Mechanical equilibrium of conformal crystals

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The relation between a conformal mapping and the geometrical features of the conformal lattice it defines is analyzed. The field created by particles interacting via inverse power law forces and located in the vortices of the conformal lattice is determined. The field of inertial forces within a rotating disk is shown to fulfill the requirements, allowing the formation of a strictly conformal crystal. Elastic properties of the conformal crystals are analyzed: their elasticity is shown to be described by a unique elastic modulus. The distortion of a strictly conformal crystal that is induced by a small deformation of the external field is calculated.

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I. INTRODUCTION

A. Gravity rainbow

Some years ago, experimenting with a system of magnetized spheres, we observed an interesting structure [1] (see Fig. 1). The experiment was performed as follows. A few hundred steel spheres, let us denote the exact number of them by N, of the diameter $\phi=1$ mm were placed within a thin, rectangular box whose lower and upper walls were made of glass. The box containing the spheres was placed within a magnetic field $\vec{H}$ produced by a pair of coils [2] (see Fig. 2). The field was perpendicular to the plane of the box, so the magnetic moments $\boldsymbol{\mu}_i$, $i=1,2,\ldots,N$ induced in each of the spheres made them repel each other with forces proportional to H^2 and inversely proportional to the fourth power of the distance r_{ij} between them. The box-coil setup was slightly tilted and the system was allowed, with the active help of the observer (delicate shaking), to find a minimum energy structure.

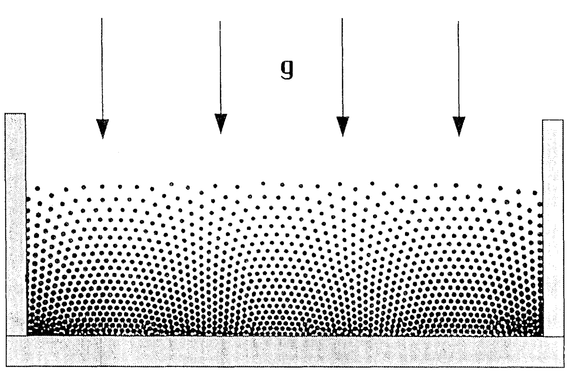

FIG. 1. Gravity rainbow: the conformal crystal observed in the system of magnetized spheres.

Figure 1 shows one of the structures we observed; keeping the terminology introduced in [1], we shall refer to it as the *gravity rainbow*. Before analyzing the message which nature sends us encoded in this picture, let us discuss the message we are given in a simpler experimental situation.

When the box-coil setup is oriented horizontally, one observes the spheres arranging themselves into imperfect, polycrystalline structures. However, even a qualitative analysis of these structures leads the observer to the conclusion that the rigorous solution to the problem of the minimum energy structure within the bulk system of two-dimensional (2D) softly repelling spheres is the perfect hexagonal lattice; hard rectangular boundary conditions (forcing formation of a few differently oriented grains) and the unsuitable number of spheres (resulting in formation of vacancies) do not allow us to observe the solution in its pure form. The perfect, hexagonal lattice could be experimentally observed only if one would be able to design periodic boundary conditions. In a laboratory we do not know how to do it. In a computer simulation the problem is easily solved.

Now, let us return to the tilted setup case. Figure 1 presents one of the structures revealed experimentally in this situation. The tilt of the experimental setup makes the gravitational field enter the plane, within which the spheres are allowed to move. This forces the system to change considerably its density over a very short distance. As a result, as discussed above, a perfect hexagonal lattice solution is no longer valid. Apparently, as seen in the figure, the new solution found by nature is different. It is the aim of the present paper to define it and indicate when, at which conditions it could be observed. The laboratory experiments we performed provide but a hint: where, among which geometrical objects one should be searching. A computer experiment should allow us to create a situation when the new perfect solution could be simulated.

To start with, let us list the basic features of the experimentally observed gravity rainbow structure presented in Fig. 1.

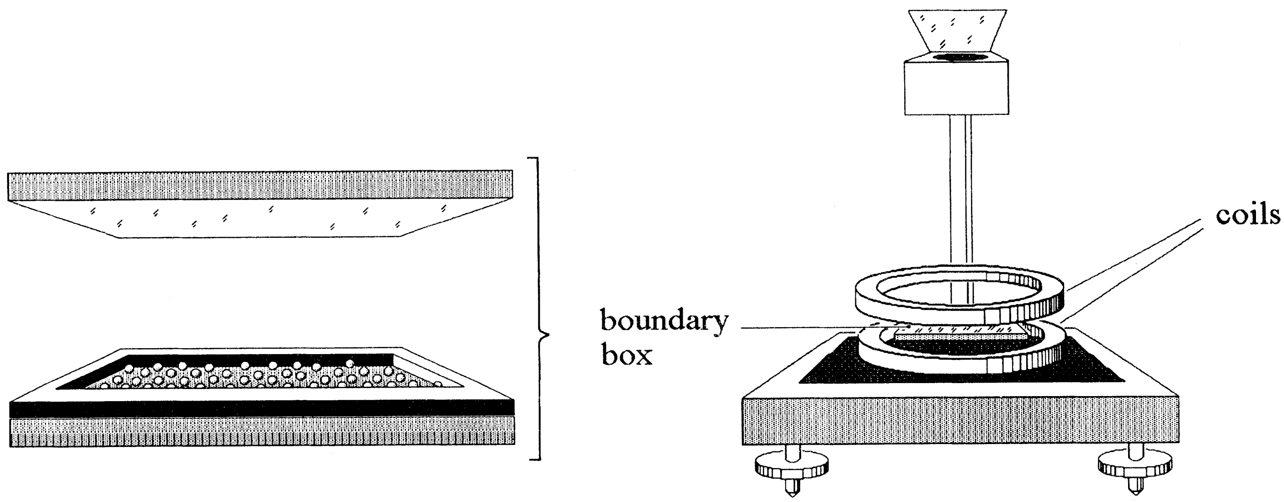

FIG. 2. Experimental setup. The steel spheres are contained within the flat boundary box whose upper and lower walls are made of glass. The box is placed within the magnetic field produced by a pair of coils. The box-coils setup is placed on the overhead projector which can be tilted.

(1) At the bottom of the box, the particles are arranged in a closely packed, hexagonal lattice whose orientation is determined by the hard bottom wall of the boundary box.

(2) Above the closely packed part, the spheres are arranged into a loose structure, which apart from places where apparently some defects are frozen in, is *locally hexagonal.* (The hexagons are not strictly regular though.)

(3) In contrast to the case of a uniform hexagonal lattice, the orientation of the hexagonal cells from the loose part of the structure, larger at the top of the sample and smaller at its bottom, changes continuously, in particular in the horizontal direction. As a result, the nearest neighbor bonds form *three sets of curved, archlike lines which cross each other at angles very close to* $\pi/3$. (The curved lines are best seen when Fig. 1 is watched at a small angle.)

The feature of the gravity rainbow structure listed as (1) results, obviously, from the presence of the hard core within the interparticle interaction and thus cannot be seen as a typical feature of the perfect solution we are looking for. Most of the defects mentioned in point (2) are found near the close packed bottom layer. Thus, they also result from the presence of a hard core within the interaction potential and/or the hard rectangular boundary conditions.

It was the observation specified in point (3) that has proven to be essential. The fact that the curved atomic lines cross each other at $\pi/3$ angle, which is characteristic of the perfect hexagonal lattice, lead us to look for the perfect solution to the tilted case among conformal mappings of a uniform hexagonal lattice. In the particular case of the gravity rainbow structure shown in Fig. 1, the complex logarithm has been indicated as the most probable one [1]. Below, we discuss the validity of this conjecture within a more general context.

B. Conformal lattices and conformal crystals

The experimentally found gravity rainbow structure points to one particular conformal mapping among many other possible ones. Other conformal mappings have proven their usefulness in a study of spiral patterns observed in the experiments on drops of magnetic fluids [3]. Why not check the properties of all conformal mappings from the point of view of their possible applicability in the problem of the minimum energy structure of 2D crystals submerged in external fields? We already addressed the question in our previous paper [4]. There remain, however, many points to be clarified which we do in the present paper.

To avoid possible ambiguities, let us define three basic notions we shall be using.

(i) Consider a conformal, i.e., angle preserving, mapping f of a plane domain D onto a plane domain D'. Let PL be a periodic, discrete, two-dimensional set of points belonging to D. (Thus, PL is a piece of a 2D lattice.) The image $\mathrm{CL}=f(\mathrm{PL})$ belongs to D'. We call it the *conformal lattice* (CL). The conformal lattice is thus a purely geometrical object, independent of any physical realization.

(ii) On the other hand, we call the *strictly conformal crystal* (SCC), a physical system consisting of particles located on the sites of a CL. In what follows we restrict ourselves to the particles interacting through power law forces.

(iii) As we shall see, the physical conditions at which one could observe a SCC are difficult to fulfill. As a consequence, the structure of a real set of particles (submerged in an external field) is very likely to deviate somewhat from that of a SCC. Systems of this kind will be referred to as *conformal crystals* (CC). The experimentally observed gravity rainbow structure makes a good example of a CC.

Our main tasks are the following.

(1) Analyze the relation between a given conformal mapping and the geometrical features of the conformal lattice it defines, in particular, the spatial distribution of its density and the orientation of its cells.

(2) Consider a strictly conformal crystal and determine the total force (stemming from the interparticle interactions) acting on each of its particles.

(3) Determine the shape of the external field which would keep the SCC considered above in a stable equilibrium.

(4) Determine the relation between the actual equilibrium structure, [i.e., CC, induced by a force field, which deviates from the exact field defined in point (3)] and the initial SCC.

The last question may seem to be hazy. In fact, as we shall demonstrate in Sec. IV, when the external force field exhibits a precise symmetry, question (4) finds a definite answer.

II. PROPERTIES OF STRICTLY CONFORMAL CRYSTALS

A. Description of conformal lattices

As mentioned above, we consider the conformal lattice (CL) as the image through a conformal map

$$w = w(z) , \tag{1}$$

of a perfect periodic lattice PL. PL and CL are discrete sets of points belonging to the complex z and w planes

$$z = x + iy \in \text{PL}, \quad w = u + iv \in \text{CL} . \tag{2}$$

$w(z)$ is conformal if and only if w is an analytic function of z, i.e., if and only if the Cauchy relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

are fulfilled, which we assume to hold in the definition domains of PL and CL. This means that (1) and its derivatives of any order can always be inverted in these domains.

As a consequence of (1) and (3), the image of two neighboring points of PL separated by the distance $ds_z = (dx^2 + dy^2)^{1/2}$ consists of two points of CL at a distance $ds_w = (du^2 + dv^2)^{1/2}$ given by

$$ds_w^2 = \left| \frac{dw}{dz} \right|^2 ds_z^2 . \tag{4}$$

The derivative of w with respect to z

$$\zeta(w) \equiv \frac{dw}{dz} \tag{5}$$

is still an analytic function; it is only its modulus that enters (4) what is a direct consequence of the Cauchy relations.

Let us now describe, in brief, the arguments given in Ref. [4] why the pointlike particles interacting through isotropic, power law forces should arrange themselves into a conformal lattice. Assume first that the external force does not contain any characteristic length. As the interaction itself does not provide any parameter of the length dimension, the only relevant length parameter is the size S of the system. Therefore, taking into account the isotropy of the interaction force, one cannot expect the relation between ds_w and ds_z to deviate much from (4). ($|dw/dz|^2$ depends only on z or, what is equivalent, on w [4].) As soon as a new length enters the description, the arguments given above are no longer valid. For instance, in the case of the gravity rainbow, the close-packed domain needs in its description a second characteristic length—the diameter of the spheres (which is irrelevant in the loose-packed part of the structure). Thus to fulfill the assumptions, under which a conformal lattice can be expected, the close-packed region must be located outside the boundary of the system taken into consideration.

Returning to (4), we notice that this relation shows that $|dw/dz|$ is a measure of a local dilation at a w belonging to the CL. Assuming that the PL under consideration is hexagonal, it contains $2/\sqrt{3}b^2$ lattice sites per unit area, where b is the distance between the nearest neighbor sites of the PL. As a consequence, the locally hexagonal lattice CL contains

$$n = \frac{2}{\sqrt{3}b^2} \left| \frac{dz}{dw} \right|^2 \tag{6}$$

lattice sites per unit area. Notice that $|dw/dz|^2$ is always finite, unless explicitly stated. This is a direct consequence of the fact that $w(z)$ can be inverted over the domains of PL and CL.

The z plane has no physical realization, so the length b is *a priori* arbitrary. In order to simplify the future discussion of the gravity rainbow structure, we put b equal to the diameter of the spheres used in the experiment. Thus, the lattice PL is locally identical with the close-packed domain of the gravity rainbow.

Consider now the analytical function $\zeta(w)$, which we write as

$$\zeta(w) = |\zeta| e^{i\varphi} . \tag{7}$$

Its argument $\varphi(w)$ has a simple interpretation. Let Δz be a vector connecting two neighboring points of the PL and let Δw be its image through $w(z)$. Writing

$$\begin{aligned} \Delta z &= |\Delta z| e^{i\varphi_z} , \\ \Delta w &= |\Delta w| e^{i\varphi_w} , \end{aligned} \tag{8}$$

we obviously have

$$\begin{aligned} |\Delta w| e^{i\varphi_w} = \Delta w &= \frac{dw}{dz} \Delta z + O(\Delta z^2) \\ &= |\zeta| |\Delta z| e^{i(\varphi + \varphi_z)} + O(\Delta z^2) . \end{aligned} \tag{9}$$

As a consequence, φ directly yields the angle a hexagon of the CL has been turned with respect to the corresponding hexagon of the PL.

Summing up, one can say that the analytical function $\zeta(w)$ contains all information about the geometry of the conformal lattice around w. Following the notation introduced in Ref. [4], $\zeta(w)$ is the *complex distortion* in the neighborhood of w, $|\zeta(w)|$ is the *dilation scale* in the neighborhood of w, $\varphi(w)$ is the *local orientation* of the lattice in the neighborhood of w, $n(w)$ is the *specific number* of particles, or the *density*.

B. Strictly conformal crystals

Starting from a CL, we transform it into a conformal crystal by setting a pointlike particle on each of its sites. Two particles, say i and j, are assumed to interact through the repulsive force, which with the use of the complex variable notation can be written as

$$F_{i\to j}=A\frac{w_j-w_i}{|w_j-w_i|^{k+2}}=F^{u}_{i\to j}+iF^{v}_{i\to j}\ , \tag{10}$$

where k is a fixed exponent and A is a positive coupling constant. ($F_{i\to j}$ is the force which particle ith exerts on particle jth.) However in order to enlarge the family of models described by the concept of conformal crystal, we may allow A to vary slowly over distances large in comparison with the interparticle distance, i.e., $b\,|dw/dz|$. In the case of the gravity rainbow, $A\propto H^2$, thus, if H is not strictly uniform, A itself is no longer constant. In what follows, unless explicitly stated, A is assumed to be a real constant. Concerning (10), it is worth noticing that despite its complex nature, $F_{i\to j}$ is obviously not an analytical function.

Let us now analyze the total force $F(w)$ which the particles located at the vortices $\{w_l\}$, $l=1,2,\dots,6\equiv 0$, of a conformally transformed hexagon exert on the particle located in the image w of its center. See. Fig. 3. (The vortices and the center of the original hexagon we denote, respectively, by $\{z_l\}$, $l=1,2,\dots,6$, and z.)

Thus, we have

$$w=w(z),\quad w_l=w(z_l),\quad l=1,2,\dots,6\ , \tag{11}$$

where

$$z_l=z+be^{i\alpha_l} \tag{12}$$

and

$$\alpha_l=\frac{l}{3}\ . \tag{13}$$

Let r be a typical characteristic variation length, i.e., a distance in the z plane over which the function $w(z)$ changes appreciably:

$$\frac{1}{|w|}\left|\frac{dw}{dz}\right|\approx\frac{1}{r}\ . \tag{14}$$

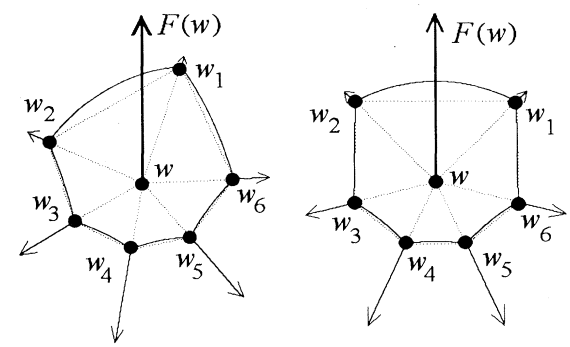

FIG. 3. Forces between the particles located at the vortices $\{w_1,w_2,\dots,w_6\}$ of conformally transformed hexagons and the particle located at the image w of their centers. $k=3$. Conformally transformed sides of the hexagons are plotted with continuous lines. Note that their curved sides meet at angles equal to $2\pi/3$. The left and right hand figures correspond to different orientations of the hexagons with respect to the direction of the resulting force.

In what follows we assume

$$\frac{b}{r}\ll 1\ . \tag{15}$$

As a consequence, defining

$$\eta\equiv\frac{d^2w}{dz^2}=|\eta|e^{i\psi} \tag{16}$$

we can write, using (12),

$$w_l-w=b\zeta e^{i\alpha_l}+\frac{b^2}{2}\eta e^{2i\alpha_l}+\cdots\ ,$$

$$\frac{1}{|w_l-w|^{k+2}}=\left[\frac{1}{b|\zeta|}\right]^{k+2}\times\left[1-\frac{k+2}{2}b\left|\frac{\eta}{\zeta}\right|\cos(\alpha_l-\psi+\varphi)\right]+\cdots\ . \tag{17}$$

The force $F_{(l)}$, which a particle located at w_l exerts on the particle located at w equals

$$F_{(l)}\equiv F_{w_l\to w}=A\frac{w-w_l}{|w-w_l|^{k+2}}\equiv F^{u}_{(l)}+iF^{v}_{(l)}\ , \tag{18}$$

thus, the total force $F_{\rm hex}$ due to all particles located on the hexagon $(w_1,w_2,\dots,w_6)$ is given by

$$F_{\rm hex}=\sum_{l=1}^{6}F_{(l)}=-A\sum_{l=1}^{6}\left[\frac{2}{b|\zeta|}\right]^{k+1}\left[e^{i(\alpha_l+\varphi)}+\frac{b|\eta|}{2|\zeta|}e^{i(\alpha_l+\psi)}\right]\left[1-\frac{k+2}{2}b\frac{|\eta|}{|\zeta|}\cos(\alpha_l+\psi-\varphi)\right]+\cdots\ . \tag{19}$$

Because $\sum_{l=1}^{6} e^{i\alpha_l} = \sum_{l=1}^{6} e^{2i\alpha_l} = 0$ the leading nonvanishing term corresponds to the product $e^{i(\alpha_l+\varphi)}\cos(\alpha_l+\varphi-\psi)$.

$$F_{\text{hex}} = A\frac{k+2}{2}\frac{|\eta|}{b^k|\zeta|^{k+2}}\sum_{l=1}^{6}\cos(\alpha_l+\psi-\varphi)e^{i(\alpha_l+\varphi)}+\cdots$$
$$= A\frac{(k+2)3}{2}\frac{|\eta|}{b^k|\zeta|^{k+2}}e^{i(2\varphi-\psi)}[1+O(b/r)] \ . \tag{20}$$

Notice, that, if the nearest neighbors of the particle located at z were located on the vortices of a regular polygon with n sides, the corresponding force $F_{n\text{-gon}}$ in the w plane would simply equal $(n/6)F_{\text{hex}}$ (assuming that their distance to z would still be given by b).

C. Influence of the orientation of the hexagon

Both φ and ψ enter (20). As a consequence, one should have the feeling that F_{hex} depends on the orientation φ of the hexagon itself. On the other hand, this dependence would be surprising since changing the orientation of the hexagon in the z plane (i.e., replacing each α_l with $\alpha_l+\alpha_0$, where α_0 is an arbitrary constant) does not change F_{hex}. In order to clear up the matter we shall rewrite (20) with the help of the density gradient of the SCC.

Let us define the density gradient as a complex number using the same construction we used defining the force $F_{\text{hex}} = F^u_{\text{hex}} + iF^v_{\text{hex}}$:

$$\nabla n \equiv \frac{\partial n}{\partial u} + i\frac{\partial n}{\partial v} \ . \tag{21}$$

Using (6) we get

$$\frac{b^2\sqrt{3}}{2}\nabla n = \frac{\partial}{\partial u}\left[\frac{dz}{dw}\left[\frac{dz}{dw}\right]^*\right] + i\frac{\partial}{\partial v}\left[\frac{dz}{dw}\left[\frac{dz}{dw}\right]^*\right] , \tag{22}$$

a^* being the complex conjugate of a. Now

$$\frac{\partial}{\partial u}\left[\frac{dz}{dw}\left[\frac{dz}{dw}\right]^*\right] = \frac{\partial w}{\partial u}\frac{d^2z}{dw^2}\left[\frac{dz}{dw}\right]^* + \text{c.c.}$$
$$= \frac{d^2z}{dw^2}\left[\frac{dz}{dw}\right]^* + \text{c.c.} \tag{23}$$

and

$$\frac{d^2z}{dw^2} = \frac{d}{dw}\left[\frac{dz}{dw}\right] = \frac{dz}{dw}\left[\frac{d}{dz}\left[\frac{dz}{dw}\right]\right]$$
$$= -\frac{d^2w}{dz^2}\left[\frac{dw}{dz}\right]^{-3} \tag{24}$$

because

$$\frac{dz}{dw} = \left[\frac{dw}{dz}\right]^{-1} . \tag{25}$$

In the same way

$$\frac{\partial}{\partial v}\left[\frac{dz}{dw}\left[\frac{dz}{dw}\right]^*\right] = i\left[\frac{d^2z}{dw^2}\left[\frac{dz}{dw}\right]^* - \text{c.c.}\right] . \tag{26}$$

Thus using (7), (16), (21), and (22), one finally gets

$$\nabla n = -\frac{4}{\sqrt{3}b^2}\frac{|\eta|}{|\zeta|}e^{i(2\varphi-\psi)} . \tag{27}$$

Comparing (27) with (20), one notices that F_{hex} can be expressed in a much more compact form:

$$\vec{F}_{\text{hex}} = -A\frac{(k+2)}{k}\frac{3^{(k+4)/2}}{2^{(k+2)/2}}\vec{\nabla} n^{k/2} , \tag{28}$$

where $\vec{a}$ is a vector with Cartesian components a_u and a_v:

$$\vec{a} \equiv (a_u, a_v) \ . \tag{29}$$

(28) shows that, apart from a purely numerical factor, $\vec{F}_{\text{hex}}$ depends only on the coupling constant A and the gradient of the density to the $k/2$ power.

Equation (28) takes into account the forces exerted on the central particle but by its six nearest neighbors. Will the result change if one takes into account the forces induced by the next (and so on. . .) nearest neighbors?

To answer the question, let us notice two facts. The set of all particles surrounding the central one can be decomposed into hexagonal subsets, i.e., 6-element subsets, whose members are located in vortices of hexagons (of different sizes and orientations but a common center). On the other hand, analyzing (28), one can see that what enters $\vec{F}_{\text{hex}}$ is only the global structure of the SCC—the local orientation of the surrounding hexagon has no influence on it. Thus the contribution of any of the hexagonal subsets of neighboring particles to the force acting on the central particle located at a given $w(z)$ will be given by an appropriately rescaled formula (28); the scaling factor depending on the size of the hexagon.

To find a simple analytical estimation of the total contribution of the hexagonal subsets, we group the particles surrounding the central one into hexagonal families H_1, H_2, H_3,... as shown in Fig. 4. Family H_1 contains but one hexagonal subset—the above considered nearest neighbors of the central particle—its contribution to the total force F^{int} acting on the central particle equals F_{hex} as given by (28). Family H_2 contains two subsets: particles of one of them are located at a distance $\sqrt{3}b$ from the central particle, while particles from the other subset are distant by $2b$ from it. Their contributions to the total force are equal, respectively, $(1/b^k 3^{k/2})F_{\text{hex}}$ and $(1/b^k 2^k)F_{\text{hex}}$. In general, family H_n contains n hexagonal subsets. Particles belonging to the subsets of a family H_n, $n>2$, are not farther than nb and not closer than $n(\sqrt{3}/2)b$ from the central particle. Each of the hexagonal subsets provides to the total force a contribution not larger than $[1/b^k(n\sqrt{3}/2)^k]F_{\text{hex}}$. Thus, denoting by $f(k)$ the factor by which the total force F^{int} is larger than F_{hex},

$$\vec{F}^{\text{int}} = f(k)\vec{F}_{\text{hex}} , \tag{30}$$

we find

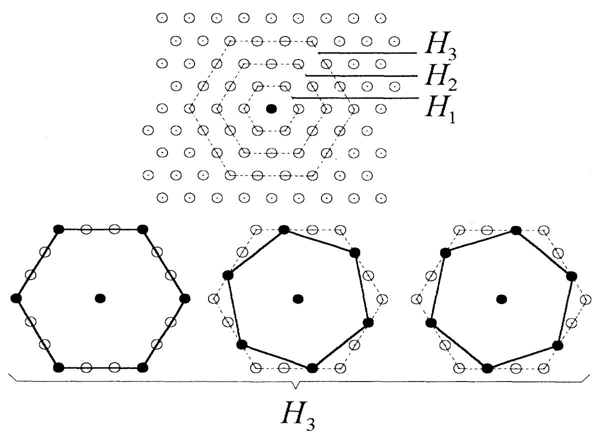

FIG. 4. Decomposition of the neighbors of the central particle into families H_n, $n=1,2,3,\ldots$. Lower part of the figure shows three hexagonal subsets which belong to the H_3 family.

$$1<f(k)<1+\frac{1}{2^k}+\frac{1}{3^{k/2}}+\left[\frac{2}{\sqrt{3}}\right]^k\sum_{n=3}^{\infty}n^{1-k}\equiv g(k) \; . \tag{31}$$

Thus, in view of (28) we simply write

$$\vec{F}^{\,\mathrm{int}}=-AJ(k)\vec{\nabla}(n^{k/2}) \; , \tag{32}$$

where

$$J(k)\cong\frac{3^{(k+4)/4}}{2^{(k+2)/2}}\frac{k+2}{k}g(k) \; . \tag{33}$$

For $k>2$, [i.e., for k values for which $g(k)$ is convergent] $J(k)$ is a monotonically decaying function of k

$$J(3)=3.879\ldots \; , \quad J(4)=2.311\ldots \; , \quad J(\infty)=0 \; . \tag{34}$$

Let us note that in the case of a system containing a finite number of particles the sum found in the definition of $g(k)$, Eq. (31), should also be finite. Coefficients $J(k)$ modified in such a way will be used in Secs. II E and II F.

D. Supplementary condition

Relation (32) is very simple. However, it was derived under the assumption that $w(z)$ (and thus dw/dz as well) is an analytical function. As n explicitly depends on dw/dz and $\vec{F}^{\,\mathrm{int}}$ explicitly depends on n, a question arises, to which conditions is the condition of analyticity of $w(z)$ equivalent as far as n and $\vec{F}^{\,\mathrm{int}}$ themselves are concerned.

Apart from a constant factor, $n=|\zeta|^{-2}$ [Eq. (6)]. Now, there must be some real function $\varphi(w)$ such that

$$|\zeta|e^{i\varphi}=\zeta(w)=(\text{an analytical function}) \; . \tag{35}$$

We know that φ describes the orientation of the hexagons in the w plane. On the other hand, we know that $|\zeta|$ cannot vanish (Sec. II A). We can therefore write the Cauchy conditions as

$$\frac{\partial|\zeta|}{\partial u}=|\zeta|\frac{\partial\varphi}{\partial v} \; , \quad \frac{\partial|\zeta|}{\partial v}=-|\zeta|\frac{\partial\varphi}{\partial v} \; . \tag{36}$$

They hold without restrictions in the whole domain of interest. Now, Eq. (32) can be written as

$$F_u=kAJ(k)\frac{1}{|\zeta|^k}\frac{\partial\varphi}{\partial v} \; , \quad F_v=-kAJ(k)\frac{1}{|\zeta|^k}\frac{\partial\varphi}{\partial u} \; . \tag{37}$$

In geometrical terms, the meaning of (37) is that in the domain of the w plane, where the strictly conformal crystal is defined, the lines of the constant phase, $\varphi=$const, coincide with the integral trajectories of

$$\frac{dv}{F_v}=\frac{du}{F_u} \; . \tag{38}$$

On the other hand, (36) leads to

$$\Delta\ln|\zeta|\equiv\frac{\partial^2}{\partial u^2}\ln|\zeta|+\frac{\partial^2}{\partial v^2}\ln|\zeta|=0 \; . \tag{39}$$

(Δ is the two-dimensional Laplace operator.) Because of (6), (39) is equivalent to

$$\Delta\ln(n)=0 \; , \tag{40}$$

which in view of (32) can be rewritten as

$$\vec{\nabla}\cdot\left[\frac{\vec{F}^{\,\mathrm{int}}}{An^{k/2}}\right]=0 \; . \tag{41}$$

Equations (40) and (41) bring the answer to the question formulated at the beginning of this section. They express conditions which the density and internal force fields must fulfill to stay in accordance with the analyticity of the conformal mapping which defines them. The conditions are of vital importance for the theoretical analysis of any experimentally observed structure. To put the structure of a strictly conformal crystal into a mechanical equilibrium, one has to equilibrate $\vec{F}^{\,\mathrm{int}}$ with an external force field $\vec{F}^{\,\mathrm{ext}}$:

$$\vec{F}^{\,\mathrm{int}}+\vec{F}^{\,\mathrm{ext}}=\vec{0} \; . \tag{42}$$

Thus in terms of the external force field $\vec{F}^{\,\mathrm{ext}}$, Eqs. (32) and (41) can be rewritten as

$$\vec{F}^{\,\mathrm{ext}}=AJ(k)\vec{\nabla}(n^{k/2}) \; , \tag{43a}$$

$$\vec{\nabla}\cdot\left[\frac{\vec{F}^{\,\mathrm{ext}}}{An^{k/2}}\right]=0 \; . \tag{43b}$$

As shown in the next sections, Eqs. (43a) and(43b) prove to be very useful in the analysis of SCCs formed in external fields of different symmetries.

E. Strictly conformal crystals in external fields of a planar symmetry

Considerations presented above are valid in the general case of any SCC, i.e., a strictly conformal crystal defined by any conformal transformation. Below we shall discuss in more detail the particular class of those SCCs which can be equilibrated with external fields whose lines of forces are parallel to a unique direction which we choose as the v axis. In what follows we shall refer to this direction as *vertical;* the direction of the perpendicular u axis will be referred to as *horizontal.* Thus we have

$$\vec{F}^{\,\text{ext}}=\vec{F}_g=(0,-F_g)\ . \tag{44}$$

As a consequence of conditions (40) and (43), which the external field must fulfill to stay in accordance with the assumed analyticity of the $z \to w$ mapping, we have

$$\frac{\partial}{\partial u}n^{k/2}=0\ , \tag{45}$$

$$F_g=-AJ(k)\frac{\partial}{\partial v}(n^{k/2})\ . \tag{46}$$

One of the conclusions to which Eqs. (45) and (46) lead is that the only component F_g of the external field is allowed to vary but along the vertical v axis, i.e., the field $\vec{F}^{\,\text{ext}}$ as a whole must have a planar symmetry. (The gravity rainbow structure we found experimentally was formed within the uniform gravitational fields. Thus, if the gravity rainbow is to be seen as a conformal crystal, its strictly conformal analog should belong to the planar symmetry class we are discussing.) Because of Eq. (40) we have

$$\frac{\partial^2}{\partial v^2}\ln n=0\ , \tag{47}$$

which is equivalent to

$$n(v)=n(0)e^{-\beta v} \tag{48}$$

and then

$$\frac{F_g}{A}=J(k)\frac{\beta k}{2}[n(0)]^{k/2}e^{-(\beta k/2)v}\ . \tag{49}$$

β is a positive constant the value of which stems from the boundary conditions. Experimental implications of Eq. (40) are rather discouraging: to equilibrate a SCC of the planar symmetry, the external force field must decay exponentially with height. In the experiments we performed, fortunately before Eq. (40) was found, the external (gravitational) field was constant. In spite of that, the structure we observed displayed basic features of a planar symmetric SCC, in particular it was clearly locally hexagonal what led us to take into consideration conformal maps. Analysis of the structures formed in fields, which do not fulfill Eq. (40), is an interesting problem. We analyze it in Sec. IV. Equation (36) can be rewritten as

$$\frac{\partial\varphi}{\partial u}=-\frac{d}{dv}\ln|\zeta|=-\frac{d}{dv}\ln(n^{-1/2})=-\frac{\beta}{2}\ . \tag{50}$$

To get as close as possible to the conditions under which conformal crystals can be observed in a laboratory experiment, let us assume that the SCC under consideration is contained within a hard boundary box whose side walls (located, respectively, at $u=0$ and $u=L$) are vertical. (Strictly speaking, hard boundary conditions are not allowed in the case of an SCC—one should rather see them as walls which induce required orientations of the adjacent hexagonal cells of the SCC under consideration.) In such boundary conditions, orientation of those hexagonal cells of the SCC, which are located at $u=0$ and $u=L$, has to be taken equal to some integer multiple of $\pi/3$. As a result, the variation of the orientation φ along the horizontal u axis must obey the following condition:

$$\varphi(L)-\varphi(0)=-\frac{\beta L}{2}=-\frac{\pi}{3}l\ , \tag{51}$$

l being an arbitrary integer. Thus the vertical density profile of the SCC

$$n(\nu)=\frac{2}{\sqrt{3}b^2}e^{-(2\pi l/3L)v}\ , \tag{52}$$

whose value at $v=0$

$$n(0)=\frac{2}{\sqrt{3}b^2}\ . \tag{53}$$

The $v=0$ horizontal line acquires a physical meaning if, crossing once more the borders of the realm of strictly conformal crystals, we assume that, as it happens in a real experiment, the inverse power law particles, from which the SCC crystal is made, contain a hard core of diameter b. Recalling that $2/\sqrt{3}b^2$ is equal to the density at which the hard-core particles become close packed, we find that $v=0$ is the horizontal boundary below which the conformal crystal turns into the close-packed one, and thus, below which (52) loses its validity.

The complex distortion of the SCCs equals

$$\frac{dw}{dz}=e^{-(i\pi l/3L)w}\ . \tag{54}$$

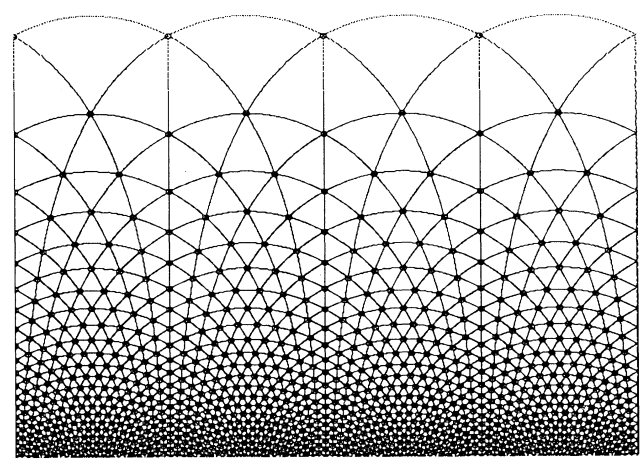

FIG. 5. Structure of a SCC formed in the external field of a planar symmetry. The structure is the image of a uniform hexagonal lattice mapped via the conformal mapping (55).

Integrating, we find the form of the conformal mapping which describes the particular kind of a SCC which can be equilibrated with a gravitational field of the shape of defined by (44)

$$w=\frac{1}{i\alpha}\ln(i\alpha z),\quad \alpha=\frac{\pi l}{3L}\ . \tag{55}$$

Figure 5 shows the conformal lattice described by Eq. (55).

The validity of the solution found above has been checked numerically. To make the test more informative we assumed that the system is cut from below by a hard wall and that at the wall the density of the SCC reaches its close-packed limit value. See Fig. 6(a). The discontinuity introduced into the SCC by the cut strongly perturbs the field of interparticle forces. We discuss the problem in more detail below, for the case of the SCC formed within the inertial field of a rotating disk.

The exponent k of the inverse power law via which the particles interact with each other was put equal to 3 (dipole-dipole interaction). Then, the total force acting on each of the particles was found numerically. Figure 6(b) presents the result.

In Fig. 7 the vertical components of the forces shown in Fig. 6(b) were plotted versus v. As expected, in the vicinity of the hard wall the numerically found values deviate strongly from those predicted by Eq. (49). Results of the simple test we performed show clearly that the equilibration of a finite conformal crystal will not be a simple task. The long range of the interparticle interaction results in strong boundary effects which certainly perturb the conformal lattice. Another problem, which needs an explanation, is why in the conditions of a constant field, which from the point of view of the theory we developed is of a wrong shape, we did observe in the laboratory a structure, which is so similar to the SCC. A partial answer to it is given in Sec. IV.

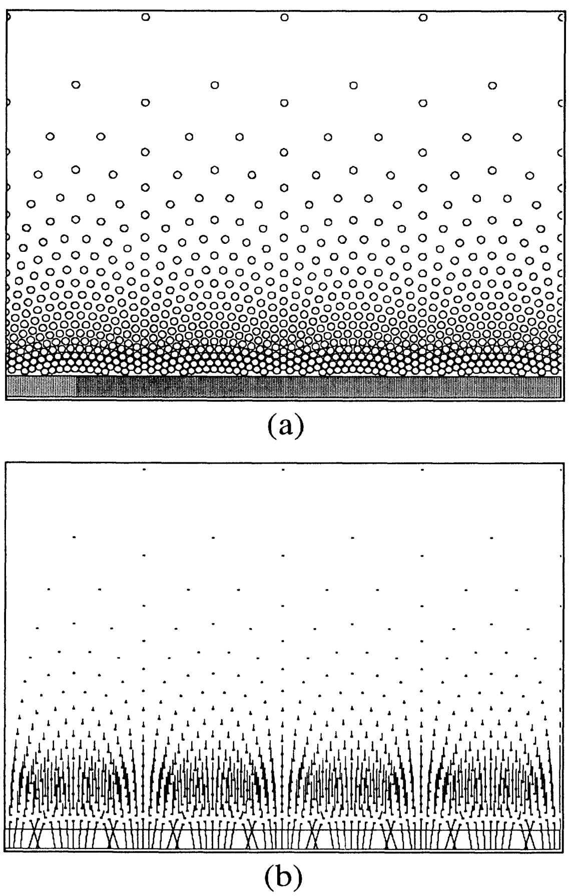

FIG. 6. (a) Configuration of a planar SCC cut from below by a hard wall. (b) Total forces acting on each of the particles shown in (a).

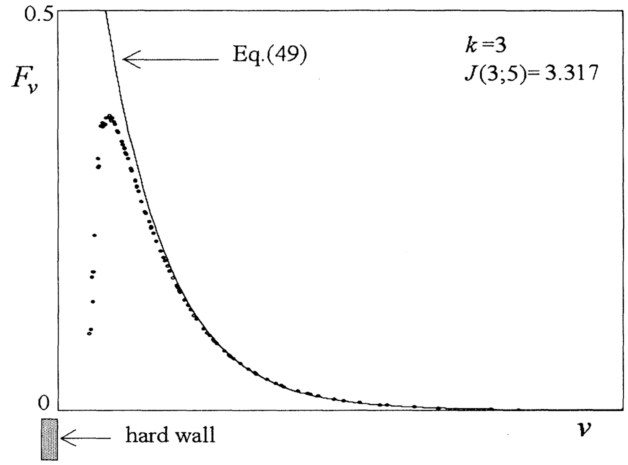

FIG. 7. The vertical component of the total force acting on each of the particles shown in Fig. 6 versus its v coordinate. The large, negative of forces acting on particles close to the hard wall are not plotted.

F. Strictly conformal crystals and conformal crystals on a rotating disk

Let us consider the case of the particular force field within which the particles of mass m are immersed when located on a disk rotating with a constant angular velocity ω

$$\vec{F}_c=m\omega^2\vec{R}\ , \tag{56}$$

where $\vec{R}$ is a vector whose origin coincides with the center of a rotating disk. ($R=|w|$.)

We now look for a conformal map $w=w(z)$ such that the SCC (i.e., the system of particles interacting via inverse power forces located in vortices of the conformal lattice), which the map defines, will stay in mechanical equilibrium when immersed in the external field given by (56).

The equilibrium conditions [(43a) and (43b)] lead to

$$\frac{m\omega^2 R}{A}=J(k)\frac{d}{dR}[n^{k/2}(R)]\ , \tag{57}$$

$$\frac{d}{dR}\left[\frac{R^2}{n^{k/2}}\right]=0\ . \tag{58}$$

We assume in what follows that $n^{1/2}(0)=0$. In this case, Eq. (58) is a mere consequence of Eq. (57): the supplementary condition a SCC has to obey is automatically fulfilled. Equation (57) leads to

$$|\zeta|^{-2}=\frac{b^2\sqrt{3}}{2}n=\left[\frac{m\omega^2}{2AJ(k)}\right]^{2/k}\frac{b^2\sqrt{3}}{2}R^{4/k} \qquad (59)$$

so that

$$\zeta(w)=\left[\frac{2aJ(k)}{m\omega^2}\right]^{1/k}\left[\frac{2}{\sqrt{3}b^2}\right]^{1/2}w^{-2/k}$$
$$=\left[\frac{2aJ(k)}{m\omega^2}\right]^{1/k}\left[\frac{2}{\sqrt{3}b^2}\right]^{1/2}R^{-2/k}e^{-i2\Phi/k}, \qquad (60)$$

Φ being the argument of w so that it coincides with the angular variable of the polar coordinate system.

Analyzing (60) one finds the general form of the conformal mapping which describes possible SCCs formed in the inertial force field of the rotating disk

$$w=az^{k/(k+2)}. \qquad (61)$$

In Sec. II A we assumed that $w=w(z)$ can be inverted anywhere in the domain of the definition of the PL and the CL. As a consequence, we shall assume that $w=0$ and $z=0$ are excluded from the definition domains. Moreover, if the inverse mapping $z=z(w)$ has to be one-to-one, we must define the PL on the Riemanian surface around $z=0$.

The boundary condition we demand is as follows: the orientation φ must change by an integer multiple of $\pi/3$ when Φ increases by 2π. (This condition is valid for a conformal lattice of a locally hexagonal symmetry. For a locally square conformal lattice, one should demand a change by an integer multiple of $\pi/2$.) With such a boundary condition there are no crystallographic defects throughout the disk. Now, (60) implies

$$\Delta\varphi=-\frac{2}{k}\Delta\Phi. \qquad (62)$$

In other words, any rotation $\Delta\Phi$ around the center of the disk induces a *disorientation* $\Delta\varphi$ (a change of the orientation), which is of the opposite sign and proportional to $\Delta\Phi$. The conformal lattice will be, as demanded, free of any defect provided

$$\Delta\varphi(2\pi)=l\frac{\pi}{3}. \qquad (63)$$

The integer l is the *disorientation number.* Comparison of (62) and (63) shows that l is negative and it depends on the inverse power law exponent k [see Eq. (10)]. Not getting into the problem of stability, we can now list the types of the strictly conformal crystals of the locally hexagonal and square symmetry which can be formed in the inertial force field on a rotating disk. See Table I. Table I shows that the structure of the SCC formed on a rotating disk is basically determined by the value of the exponent k. Analyzing the physical meaning of Eq. (61) one finds that the density of such a crystal drops to zero in the center of the disk. Locally hexagonal conformal lattices defined by Eq. (61) are shown in Fig. 8.

TABLE I. List of possible strictly conformal crystals formed on a rotating disk.

Local symmetry of the lattice	Power law exponent k	Disorientation number l
Hexagonal	1 (Coulomb interaction)	−12
	2	−6
	3 (dipole-dipole)	−4
Square	1 (Coulomb interaction)	−8
	2	−4
	3 (dipole-dipole)	no solution

Looking for a possible experimental setup, within which a conformal crystal of the type discussed above could be observed, we encounter a problem of the boundary box. The most natural boundary, a hard (circular) wall located at a radius R_b, is not best, since it does not correspond to any of the constant phase lines. Thus performing an experiment within such a circular boundary box one should expect formation of defects in its vicinity. (The situation is even worse than in the case of the planar symmetry field, since there a hard bottom wall, although being neither a constant phase line, allows at least formation of a close packed structure.) Below, we present results of a simple numerical simulation of a SCC within such a circular boundary box.

Figure 9(a) presents the initial configuration of $N=600$ particles within a circular boundary box. The configuration is cut out from an infinite conformal lattice defined by Eq. (61). We assume that the diameter b of their hard core equals 1, and that their mass $m=1$. The radius of the box was set to such a value $R_b=16.8$ that at the fixed N, the particles which touch the boundary wall,

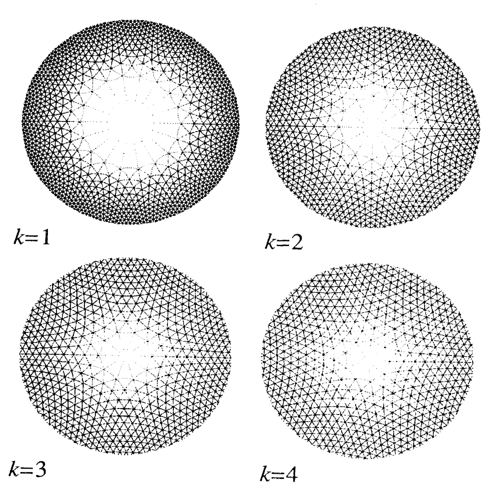

FIG. 8. Locally hexagonal conformal lattices defined by mapping (61) for $k=1$, 2, 3, and 4.

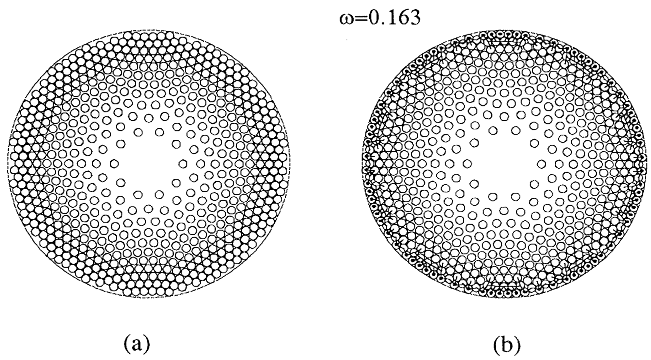

FIG. 9. (a) Initial configuration: a piece of a SCC. (b) Final configuration: a conformal crystal.

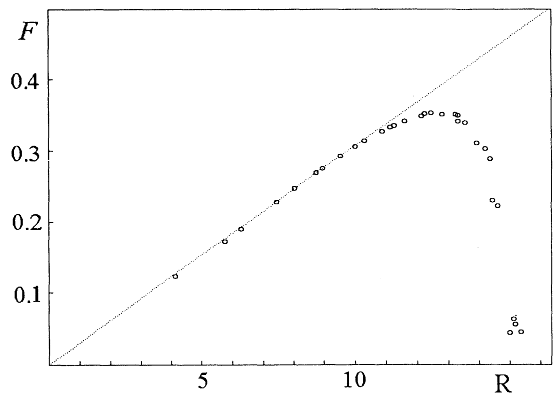

FIG. 11. The magnitude of forces acting on the particles shown in Fig. 10(a) plotted versus their radial coordinate. The dotted line presents the theoretically predicted dependence (valid for an infinite crystal) with $J(k)=3.317$. $k=3$, see text.

also touch their neighbors.

Assuming that the particles interact with each other via forces given by Eq. (10) with $k=3$ and $A=1$, we calculated numerically the total force acting on each particle. The forces are shown in Fig. 10(a).

For particles located close to the box center (i.e., far from the boundary wall), the forces are directed to a great accuracy towards the center of the disk, their magnitude growing with the distance from the center. Closer to the boundary, the forces diminish and eventually change their direction. Such is the result of the discontinuity introduced within the SCC by a finite boundary box. To check the accuracy of the theoretical predictions presented above, we plotted the magnitudes of calculated forces versus the radial coordinate of the particles see Fig. 11.

The excellent accord with theoretical predictions, observed for particles located close to the center of the box, is obtained for a value $J(k)=3.317$, which one finds limiting the summation from Eq. (31) but to 5. (The finite size of the considered sample well justifies the choice.)

Putting the circular box into steady rotation around its center allows one to compensate the interparticle forces with the centrifugal force. For an infinite crystal, at a properly chosen angular velocity, the compensation would be complete for all particles. In the finite case, we consider, this is obviously no longer possible. Figure 10(b) shows the result of such an attempt, with the angular velocity of the disk rotation chosen in such a manner, that the interparticle forces acting on the particles closest to the disk center are fully equilibrated. Obviously, the particles which at $\omega=0$ were already pressed against the boundary wall, are now pressed against it even stronger.

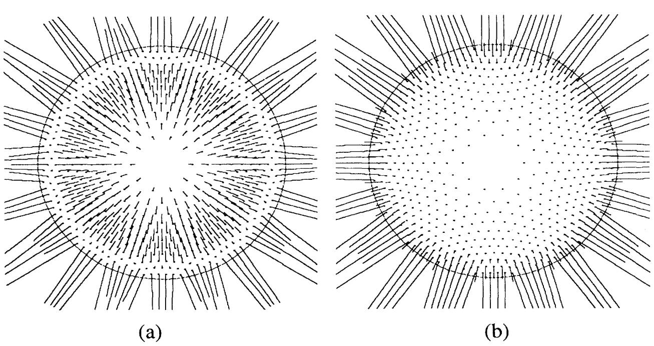

FIG. 10. (a) The field of interparticle forces calculated for the initial configuration shown in Fig. 9. (b) Total (interparticle + centrifugal) forces calculated for the angular velocity of the disk $\omega=0.172$.

It is interesting to check, if the conformal structure does not break down, when the particles, which are not equilibrated by the centrifugal force are allowed to move. Below we present results of such a simple numerical experiment. Its rules were as follows.

(1) To prevent formation of unnecessary defects we assumed that the boundary wall is sticky, i.e., each particle which touched it became immobilized.

(2) All other particles were allowed to move freely in the directions and with velocities determined by the total forces which act on them. The interparticle forces were calculated by summing up forces stemming from particles found within the range of five diameters around the particle in question.

(3) Overlappings, which occurred as a result of this procedure, were removed simply by pushing the overlapping particles apart.

Figure 9(b) presents the final configuration obtained in a computer experiment carried out according to these simple rules at $\omega=0.163$, i.e., slightly below the value which equilibrates the central particles of the sample. We lowered the angular velocity to prevent formation of a close-packed layer at the boundary wall. Particles which were immobilized by the sticky boundary wall are marked with black dots. Particles which touch each other are marked with lines joining their centers.

Obviously, except for particles which were stopped by the boundary wall and a few more, which came into close-packed contact with the latter, all particles of the final configuration are in equilibrium, i.e., the interparticle and centrifugal forces which act on them are in equilibrium. The structure shown in Fig. 9(b) is not a strictly conformal crystal. According to the terminology introduced in Sec. I it should be considered as a conformal crystal.

G. Energy of a strictly conformal crystal

Let us consider the case where the external force $\vec{F}^{\,\mathrm{ext}}$ is conservative, i.e., can be seen as a gradient of the potential energy U:

$$\vec{F}^{\,\mathrm{ext}} = -\vec{\nabla} U \ . \tag{64}$$

We can then express the equation of mechanical equilibrium (43a) as an equation of conservation of energy for a particle submitted both to the external and interparticle forces

$$U + AJ(k)n^{k/2} = U_0 = \mathrm{const} \ . \tag{65}$$

From Eq. (65) one can derive the density of energy

$$u = nU + \tfrac{1}{2}AJ(k)n^{(k+2)/2} \ , \tag{66}$$

so that the total energy of the system

$$E^{\mathrm{tot}} = \sum_{\substack{\text{occupied}\\ \text{sites}}} [U(w) + \tfrac{1}{2}AJ(k)n^{k/2}(w)]$$

$$\cong \int_{\mathrm{CC}} d\sigma \, n[U + \tfrac{1}{2}AJ(k)n^{k/2}] \ , \tag{67}$$

where the integration range covers the area occupied by the CC. The 1/2 factor takes into account the fact that since the summation is carried out over all particles, thus a contribution provided by a single pair or particle is counted twice.

As an example, let us consider the SCC on a rotating disk. The centrifugal force $\vec{F}_c$ is given by

$$\vec{F}_c = m\omega^2 \vec{R} = -\vec{\nabla} U \ . \tag{68}$$

In the reference frame rotating with the disk, the potential energy equals

$$U = -m\omega^2 \frac{R^2}{2} \tag{69}$$

thus the total energy

$$E^{\mathrm{tot}}_{\mathrm{rot}} = 2\pi \int_0^a dR \, Rn[U + \tfrac{1}{2}AJ(k)n^{k/2}] \ , \tag{70}$$

where a is the radius of the disk.

In the motionless (laboratory) frame, the total energy E^{tot} is given by [6]

$$E^{\mathrm{tot}} = E^{\mathrm{tot}}_{\mathrm{rot}} + \vec{L}\cdot\vec{\omega} \ , \tag{71}$$

where the angular momentum of the system can be written as

$$L = 2\pi \int_0^a dR \, m\omega R^3 n \ , \tag{72}$$

so that finally

$$E^{\mathrm{tot}} = 2\pi \int_0^a dR \, Rn \left[\frac{m\omega^2 R^2}{2} + \frac{1}{2} AJ(k)n^{k/2} \right] . \tag{73}$$

To fix the upper limit of integration, let us assume that in the limit the density of the SCC reaches its close-packing value $2/\sqrt{3}b^2$. Thus since according to (59)

$$n = \left[\frac{m\omega^2}{2AJ(k)} \right]^{2/k} R^{4/k} \ , \tag{74}$$

we find

$$n(a) = \frac{2}{\sqrt{3}b^2} = \left[\frac{m\omega^2}{2AJ(k)} \right]^{2/k} a^{4/k} \ . \tag{75}$$

Eventually, the total energy of the SCC contained within the disk of radius a is easily calculated if one notices that

$$n\frac{m\omega^2}{2}R^2 = \left[\frac{m\omega^2}{2AJ(k)} \right]^{2/k} \frac{m\omega^2}{2} R^{(2k+4)/k}$$

$$= 2[\tfrac{1}{2}AJ(k)n^{(k/2+1)}] \ , \tag{76}$$

i.e., the second term of the integral (73) is twice as big as the first one. The integration gives

$$E^{\mathrm{tot}} = \frac{3\pi}{8}\frac{k}{k+1}\frac{(m\omega^2)^{(k+2)/k}}{[AJ(k)]^{2/k}} a^{4(k+1)/k}$$

$$= \frac{3\pi}{8}\frac{k}{k+1} m\omega^2 a^4 n(a) \ . \tag{77}$$

On the other hand, the number of particles N inside the disk of a diameter a is given by

$$N = 2\pi \int_0^a dR \, Rn = 2\pi \frac{k}{2(k+2)} \left[\frac{m\omega^2}{2AJ(k)} \right]^{2/k} a^{[2(k+2)]/k}$$

$$= \pi \frac{k}{2+k} n(a) a^2 \tag{78}$$

thus, the total energy of the SCC within the disk of a diameter a, at which the SCC becomes close packed, equals

$$E^{\mathrm{tot}} = \frac{3\sqrt{3}}{16\pi}\frac{(k+2)^2}{k(k+1)} m\omega^2 (Nb)^2 \ . \tag{79}$$

Notice, that because of Eq. (75) $\omega^2 \propto a^{-2} \propto 1/N$, the total energy E^{tot} is eventually proportional to N.

III. THE STRICTLY CONFORMAL CRYSTAL AS A 2D SOLID WITH A UNIQUE ELASTICITY MODULE

In the classical theory of elasticity, simple fluids are characterized by a unique elastic modulus, namely, the compression modulus $E_p = -V(\partial p/\partial V)$. This is obvious, because simple fluids cannot resist shear. On the contrary, solids have at least two elastic moduli: isotropic solids are characterized by Young's modulus and the Poisson ratio, whereas solids with a lower symmetry exhibit at least three independent parameters describing their equilibrium behavior in the presence of static elastic constraints. Even isotropic 2D solids are characterized by two elastic moduli. Using a comparison with the classical theory of linear elasticity we shall show here that elastic properties of a strictly conformal crystal are described with a unique elastic modulus. Thus, let us consider a SCC submitted to an external force field $\vec{F}_{\rm ext}$. ($\vec{F}_{\rm ext}$ is the force exerted by the external field on a single particle.) In the case of a planar SCC, this would be a gravitational force decaying exponentially with height, whereas, in the case of a radial SCC, the necessary force could be identified with the inertial (centrifugal) force present within the coordinate frame of a rotating disk.

In the case of a SCC, (43a) would read

$$\vec{F}_{\rm ext}=AJ(k)\vec{\nabla}(n^{k/2}) . \tag{80}$$

Now, let us rewrite (4) in terms of the classical theory of elasticity

$$ds_w^2=ds_z^2+2\sum_{\alpha,\beta=1}^{2}\varepsilon_{\alpha\beta}dx_\alpha dx_\beta , \tag{81}$$

where, for the sake of simplicity (x,y) has been replaced with (x_1,x_2). $\varepsilon_{\alpha\beta}$ is the strain tensor [7].

In the linear approximation

$$\varepsilon_{\alpha\beta}=\frac{1}{2}\left[\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right] \tag{82}$$

while, outside the range of the linear theory, $\varepsilon_{\alpha\beta}$ takes the form [6]

$$\varepsilon_{\alpha\beta}=\frac{1}{2}\left[\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}+\sum_{\gamma=1}^{2}\frac{\partial u_\gamma}{\partial x_\alpha}\frac{\partial u_\gamma}{\partial x_\beta}\right] . \tag{83}$$

(83) is a direct consequence of the definition (81) and the form of $ds_w^2=dx^2+dy^2$ and $ds_z^2=du^2+dv^2$. On the other hand, the displacement vector $\vec{u}$ may be expressed as

$$\vec{u}=(u(x,y)-x,v(x,y)-y) . \tag{84}$$

It is easy to rewrite (83) as

$$\varepsilon=\begin{bmatrix}\frac{1}{2}\left[\left[\frac{\partial u}{\partial x}\right]^2+\left[\frac{\partial v}{\partial x}\right]^2\right] & \frac{1}{2}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]\\ \frac{1}{2}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right] & \frac{1}{2}\left[\left[\frac{\partial u}{\partial y}\right]^2+\left[\frac{\partial v}{\partial y}\right]^2\right]\end{bmatrix} \tag{85}$$

(see Ref. [4]). Being conformal, the mapping $(x,y)\rightarrow(u,v)$ fulfills the Cauchy relations (3). As a consequence

$$\varepsilon=\begin{bmatrix}\frac{1}{2}\left[\left[\frac{\partial u}{\partial x}\right]^2+\left[\frac{\partial v}{\partial x}\right]^2\right] & 0\\ 0 & \frac{1}{2}\left[\left[\frac{\partial u}{\partial y}\right]^2+\left[\frac{\partial v}{\partial y}\right]^2\right]\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\left|\frac{dw}{dz}\right|^2 & 0\\ 0 & \left|\frac{dw}{dz}\right|^2\end{bmatrix} \tag{86}$$

[compare (4) and (81)]. Equation (6) allows us to write

$$\varepsilon=\frac{1}{\sqrt{3}b^2}\begin{bmatrix}n^{-1} & 0\\ 0 & n^{-1}\end{bmatrix} \tag{87}$$

from which we find

$$\mathrm{tr}\varepsilon\equiv\varepsilon_{11}+\varepsilon_{22}=\frac{2}{\sqrt{3}b^2}n^{-1} . \tag{88}$$

Now, because of (83), we have

$$\mathrm{tr}\varepsilon=\vec{\nabla}\cdot\vec{u}+\frac{1}{2}\sum_{\alpha,\beta=1}^{2}\left[\frac{\partial u_\alpha}{\partial x_\beta}\right]^2\equiv\mathrm{Div}\,\vec{u} . \tag{89}$$

The right hand side of (89) is a nonlinear differential operator acting upon the displacement vector $\vec{u}$. We denote it Div $\vec{u}$, since in the linear approximation it reduces to the conventional divergence operator

$$\vec{\nabla}\cdot\vec{u}\equiv\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2} . \tag{90}$$

The important point here is that the definition (89) enables one to rewrite (80) in a compact form as

$$E(k)\vec{\nabla}[\mathrm{Div}\,\vec{u})^{-k/2}]=\vec{F}_{\rm ext} , \tag{91}$$

where the "elastic modulus" $E(k)$ replaces all constant factors appearing both in Eq. (80) and (88):

$$E(k)=AJ(k)\left[\frac{2^{1/2}}{b3^{1/4}}\right]^k . \tag{92}$$

On the other hand, in the linear theory of elasticity, an isotropic body immersed within an external force $\vec{f}$ per unit volume satisfies the following equation:

$$\Delta\vec{u}+\frac{1}{1-2\sigma}\vec{\nabla}\vec{\nabla}\cdot\vec{u}=-\vec{f}\frac{2(1+\sigma)}{E} , \tag{93}$$

where E is the Young modulus and σ is the Poisson ratio. Comparison of (91) and (93) is very informative.

In the case of a SCC, the relation between $\vec{F}_{\rm ext}$ and $\vec{u}$ is strongly nonlinear: first of all Div $\vec{u}$ is a nonlinear operator, moreover $\vec{F}_{\rm ext}$ and $\vec{u}$ are submitted to the complementary condition (43b). In the linear theory of elasticity (93), the relation between $\vec{g}$ and $\vec{u}$ depends on two independent elastic moduli E and σ, while in the case of a

SCC the analogous relation contains only one elastic modulus $E(k)$. In a certain sense the SCC case is much more isotropic: large distortions and a strong departure from the linear approximation allow the SCC to behave within the gravitational field as isotropically as possible—all due to appropriate variations both of the size and orientation of its hexagonal unit cells.

IV. CONFORMAL CRYSTALS: A MORE GENERAL CONCEPT

Section II was devoted to the concept of strictly conformal crystals. Looking at the gravity rainbow structure, one can immediately notice that the concept of a strictly conformal crystal is too narrow. Consideration presented in Sec. II led to the conclusion that the only external field within which a planar SCC can reach the equilibrium state must decay exponentially with height. On the other hand, comparison between the gravity rainbow, i.e., the structure formed in a constant (gravitational) field Fig. 1 and a part of the SCC shown in Fig. 5, leads to the conclusion, that they do not differ much [4]. The question arises: what happens to the SCC when the external field within which it was built changes its v dependence? Below we shall try to answer it. To fix attention we shall discuss a SCC formed in an external field of the planar symmetry.

Thus let us imagine that in an external field $\vec{F}^{\,\mathrm{ext}}$ of the proper, exponentially decaying shape a SCC was built and then, preserving the planar symmetry, the field changed slightly its v dependence. Let us assume, that as a result of the change, the structure of the crystal was also slightly deformed: leaving the exclusive world of strictly conformal crystals, the structure entered a much broader domain of conformal crystals [5].

Consider a particle p from the CC and compare two forces acting on it.

(1) F_{nn}—the force due to one of the nearest neighbors located on the hexagon surrounding p; its magnitude is given by

$$F_{\mathrm{nn}} \propto A\frac{1}{(b|\zeta|)^{k+1}} \; . \tag{94}$$

(2) F_{hex}—the total force due to all of the nearest neighbors of p located on the same hexagon; in view of Eq. (20) its magnitude is given by

$$F_{\mathrm{hex}} \propto A\frac{|\eta|}{b^k|\zeta|^{k+2}} \; . \tag{95}$$

Thus, because of Eq. (15)

$$\frac{F_{\mathrm{hex}}}{F_{\mathrm{nn}}} \propto \frac{|\eta| b}{|\zeta|} \ll 1 \; . \tag{96}$$

Let us discuss in more detail the sense and significance of this simple result. Any conformal map transforms a circle into a circle, providing the radius of the mapped circle is vanishingly small. This means, that any regular hexagon, of a vanishingly small size, is mapped onto another regular hexagon, thus $F_{\mathrm{hex}}/F_{\mathrm{nn}} \to 0$. As a consequence, a small uniaxial distortion of the SCC structure in the direction of its density gradient can easily change in suitable proportions the magnitude of $\vec{F}_{\mathrm{hex}}$ without changing its direction and without introducing any topological defects into the structure. Let us emphasize that the reasoning presented above is valid for any conformal map.

According to Eqs. (32) and (38) also the total force $\vec{F}_{\mathrm{int}}$ is parallel to the direction of density gradient and thus to the direction of the lines of constant orientation φ, independently of its value. As a consequence, shifting all hexagons along those lines on a small distance proportional to the local value of the density is likely to (i) modify the magnitude of $\vec{F}_{\mathrm{int}}$ without changing its direction (or sense), (ii) preserve the existing distribution of the orientation throughout the whole structure.

Summing up, one can expect, that when the external field $\vec{F}^{\,\mathrm{ext}}$, under which a SCC stays in equilibrium, is slightly distorted in the direction of its force lines, then a small, appropriate uniaxial distortion of the SCC structure itself, in the direction of its density gradient, can change the distribution of the $\vec{F}_{\mathrm{int}}$ forces in such a way, that the distorted structure will reach a new equilibrium with the distorted field. Below we study the problem in quantitative terms.

Looking for a conformal map associated with a given force field

Assume that a force field $\vec{F}\equiv(F_u,F_v)$ is given. Is it possible to build a conformal crystal satisfying the necessary (weak) condition, that the lines of constant orientation, $\varphi(u,v)=\mathrm{const}$, coincide with the integral trajectories of Eq. (38)? Requirement (38) is equivalent to the following one: there exists a function $\lambda(u,v)$, such that

$$d\varphi(u,v)=\lambda F_v du-\lambda F_u dv \tag{97}$$

is an exact differential form which vanishes along the integral trajectories of Eq. (38). The existence of λ is guaranteed because the problem is two dimensional: λ is the integrating factor of the differential form $F_v du - F_u dv$. Equation (97) can be expressed as

$$\frac{\partial\varphi}{\partial u}=\lambda(u,v)F_v, \quad \frac{\partial\varphi}{\partial v}=-\lambda(u,v)F_u \; . \tag{98}$$

On the other hand

$$\frac{\partial^2\varphi}{\partial u\,\partial v}=\frac{\partial^2\varphi}{\partial v\,\partial u} \; ,$$

thus

$$\vec{\nabla}\cdot\lambda\vec{F}=\frac{\partial}{\partial u}(\lambda F_u)+\frac{\partial}{\partial v}(\lambda F_v)=0 \; . \tag{99}$$

Now, if φ is to be interpreted as the phase, then it must be harmonic

$$\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}=\frac{\partial}{\partial u}(\lambda F_v)-\frac{\partial}{\partial v}(\lambda F_u)=0 \tag{100}$$

or

$$\vec{\mathrm{rot}}\,\lambda\vec{F}=0 \; , \tag{101}$$

which is equivalent to the demand that

$$\lambda\vec{F}(u,v)=-\vec{\nabla}\gamma \ . \tag{102}$$

Equations (98) and (102) lead to

$$\frac{\partial\varphi}{\partial u}=-\frac{\partial\gamma}{\partial v}, \quad \frac{\partial\varphi}{\partial v}=\frac{\partial\gamma}{\partial u} \ . \tag{103}$$

Thus the phase φ and function γ defined by Eq. (102) are *conjugate harmonic functions:* both satisfy the Laplace's equation and the Cauchy-Riemann condition. As a consequence

$$s(w)\equiv\gamma(u,v)+i\varphi(u,v) \tag{104}$$

is also an analytical function. The existence of $s(w)$ is not guaranteed: Eq. (101) imposes a condition on the force field $\vec{F}$, thus, if $\vec{F}$ is given, it should be checked to fulfill simultaneously Eqs. (99) and (101). If this is the case, one can make use of Eq. (36) which leads to

$$\frac{\partial}{\partial u}\ln|\zeta|=\frac{\partial\gamma}{\partial u}, \quad \frac{\partial}{\partial v}\ln|\zeta|=\frac{\partial\gamma}{\partial v} \ . \tag{105}$$

As a consequence,

$$\frac{dw}{dz}(w)\equiv\zeta(w)=|\zeta|e^{i\varphi}=\text{const}\times e^{s(w)} \ , \tag{106}$$

which can be solved to yield

$$z(w)=\text{const}\times\int dw\, e^{-s(w)} \ . \tag{107}$$

Examples. (a) The gravity rainbow is characterized by a force field $\vec{F}=(0,-mg)$. Equations (99) and (101) can be satisfied in a trivial manner if one writes

$$\varphi=-\lambda mgu \ , \tag{108}$$

whereby λ is now a small arbitrary constant. Therefore

$$\gamma=\lambda mgv, \quad s(w)=-i\lambda mgw \tag{109}$$

and

$$z(w)=\text{const}\times\int dw\, e^{-i\lambda mgw}\propto e^{-i\lambda mgw} \ . \tag{110}$$

(b) In the case of an arbitrary radial field

$$\vec{F}=(h(R)u,h(R)v), \quad R=(u^2+v^2)^{1/2} \ , \tag{111}$$

where $h(R)$ is a regular function everywhere, with a possible exception of the origin. Application of the procedure described above is straightforward if one looks for functions λ and γ depending on R alone. As a result one obtains

$$s(w)=\alpha(\ln R+i\Phi) \tag{112}$$

and

$$z(w)=\begin{cases}\text{const}\times w^{\alpha+1}, & \alpha\neq-1\\ \text{const}\times\ln w, & \alpha=-1 \ .\end{cases} \tag{113}$$

The case, where $\alpha=-1$ corresponds to a spiral lattice [4].

V. CONCLUSIONS

The theory of analytical functions has many applications in elementary physics. For instance, in two-dimensional electrostatics the distribution of the electric field between conductors of simple shapes can be visualized with the help of a suitable analytical function.

In this paper we discussed another type of such an application. Pointlike particles interacting through inverse power law forces and immersed in a static external field are shown to build two-dimensional structures, called conformal crystals, which can be well described by analytical function images of a perfect periodic lattice.

Strictly conformal crystals, i.e., conformal crystal in their pure form, cannot be built in external fields of any shape. If one assumes a certain symmetry of the external field, e.g., planar or radial, then its dependence on the variable along which the field is allowed to change proves to be completely determined by the exponent of the inverse power law. Simultaneously, the analytical function, which describes the structure of the SCC built in this field, is also given.

The power law forces, which allow existence of the conformal crystals are long range. As a consequence, strong boundary effects cannot be avoided. In the ordinary condensed matter, the interparticle forces are short range. As a consequence, the boundary effects can be safely accounted for by the introduction of the surface tension. This is no longer possible in the case of conformal crystals.

Another important problem one has to cope with is that any realistic model of the interparticle interaction has to contain a certain kind of a hard core of a finite radius. As a consequence, realistic conformal crystals always meet in their density an upper limit above which they are forced to change their structure into a close-packed one.

A certain drawback of the concept of a strictly conformal crystal can be found in the mathematics itself: the set of analytical functions is not rich enough to describe shapes and symmetries one can find in nature. Thus given a realistic, possible to create in a laboratory, external field of force, there is only a small chance that we shall be able to built within it a strictly conformal crystal. On the other hand, as we know now, the chance is nonzero: the field of the centrifugal force found within a rotating disk proves to fit perfectly well the requirements of the theory we developed. A laboratory experiment remains still to be done.

A reasonable weakening of the theoretical requirements, which we discussed in Sec. III, allows one to consider structures which being not strictly conformal crystals are topologically so similar to them that they deserve the name—conformal. The gravity rainbow structure observed in a uniform gravitational field makes a good example.

Among the essential conditions, under which the theory of strictly conformal crystals preserves its validity, we find the requirement that the interparticle interaction is described by a power law. Forces of this type are invariant with respect to a scale transformation. The connection between the scale invariance and the conformal mapping is well known in physics, notably in the theory of the critical phenomena [8]. When the forces do not exhibit scale invariance, the ordered state which may appear is not likely to display conformal structure.

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[1] Piotr Pierański, in *Phase Transitions in Soft Condensed Matter,* Vol. 211 of *NATO Advanced Study Institute, Series B: Physics,* edited by T. Riste and D. Sherington (Plenum, New York, 1989).

[2] To avoid permanent magnetization of the spheres, we used an ac, 50 Hz, power supply.

[3] L. S. Levitov, Europhys. Lett. **14**, 533 (1991); S. Douady and Y. Couder, Phys. Rev. Lett. **68**, 2098 (1992).

[4] F. Rothen, P. Pierański, N. Rivier, and A. Joyet, Eur. J. Phys. **14**, 227 (1993).

[5] N. Rivier, P. Pierański, and F. Rothen, *Aperiodic'94,* edited by G. Chapuis and W. Paciorek (World Scientific, Singapore, 1995).

[6] L. D. Landau and E. M. Lifshitz, *Physique Statistique* (Mir, Moscow, 1967).

[7] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, New York, 1959).

[8] M. E. Fisher, *Spatial Symmetries and Critical Phenomena,* The Fifth Philip Morris Science Symposium, Richmond, VA, 1985, edited by C. Greenslit Lunsford (Philip Morris Inc., New York, 1987).