$ in Eq. (1.5) and the quantity in square brackets is the function that was denoted there by $\mathcal{D}(m\ell, v)$. This expression is accurate to lowest nonvanishing order in $m\ell$ and for arbitrary v in the range $0 < v < v_R$. In the limit $v \to 0$, $\mathcal{D}_0 = \mathcal{D}_1 = [(\kappa - 1)/2\kappa^2]^{1/2}$.

In interpreting (8.1), it will be useful first to show how it relates to other results in this field. One of the most attractive and pervasive concepts in fracture mechanics is the idea that it might be possible to determine the direction of crack extension by examining not the detailed dynamics at the crack tip as has been attempted here but simply the far-field stress, that is, by computing only the stress-intensity tensor associated with a crack tip. In particular, if a crack initially in mode I follows a trajectory along which the release of elastic energy per unit length is always a local maximum, then its tip supposedly will move at any instant in a direction such that the mode II stress-intensity factor $K_{\rm II}$ vanishes. This $K_{\rm II} = 0$ law is strictly correct, however, only in the limit of zero curvature of the crack trajectory.

By definition, the far-field stresses are those at distances from the tip much larger than ℓ but still much smaller than macroscopic lengths such as W. It is ℓ that sets the scale of the process zone and the size of the region in which the applied stresses are concentrated. Accordingly, we ought to be able to recover results of previous far-field calculations simply by taking the limit $\ell \to 0$ in (8.1). Indeed, this is precisely what happens. The $\ell \to 0$, "far-field" limit of (8.1) is obtained simply by dropping the cohesive shear term, i.e., the second term in the square brackets, because it contains an explicit extra factor $m\ell$. (We already have dropped corrections of relative order $m\ell$ in $\mathcal{D}_0$.)

To see that this is the same as a $K_{\rm II} = 0$ theory, we can go back to the formula for the shear stress in (5.15) and note that, had we been calculating $K_{\rm II}$ for a geometrically sharp crack on the curve $y = Y_{\rm cen}(x)$, we simply would have omitted the cohesive-shear term $\hat{\Lambda}^{(-)}_{cS1}(k)$ and computed the coefficient of $k^{1/2}$ on the right-hand side in the limit of large k. If we then set $K_{\rm II} = 0$, all the rest of our analysis would have remained unchanged except that we would have been missing the cohesive-shear term in (8.1). In a sense, we have "derived" the $K_{\rm II} = 0$ theory for situations in which the cohesive-shear stress is not important.

Our result without the cohesive-shear stress is closely related to the stability theory of Cotterell and Rice [16]. The Cotterell-Rice (CR) theory is a quasistatic theory and therefore we also should take the limit $v \to 0$, but only after letting ℓ vanish in order to remove the cohesive-shear term. Specifically, we have

$$\lim_{v\to 0}\lim_{m\ell\to 0} \frac{-1}{im\hat{\chi}_Y(m,v)}$$

$$= \Delta\varepsilon_\infty + (-imW)^{1/2}\Sigma_{N\infty}\left(\frac{\kappa-1}{2\kappa^2}\right)^{1/2}. \quad (8.2)$$

To emphasize the relationship with the CR theory, we recast this expression in the form

$$-im\hat{\chi}_{Y\rm CR} = \frac{1}{\tilde{K}_I(-im)^{1/2} - T}, \quad (8.3)$$

where

$$\tilde{K}_{\rm I} = \Sigma_{N\infty} W^{1/2}\left(\frac{\kappa-1}{2\kappa^2}\right)^{1/2} \quad (8.4)$$

is proportional to the mode I stress-intensity factor and

$$T = -\Delta\varepsilon_\infty = \Sigma_{T\infty} - \Sigma_{N\infty} \quad (8.5)$$

is the CR "T" stress that is appropriate for this situation.

Equation (8.3) has the same properties as those found by Cotterell and Rice. For positive T, $\hat{\chi}_Y$ has a pole at $m = m_s = i(T/\tilde{K}_{\rm I})^2$, which would correspond to an unstable trajectory of the form

$$Y_{\rm cen} \propto e^{im_s x} \propto \exp[-(T/\tilde{K}_{\rm I})^2 x]. \quad (8.6)$$

This is a growing exponential because the crack is moving in the $-x$ direction. For negative T, on the other hand, the system appears to be stable; the only singularity on the physical sheet of the complex m plane is the branch point at the origin. The associated branch cut should be drawn along the negative imaginary m axis. In this situation, Cotterell and Rice compute the trajectory of a crack tip that initially points away from the original direction of fracture. We obtain their result by computing the response of our crack to a localized patch of shear stress. That is, we write the perturbing stress as a linear superposition of our Fourier modes $\hat{\varepsilon}_m \exp(imx)$ and, for simplicity, let $\hat{\varepsilon}_m = \bar{\varepsilon}$ be a constant so that

$$\varepsilon_{\rm shear}(x) = \int \frac{dm}{2\pi}\,\hat{\varepsilon}_m\, e^{imx} = \bar{\varepsilon}\,\delta(x). \quad (8.7)$$

We then find, for $x = -|x| < 0$,

$$\frac{dY_{\rm cen}}{dx} = -\bar{\varepsilon}\int \frac{dm}{2\pi}\,\frac{e^{-im|x|}}{|T| + \tilde{K}_{\rm I}(-im)^{1/2}}$$
$$= -\frac{\bar{\varepsilon}}{\pi\tilde{K}_{\rm I}}\int_{-\infty}^{+\infty} dw\,\frac{w^2\,e^{-w^2|x|}}{w^2 + \frac{T^2}{\tilde{K}_{\rm I}^2}}. \quad (8.8)$$

This formula can be rewritten in terms of an error function, but it is easier to see what is happening in this integral representation. The perturbation is at $x = 0$. As the crack moves from right to left past this point, its initial trajectory is $Y_{\rm cen} \approx (2\bar{\varepsilon}/\tilde{K}_{\rm I})\sqrt{|x|/\pi}$. After moving a distance of order W, this trajectory is again parallel to the x axis but is displaced by an amount $\bar{\varepsilon}/|T|$.

Our analysis of the tip stresses in Sec. III, however, indicates that the CR crack must be strongly unstable, independent of the sign of T, because there is no cohesive-shear stress available to resist the strong tangential stresses that favor bending into mode II. To see what is happening, we return to (8.1) and look more closely at its behavior near threshold, that is, near $v = 0$, where the singularity in the cohesive-shear term may make it relevant even though the term as a whole is explicitly proportional to $m\ell$.

Before doing this, however, we argue that we can drop $\Delta\varepsilon_\infty = -T$ in (8.1). That term has been useful up to now in helping us make contact with the analysis of Cotterell and Rice, but keeping it is unnecessary for understanding the dynamic instabilities in this model. It is not even clear that this term is consistent with the basic assumptions of our calculations. In principle, $\Delta\varepsilon_\infty$ could be comparable in magnitude with the other terms on the right-hand side of (8.1) if $(mW)^{1/2}$ were of order unity, that is, if the wavelength of the perturbed crack were comparable to the width of the strip. But then we would have to reexamine our large-W approximations in Sec. II. Note also that the exponential instability in (8.6) has the form $Y_{\rm cen} \propto \exp(-{\rm const}\times x/W)$. This instability becomes arbitrarily weak in the limit of a large system even when the stress-intensity factor at the crack tip is kept fixed.

When we drop $\Delta\varepsilon_\infty$ in (8.1) and expand to lowest relevant order in v, we find

$$\frac{1}{-im\hat{\chi}_Y(m,v)} \cong (-imW)^{1/2}\Sigma_{N\infty}\left(\frac{\kappa-1}{2\kappa^2}\right)^{1/2} f(m\ell,v), \tag{8.9}$$

where

$$f(m\ell,v)=\frac{(1-\frac{1}{\kappa})\frac{v^2}{2}+\frac{1}{2}\,im\ell}{(1-\frac{1}{\kappa})\frac{v^2}{2}-im\ell}. \tag{8.10}$$

The denominator in this expression is the quantity that vanishes at the singularity of the cohesive-shear term. The quantity $(1-\frac{1}{\kappa})\frac{v^2}{2}$ comes from the factor β_t/β_l. It cannot be neglected because it is the lowest power of v that remains when $m\ell=0$. We needed it, in fact, when we recovered the CR theory by taking the far-field limit

$$\lim_{v\to 0}\lim_{m\ell\to 0} f(m\ell,v)=1. \tag{8.11}$$

We maintain, however, that the correct way of deriving a quasistatic stability theory is to keep $m\ell$ fixed while letting v go to zero. Even if we are interested only in very small values of $m\ell$, we should take the limits in the opposite order

$$\lim_{m\ell\to 0}\lim_{v\to 0} f(m\ell,v)=-\frac{1}{2}. \tag{8.12}$$

Equation (8.12) is a physically implausible result. It would give us the wrong sign of the deflection in (8.8) and would, if we restored $\Delta\varepsilon_\infty$ in (8.9), give the wrong sign of the response to the T stress. The reason for this behavior is that the crack, even with the cohesive-shear stress, is only marginally stable at $v=0$ and is unstable at arbitrarily small but nonzero v. The function $f(m\ell,v)$ vanishes and therefore $\hat{\chi}_Y(m,v)$ has a pole, at $m=m_s=i(1-\frac{1}{\kappa})\frac{v^2}{\ell}$, which, for any nonzero v is in the upper half of the m plane. Unlike the weakly unstable CR trajectory in (8.6), however, this instability is very strong:

$$Y_{\rm cen}\propto e^{im_s x}\propto \exp\left[-\left(1-\frac{1}{\kappa}\right)v^2\left(\frac{x}{\ell}\right)\right]. \tag{8.13}$$

Thus it is the very small length ℓ, not the macroscopic length W, that sets the scale for unstable motion away from a straight trajectory. This analysis also tells us what happened when, in the far-field theory, we let ℓ become small at fixed v. The pole at m_s moved indefinitely far up the positive imaginary m axis, where it became mathematically invisible but nevertheless implied strong physical instability.

A next obvious question is whether there is any modification of this model that can produce stability in isotropic materials, at least at small speeds. That question is large and complicated, far too large to be addressed in this paper in any generality. But there is one simple modification that is suggested by physical considerations and, when implemented, helps us to understand the significance of our results. Once one accepts the implications of the tip-stress analysis, it seems clear that stability against deflection requires that there be a strong enough cohesive-shear stress at the crack tip to counteract the tangential stresses and suppress the growth of mode II fracture. Our special choice of the cohesive-shear stress, based on our picture of central forces acting between the newly opened fracture surfaces, puts us exactly at the marginal point for quasistatic motion and fails to stabilize the crack at any nonzero speed.

It is easy to see what happens if we move just slightly away from this point. The cohesive-shear stress is defined in (2.23b). We can change the central-force assumption by multiplying the right-hand side of this equation by a factor different from unity, say, $1+\rho$. Let us also say that ρ is a small quantity. This changes our final result for $\hat{\chi}_Y(m,v)$, Eq. (8.1), by inserting factors $1+\rho$ in front of the functions $\mathcal{D}_1(v)$ and $\mathcal{C}_1(m\ell,v)$. Only the latter change is significant to lowest order in ρ because it shifts the pole in the m plane. Specifically, in (8.9), we now have

$$f(m\ell,v)=\frac{(1-\frac{1}{\kappa})\frac{v^2}{2}+\frac{1}{2}\,im\ell-\rho}{(1-\frac{1}{\kappa})\frac{v^2}{2}-im\ell-\rho} \tag{8.14}$$

and the pole is now at $m_s=-2i\frac{\rho}{\ell}+i(1-\frac{1}{\kappa})\frac{v^2}{\ell}$. If ρ is negative, that is, if the mode II cohesive stress is less than the mode I stress, then the only change is that m_s remains in the upper half plane at $v=0$. As expected, the crack with less than marginal resistance to bending into mode II is unstable even in the quasistatic limit.

When ρ is positive, on the other hand, the pole is at $m_s=-2i\rho/\ell$ for $v=0$ and it moves into the unstable upper half plane at a nonzero critical velocity $v_c=\sqrt{2\rho/(1-1/\kappa)}$. For $v<v_c$, we can compute the response to perturbations that are slowly varying on the scale of ℓ by taking the limit $m\ell\to 0$. In this way, we recover the CR theory in all respects. In particular, (8.11) is correct. But such a calculation is entirely incapable of telling us whether or not the crack is stable and thus whether or not a CR calculation of this kind is physically sensible. To determine stability, we must look in detail at the cohesive forces acting at the crack tip and try to understand the physical mechanisms that might cause the cohesive-shear stress to be larger than its central force value.

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