

## Negative-energy perturbations in circularly cylindrical equilibria within the framework of Maxwell-drift kinetic theory

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The conditions for the existence of negative-energy perturbations (which could be nonlinearly unstable and cause anomalous transport) are investigated in the framework of linearized collisionless Maxwell-drift kinetic theory for the case of equilibria of magnetically confined, circularly cylindrical plasmas and vanishing initial field perturbations. For wave vectors with a nonvanishing component parallel to the magnetic field, the plane equilibrium conditions (derived by Throumoulopoulos and Pfirsch [Phys Rev. E **49**, 3290 (1994)]) are shown to remain valid, while the condition for perpendicular perturbations (which are found to be the most important modes) is modified. Consequently, besides the tokamak equilibrium regime in which the existence of negative-energy perturbations is related to the threshold value of  $2/3$  of the quantity  $\eta_\nu = \partial \ln T_\nu / \partial \ln N_\nu$ , a new regime appears, not present in plane equilibria, in which negative-energy perturbations exist for *any* value of  $\eta_\nu$ . For various analytic cold-ion tokamak equilibria a substantial fraction of thermal electrons are associated with negative-energy perturbations (active particles). In particular, for linearly stable equilibria of a paramagnetic plasma with flat electron temperature profile ( $\eta_e = 0$ ), the entire velocity space is occupied by active electrons. The part of the velocity space occupied by active particles increases from the center to the plasma edge and is larger in a paramagnetic plasma than in a diamagnetic plasma with the same pressure profile. It is also shown that, unlike in plane equilibria, negative-energy perturbations exist in force-free reversed-field pinch equilibria with a substantial fraction of active particles. The present results, in particular the fact that a threshold value of  $\eta_\nu$  is not necessary for the existence of negative-energy perturbations, enhance even more the relevance of these modes.

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### I. INTRODUCTION

Negative-energy perturbations are potentially dangerous because they may become nonlinearly unstable and cause anomalous transport [1–15]. Conditions for the existence of perturbations of this kind can be obtained on the basis of the expressions for the second variation of the free energy which were derived by Pfirsch and Morrison [6] for arbitrary perturbations of general equilibria within the framework of collisionless Maxwell-Vlasov and Maxwell-drift kinetic theories.

For homogeneous, magnetized plasmas and vanishing initial field perturbations they found that negative-energy perturbations exist for any wave vector  $\mathbf{k}$  having a nonvanishing component parallel to the magnetic field (parallel and oblique modes) whenever the condition

$$v_{\parallel} \frac{\partial f_{g\nu}^{(0)}}{\partial v_{\parallel}} > 0 \quad (1)$$

holds for the equilibrium guiding center distribution function  $f_{g\nu}^{(0)}$  for some particle species  $\nu$  and parallel velocity  $v_{\parallel}$  in the frame of lowest equilibrium energy. For inhomogeneous magnetically confined plasmas with equilibria depending on just one Cartesian coordinate  $y$ , Throumoulopoulos and Pfirsch [14] showed that, in addition to parallel and oblique modes, for which condition (1) also applies, perpendicular modes have negative energies if

$$\frac{dP^{(0)}}{dy} \frac{\partial f_{g\nu}^{(0)}}{\partial y} < 0 \quad (2)$$

holds, where  $P^{(0)}$  is the equilibrium plasma pressure. For tokamaklike equilibria, condition (2) implies a threshold value of  $2/3$  of the quantity  $\eta_\nu = \partial \ln T_\nu / \partial \ln N_\nu$ , where  $T_\nu$  is the temperature and  $N_\nu$  the density of particle species  $\nu$ . These investigations are extended in this paper to the more interesting case of circularly cylindrical plasmas. The method of investigation consists in evaluating the general expression for the second-order perturbation energy obtained by Pfirsch and Morrison within the framework of the linearized collisionless Maxwell-drift kinetic theory. The most important conclusions are (i) condition (1) for the existence of parallel and oblique modes remains valid, (ii) for tokamak and reversed-field pinch cold-ion equilibria a new regime appears, not present in plane equilibria, in which perpendicular negative-energy perturbations exist without restriction on the values of  $\eta_\nu$ .

The equilibrium properties of the circularly cylindrical plasmas under consideration are discussed in Sec. II. The second-order perturbation energy for vanishing initial field perturbations is derived in Sec. III. Some of the relevant lengthy calculations are presented in Appendix A. The conditions for the existence of negative-energy perturbations are obtained in Sec. IV. The cases of parallel, oblique, and perpendicular wave propagation are

examined separately. The consequences of the condition for the existence of perpendicular negative-energy perturbations in straight tokamak and reversed-field pinch equilibria are discussed in Sec. V. For various analytic cold-ion equilibria with non-negative and negative  $\eta_e$  values, the part of the velocity space occupied by electrons associated with negative-energy perturbations is also obtained. An example is presented in Appendix B. The main results are summarized in Sec. VI.

## II. EQUILIBRIUM

The collisionless Maxwell-drift kinetic theory applied in the present paper is based on Littlejohn's Lagrangian formulation of the guiding center theory [16] in the form given by Wimmel [17]. A brief review of this theory is given in the first paragraph of Sec. III. More details can be found in Ref. [6] and in Sec. II of Ref. [14].

For a magnetically confined, circularly cylindrical plasma the equilibrium vector potential and magnetic field are given by

$$\mathbf{A}^{(0)} = A_\theta^{(0)}(r)\mathbf{e}_\theta + A_z^{(0)}(r)\mathbf{e}_z \quad (3)$$

and

$$\mathbf{B}^{(0)} = B_\theta^{(0)}(r)\mathbf{e}_\theta + A_z^{(0)}(r)\mathbf{e}_z, \quad (4)$$

with

$$\frac{1}{r} \frac{d}{dr} (rA_\theta^{(0)}) = B_z^{(0)}, \quad -(A_z^{(0)})' = B_\theta^{(0)}. \quad (5)$$

Here,  $r$ ,  $\theta$ ,  $z$  are cylindrical coordinates with unit base vectors  $\mathbf{e}_r$ ,  $\mathbf{e}_\theta$ ,  $\mathbf{e}_z$  and the prime denotes differentiation with respect to  $r$ . It is assumed that there is no equilibrium electric field. To calculate the guiding center velocity, Eq. (25) below, one needs the following quantities:

$$\mathbf{b}^{(0)} = \frac{\mathbf{B}^{(0)}}{B^{(0)}} = \frac{B_\theta^{(0)}}{B^{(0)}}\mathbf{e}_\theta + \frac{B_z^{(0)}}{B^{(0)}}\mathbf{e}_z = b_\theta^{(0)}\mathbf{e}_\theta + b_z^{(0)}\mathbf{e}_z, \quad (6)$$

$$\mathbf{A}_\nu^{*(0)} = \mathbf{A}^{(0)} + \frac{m_\nu c}{e_\nu} v_\parallel \mathbf{b}^{(0)}, \quad (7)$$

$$e_\nu \phi_\nu^{*(0)} = \mu B^{(0)} + \left(\frac{m_\nu}{2}\right) v_\parallel^2, \quad (8)$$

$$\mathbf{v}_E^{(0)} = c \frac{\mathbf{E}^{(0)} \times \mathbf{B}^{(0)}}{(B^{(0)})^2} = \mathbf{0}, \quad (9)$$

$$\mathbf{E}_\nu^{*(0)} = -\nabla \phi_\nu^{*(0)} = -\frac{\mu}{e_\nu} (B^{(0)})' \mathbf{e}_r, \quad (10)$$

and

$$\begin{aligned} \mathbf{B}_\nu^{*(0)} &= \nabla \times \mathbf{A}_\nu^{*(0)} \\ &= B_\nu^{*(0)} \mathbf{b} + \frac{m_\nu c}{e_\nu} v_\parallel \frac{(b_\theta^{(0)})^2}{r} (\mathbf{e}_r \times \mathbf{b}^{(0)}), \end{aligned} \quad (11)$$

with

$$B_\nu^{*(0)} = \mathbf{B}_\nu^{*(0)} \cdot \mathbf{b}^{(0)} = B^{(0)} + \frac{m_\nu c}{e_\nu} v_\parallel Y_{\theta z} \quad (12)$$

and

$$\begin{aligned} Y_{\theta z}(r) &\equiv \mathbf{b}^{(0)} \cdot (\nabla \times \mathbf{b}^{(0)}) \\ &= (b_\theta^{(0)})' b_z^{(0)} - (b_z^{(0)})' b_\theta^{(0)} + \frac{b_\theta^{(0)} b_z^{(0)}}{r}. \end{aligned} \quad (13)$$

With the aid of Eqs. (6)–(13) the guiding center velocity takes the form

$$\begin{aligned} \mathbf{v}_{g\nu}^{(0)} &= v_\parallel \mathbf{b}^{(0)} - \frac{\mu c}{e_\nu B_\nu^{*(0)}} \frac{dB^{(0)}}{dr} (\mathbf{e}_r \times \mathbf{b}^{(0)}) \\ &\quad + \frac{v_\parallel^2}{\omega_\nu^*} \frac{(b_\theta^{(0)})^2}{r} (\mathbf{e}_r \times \mathbf{b}^{(0)}), \end{aligned} \quad (14)$$

with  $\omega_\nu^* \equiv e_\nu B_\nu^{*(0)} / cm_\nu$ . The first, second, and third terms in (14) are the component of  $\mathbf{v}_{g\nu}^{(0)}$  parallel to  $\mathbf{B}^{(0)}$ , the grad-B drift, and the curvature drift.  $\mathbf{v}_{g\nu}^{(0)}$  has no  $r$ -component and therefore  $r$  is a constant of motion. Since there is also no force parallel to  $\mathbf{B}^{(0)}$ , another constant of motion is the parallel guiding center velocity  $v_\parallel$ . The guiding center distribution functions  $f_{g\nu}^{(0)}$ , are therefore functions of  $r$ ,  $v_\parallel$ , and the magnetic moment  $\mu$ .

To calculate the current density  $\mathbf{J}^{(0)}$ , we apply the general formula (8.15) of Ref. [18], which was derived in the context of collisionless Maxwell-drift kinetic theory. The result is

$$\begin{aligned} \mathbf{J}^{(0)} &= \frac{c}{4\pi} \nabla \times \mathbf{B}^{(0)} \\ &= \sum_\nu e_\nu \int dv_\parallel d\mu B_\nu^{*(0)} f_{g\nu}^{(0)} \mathbf{v}_{g\nu} \\ &\quad - \sum_\nu c \nabla \times \int dv_\parallel d\mu \left\{ B_\nu^{*(0)} f_{g\nu}^{(0)} \right. \\ &\quad \left. \times \left( \mu \mathbf{b} - \frac{m_\nu}{B} v_\parallel \mathbf{v}_{g\nu\perp} \right) \right\}, \end{aligned} \quad (15)$$

where  $\mathbf{v}_{g\nu\perp} = \mathbf{v}_{g\nu} - v_\parallel \mathbf{b}$ . The first and second sums in (15) represent, respectively, the guiding center and the magnetization contributions to  $\mathbf{J}^{(0)}$ . Taking the cross product of Eq. (15) with  $\mathbf{B}^{(0)}$ , using Ampere's law on the left-hand side of the resulting equation and doing some straightforward algebraic manipulations yields

$$\frac{d}{dr} \left[ P^{(0)} + \frac{B^{(0)}}{8\pi} \right] + \frac{(B_\theta^{(0)})^2}{4\pi r} + \Pi(r) = 0, \quad (16)$$

with

$$P^{(0)} = \sum_\nu \int dv_\parallel d\mu \mu B^{(0)} B_\nu^{*(0)} f_{g\nu}^{(0)} \quad (17)$$

and

$$\begin{aligned} \Pi(r) \equiv & \sum_{\nu} \int dv_{\parallel} d\mu B_{\nu\parallel}^{*(0)} \frac{(b_{\theta}^{(0)})^2}{r} (\mu B^{(0)} - m_{\nu} v_{\parallel}^2) f_{g\nu}^{(0)} \\ & - 2 \sum_{\nu} \frac{m_{\nu} c}{e_{\nu}} \int dv_{\parallel} d\mu v_{\parallel} \left\{ \frac{b_{\theta}^{(0)} b_z^{(0)}}{r} \left( \mu (B^{(0)})' \right. \right. \\ & \left. \left. - m_{\nu} v_{\parallel}^2 \frac{(b_{\theta}^{(0)})^2}{r} \right) \right\}. \end{aligned} \quad (18)$$

Relation (16) can also be derived by the momentum-conservation relation  $T^{\mu}_{\mu} = 0$  with the tensor  $T^{\mu}_{\rho}$  given in explicit form by Eq. (76) of Ref. [19]. (The comma in the subscript denotes covariant derivative.) For Maxwellian

distribution functions it holds that  $\Pi_{\nu} = 0$ , and Eq. (16) reduces to the known MHD equilibrium relation.

### III. SECOND-ORDER PERTURBATION ENERGY

The second-order energy of perturbations around an equilibrium state is given by

$$F^{(2)} = \int d^3x T_0^{(2)0}, \quad (19)$$

where  $T_0^{(2)0}$  is the energy component of the second-order energy-momentum tensor [6]

$$\begin{aligned} T_{\rho}^{(2)\lambda} = & - \sum_{\nu} \int d\hat{q} d\tilde{P} \left( \frac{\partial S_{\nu}^{(1)}}{\partial \hat{q}^{\rho}} - \frac{e_{\nu}}{c} A_{\rho}^{(1)} \right) \left[ f_{\nu}^{(0)} \left( \frac{\partial S_{\nu}^{(1)}}{\partial \hat{q}^{\kappa}} - \frac{e_{\nu}}{c} A_{\kappa}^{(1)} \right) \frac{\partial^2 \mathcal{H}_{\nu}^{(0)}}{\partial \tilde{P}_{\lambda} \partial \tilde{P}_{\kappa}} \right. \\ & \left. + f_{\nu}^{(0)} F_{\tau\sigma}^{(1)} \frac{\partial^2 \mathcal{H}_{\nu}^{(0)}}{\partial \tilde{P}_{\lambda} \partial F_{\tau\sigma}^{(0)}} + \left( f_{\nu}^{(0)} \frac{\partial S_{\nu}^{(1)}}{\partial \tilde{P}_i} \right)_{,i} \frac{\partial \mathcal{H}_{\nu}^{(0)}}{\partial \tilde{P}_{\lambda}} \right] \\ & - 2 F_{\mu\rho}^{(1)} \sum_{\nu} \int d\hat{q} d\tilde{P} \left[ f_{\nu}^{(0)} \left( \frac{\partial S_{\nu}^{(1)}}{\partial \hat{q}^{\kappa}} - \frac{e_{\nu}}{c} A_{\kappa}^{(1)} \right) \frac{\partial^2 \mathcal{H}_{\nu}^{(0)}}{\partial \tilde{P}_{\kappa} \partial F_{\mu\lambda}^{(0)}} + f_{\nu}^{(0)} F_{\sigma\tau}^{(1)} \frac{\partial^2 \mathcal{H}_{\nu}^{(0)}}{\partial F_{\mu\lambda}^{(0)} \partial F_{\sigma\tau}^{(0)}} \right] - \frac{1}{4\pi} F_{\mu\rho}^{(1)} F^{(1)\mu\lambda} \\ & + \delta_{\rho}^{\lambda} \left( \sum_{\nu} \int d\hat{q} d\tilde{P} f_{\nu}^{(0)} (\mathcal{H}_{\nu}^{(2)} - \mathcal{H}_{\nu}^{(0)(2)}) + \frac{1}{16\pi} F_{\tau\sigma}^{(1)} F^{(1)\tau\sigma} \right). \end{aligned} \quad (20)$$

Here, the superscripts (0), (1), and (2) denote, respectively, equilibrium first- and second-order quantities;  $A_{\rho} = (-\phi, \mathbf{A})$ , where  $\phi$  is the scalar potential and  $\mathbf{A}$  the vector potential of the electromagnetic field;  $F_{\mu\nu}$  is the electromagnetic tensor;  $S_{\nu}^{(1)}$  are generating functions associated with the perturbations; the scalar quantity  $[f_{\nu}^{(0)} (\partial S_{\nu}^{(1)} / \partial \tilde{P}_i)]_{,i}$  results from the contraction in the second-order tensor  $[f_{\nu}^{(0)} (\partial S_{\nu}^{(1)} / \partial \tilde{P}_i)]_{,j}$ ; the rest of the notation is defined on page 273 of Ref. [6]. In expression (20) the time derivatives  $\partial S_{\nu}^{(1)} / \partial t$  are given by

$$\begin{aligned} \frac{\partial S_{\nu}^{(1)}}{\partial t} - e_{\nu} A_0^{(1)} = & - [S_{\nu}^{(1)}, H_{\nu}^{(0)}] + \frac{e_{\nu}}{c} \mathbf{A}^{(1)} \cdot \frac{\partial H_{\nu}^{(0)}}{\partial \mathbf{P}} \\ & - F_{\mu\lambda}^{(1)} \frac{\partial H_{\nu}^{(0)}}{\partial F_{\mu\lambda}^{(0)}}, \end{aligned} \quad (21)$$

where the mixed variable Poisson bracket is defined as

$$[a, b] = \frac{\partial a}{\partial \hat{q}_i} \frac{\partial b}{\partial \tilde{P}_i} - \frac{\partial a}{\partial \tilde{P}_i} \frac{\partial b}{\partial \hat{q}_i}.$$

The Hamiltonian for the guiding center motion of particle species  $\nu$  is obtained from the Lagrangian

$$L_{\nu} = \left( \frac{e_{\nu}}{c} \right) \mathbf{A}_{\nu}^* \cdot \dot{\mathbf{x}} - e_{\nu} \phi_{\nu}^* \quad (22)$$

with

$$\mathbf{A}_{\nu}^* = \mathbf{A} + \frac{m_{\nu} c}{e_{\nu}} q^4 \mathbf{b},$$

$$e_{\nu} \phi_{\nu}^* = e_{\nu} \phi + \mu B + \frac{m_{\nu}}{2} [(q^4)^2 + \mathbf{v}_E^2],$$

$$\mathbf{v}_E = c \frac{\mathbf{E} \times \mathbf{B}}{B^2},$$

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{b} = \frac{\mathbf{B}}{B}.$$

This Lagrangian is defined in terms of the variables

$$t, \quad \mathbf{x} = \mathbf{x}(q^1, q^2, q^3), \quad \text{and } q^4.$$

Here,  $q^1, q^2, q^3$  are generalized coordinates in normal space and  $q^4$  is an additional independent variable for which one of the Lagrangian equations yields the relation  $q^4 = \mathbf{v} \cdot \mathbf{b} = v_{\parallel}$ . The momenta canonically conjugated to  $\mathbf{x}$  and  $q^4$  follow from (22) as

$$\mathbf{p} = \frac{\partial L_{\nu}}{\partial \dot{\mathbf{x}}} = \frac{\partial L_{\nu}}{\partial \dot{q}^l} \mathbf{e}^l = \frac{e_{\nu}}{c} \mathbf{A}_{\nu}^*, \quad p_4 = \frac{\partial L_{\nu}}{\partial \dot{q}^4} = 0, \quad (23)$$

where  $\mathbf{e}^l$  are the reciprocal base vectors. Since Eqs. (23) do not contain  $\dot{\mathbf{x}}$  and  $\dot{q}^4$ , they are constraints between the momenta and the coordinates. It therefore follows that Hamilton's equations based on the usual Hamilto-

nian corresponding to the above nonstandard Lagrangian are not the equations of motion. To overcome this difficulty, Dirac's theory of constrained dynamics [20] is applied, which yields the Dirac Hamiltonians:

$$H_\nu = e_\nu \phi_\nu^* + \mathbf{v}_{g\nu} \cdot [\mathbf{p} - (e_\nu/c) \mathbf{A}_\nu^*] + V^4 p_4, \quad (24)$$

from which

$$\dot{\mathbf{x}} = \mathbf{v} = \frac{\partial H_\nu}{\partial \mathbf{p}} = \mathbf{v}_{g\nu} (t, \mathbf{x}, q^4) = \frac{q^4}{B_{\nu\parallel}^*} \mathbf{B}_\nu^* + \frac{c}{B_{\nu\parallel}^*} \mathbf{E}_\nu^* \times \mathbf{b} \quad (25)$$

and

$$\dot{q}^4 = \frac{\partial H_\nu}{\partial p_4} = V^4 (t, \mathbf{x}, q^4) = \frac{e_\nu}{m_\nu} \frac{1}{B_{\nu\parallel}^*} \mathbf{E}_\nu^* \cdot \mathbf{B}_\nu^* \quad (26)$$

follow. [Here,  $\mathbf{E}_\nu^* \equiv \nabla \phi_\nu^* - (1/c) \partial \mathbf{A}_\nu^* / \partial t$ ,  $\mathbf{B}_\nu^* \equiv \nabla \times \mathbf{A}_\nu^*$ , and  $B_{\nu\parallel}^* = \mathbf{B}_\nu^* \cdot \mathbf{b}$ .] Special solutions of the equations of motion following from the Hamiltonians (24) are the constraints (23). The distribution functions  $f_\nu(\mathbf{x}, q^4, \mathbf{p}, p_4, t)$  must guarantee that these constraints are satisfied. As concerns this requirement, it is important to note that  $\mathbf{p} - (e_\nu/c) \mathbf{A}_\nu^* = 0$  and  $p_4 = 0$  do not represent special values of some constants of motion. Therefore,  $\delta$  functions of the constraints are not constants of motion either. But  $f_\nu$  must be proportional to such  $\delta$  functions and, at the same time, also a constant of motion. Both conditions are uniquely satisfied by

$$f_\nu = \delta(p_4) \delta\left(\mathbf{p} - \frac{e_\nu}{c} \mathbf{A}_\nu^*\right) B_{\nu\parallel}^* f_{g\nu}(\mathbf{x}, q^4, \mu, t), \quad (27)$$

where the guiding center distribution functions  $f_{g\nu}$  are constants of motion and solutions of the drift kinetic differential equations

$$\frac{\partial f_{g\nu}}{\partial t} + \mathbf{v}_{g\nu} \cdot \frac{\partial f_{g\nu}}{\partial \mathbf{x}} + V^4 \frac{\partial f_{g\nu}}{\partial q^4} = 0. \quad (28)$$

In the present paper, the second-order perturbation energy is calculated for the case of equilibria defined in Sec. II and for initial perturbations  $\mathbf{A}^{(1)} = \dot{\mathbf{A}}^{(1)} = \mathbf{0}$ . It is also shown *a posteriori* that one can choose initial perturbations without changing the particle contribution to the energy such that the corresponding charge density  $\rho^{(1)}$  vanishes. Therefore, choosing initial perturbations of this kind, we put from the outset

$$F_{\mu\lambda}^{(1)} \equiv 0, \quad A_\rho^{(1)} \equiv 0. \quad (29)$$

Equation (21) then reduces to

$$\frac{\partial S_\nu^{(1)}}{\partial t} = - [S_\nu^{(1)} H_\nu^{(0)}], \quad (30)$$

the Dirac Hamiltonians to

$$H_\nu^{(0)} = e_\nu \phi_\nu^{*(0)} + \mathbf{v}_{g\nu}^{(0)} \cdot \left[ \mathbf{P} - \frac{e_\nu}{c} \mathbf{A}_\nu^{*(0)} \right], \quad (31)$$

and Eq. (19), after integration by parts of the term which contains derivatives of  $f_\nu^{(0)}$ , i.e.,  $(f_\nu^{(0)} \partial S_\nu^{(1)} / \partial \tilde{P}_i)_{,i}$ , takes the form

$$F^{(2)} = \sum_\nu \int d^3 x d q_4 d \tilde{P} f_\nu^{(0)} \mathcal{A}. \quad (32)$$

Here,

$$\begin{aligned} \mathcal{A} = & \frac{1}{2} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i} \frac{\partial}{\partial q^i} \left( \frac{\partial H_\nu^{(0)}}{\partial \tilde{q}^j} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_j} \right) \\ & - \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i} \frac{\partial}{\partial \tilde{q}^i} \left( \frac{\partial S_\nu^{(1)}}{\partial \tilde{q}^j} \frac{\partial H_\nu^{(0)}}{\partial \tilde{P}_j} \right) \end{aligned} \quad (33)$$

with  $i, j = 1, \dots, 4$ . As is shown in Appendix A, one further obtains

$$\begin{aligned} \mathcal{A} = & -\frac{1}{2} \frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^1} (\xi^1)^2 - \frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial (A_{g\nu}^{*(0)})_l}{\partial q^4} \xi^1 \xi^4 \\ & - \frac{1}{2} \frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^4} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^4} (\xi^4)^2 + \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial \hat{S}_\nu^{(1)}}{\partial q^l} \xi^1 + (v_{g\nu}^{(0)})^l \frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial q^k \partial q^l} \xi^k + \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^4} \frac{\partial \hat{S}_\nu^{(1)}}{\partial q^l} \xi^4 \\ & + (v_{g\nu}^{(0)})^l \frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial q^4 \partial q^l} \xi^4 + \frac{e_\nu}{c} (v_{g\nu}^{(0)})^k \frac{\partial (A_\nu^{*(0)})^l}{\partial q^1} \frac{\partial \xi^l}{\partial q^k} \xi^1 + \frac{e_\nu}{c} (v_{g\nu}^{(0)})^k \frac{\partial (A_\nu^{*(0)})_l}{\partial q^1} \frac{\partial \xi^l}{\partial q^k} \xi^1 + \frac{e_\nu}{c} (v_{g\nu}^{(0)})^k \frac{\partial (A_\nu^{*(0)})_l}{\partial q^4} \frac{\partial \xi^l}{\partial q^k} \xi^4, \end{aligned} \quad (34)$$

with  $k, l = 1, \dots, 3$ . Since the equilibrium is independent of  $q^2$  and  $q^3$ , an appropriate ansatz for the functions  $\hat{S}_\nu^{(1)}$  is

$$\hat{S}_\nu^{(1)} \equiv G_\nu^{(1)}(q^1, q^4, \mu) e^{i(\mathbf{k}_{23} \cdot \mathbf{x})}. \quad (35)$$

The wave vector  $\mathbf{k}_{23} = \mathbf{k}_{\theta z}$  introduced here has constant covariant components  $k_2$  and  $k_3$  and physical components  $k_\theta$  and  $k_z$ :

$$\mathbf{k}_{23} = k_2 \mathbf{e}^2 + k_3 \mathbf{e}^3 = k_\theta \mathbf{e}_\theta + k_z \mathbf{e}_z = \mathbf{k}_{\theta z}. \quad (36)$$

Therefore, it lies on magnetic surfaces. Substituting the integral over the momentum space according to the rule

$\int d\tilde{P}f_{\nu}^{(0)} \dots \rightarrow \int d\mu B_{\nu\parallel}^{*(0)} f_{g\nu}^{(0)} \dots$  (a proof is given in Ref. [18]), introducing real quantities by the rule  $AB \rightarrow \frac{1}{2}\Re A^*B$ , inserting Eqs. (37)–(40) of Ref. [14] for  $\xi^i$  and Eq. (36) into Eq. (32) and integrating in Eq. (32) the terms in which  $\partial|G^{(1)}|^2/\partial q^4$  and  $\partial|G^{(1)}|^2/\partial q^1$  appear yields (after a lengthy algebra) the concise form

$$F^{(2)} = -S \sum_{\nu} \int r dr dv_{\parallel} d\mu \left\{ \frac{B_{\nu\parallel}^{*(0)}}{m_{\nu}} |G_{\nu}^{(1)}|^2 (\mathbf{k}_{\theta z} \cdot \mathbf{v}_{g\nu}^{(0)}) \left[ \left( k_{\parallel} + k_{\perp} \frac{v_{\parallel}}{\omega_{\nu}^{*(0)}} \frac{(b_{\theta}^{(0)})^2}{r} \right) \frac{\partial f_{g\nu}^{(0)}}{\partial v_{\parallel}} - k_{\perp} \frac{1}{\omega_{\nu}^{*(0)}} \frac{\partial f_{g\nu}^{(0)}}{\partial r} \right] \right\}. \quad (37)$$

(Here,  $S$  is a normalization surface  $= 2\pi r_0 L_z$ ,  $r_0 =$  plasma radius, and  $L_z =$  length of plasma column to which  $F^{(2)}$  refers.) We note that  $F^{(2)}$  depends on  $G_{\nu}^{(1)}$  only via  $|G_{\nu}^{(1)}|^2$ .

Since the first-order charge density  $\rho^{(1)}$  is a  $v_{\parallel}, \mu$  integral over an expression that is linear in  $S_{\nu}^{(1)}$  and therefore also linear in  $G_{\nu}^{(1)}$ , one can satisfy the relation  $\rho^{(1)} = 0$  (invoked at the beginning of the second paragraph of this section) by a proper distribution of positive and negative values of  $G_{\nu}^{(1)}$ , on which  $F^{(2)}$  does not depend.

For vanishing field line curvature ( $B_{\theta}^{(0)} = 0$  or  $r \rightarrow \infty$ ), Eq. (37) reduces to the  $F^{(2)}$  expression for plane equilibria which was derived by Throumoulopoulos and Pfirsch [14] [Eq. (82) therein]. New terms here are the curvature-drift component of  $\mathbf{v}_{g\nu}^{(0)}$ , and  $k_{\perp} v_{\parallel} / \omega_{\nu}^{*(0)} (b_{\theta}^{(0)})^2 / r \partial f_{g\nu}^{(0)} / \partial v_{\parallel}$ . The latter term signifies that  $\partial f_{g\nu}^{(0)} / \partial v_{\parallel}$  plays a role for perturbations propagating not parallel to  $\mathbf{B}^{(0)}$  ( $k_{\perp} \neq 0$ ), a property deriving from the fact that the curvature drift component of  $\mathbf{v}_{g\nu}^{(0)}$  depends (quadratically) on the parallel velocity  $v_{\parallel}$ .

#### IV. CONDITION FOR THE EXISTENCE OF NEGATIVE-ENERGY PERTURBATIONS

First it is again noted that the conditions for the existence of negative-energy perturbations hold if the chosen frame of reference is that of minimum energy. Perturbations propagating parallel, obliquely, and perpendicularly to  $\mathbf{B}^{(0)}$  are separately considered.

##### A. Parallel modes ( $k_{\perp} = 0$ )

In this case Eq. (37) reduces to

$$F^{(2)} = -S \sum_{\nu} \int r dr dv_{\parallel} d\mu \left[ \frac{B_{\nu\parallel}^{*(0)}}{m_{\nu}} |G_{\nu}^{(1)}|^2 k_{\parallel}^2 v_{\parallel} \frac{\partial f_{g\nu}^{(0)}}{\partial v_{\parallel}} \right]. \quad (38)$$

Thus, one obtains  $F^{(2)} < 0$  if

$$v_{\parallel} \frac{\partial f_{g\nu}^{(0)}}{\partial v_{\parallel}} > 0 \quad (39)$$

holds for some  $r$ ,  $v_{\parallel}$ , and  $\mu$  for any particle species  $\nu$ . Condition (39), first derived by Pfirsch and Morrison [6] for a homogeneous, magnetized plasma, guarantees the existence of negative-energy perturbations without any

restrictions on the magnitude or orientation of the wave vector other than  $k_{\parallel} \neq 0$ : it suffices to localize  $G_{\nu}^{(1)}$  to the region in  $r$ ,  $v_{\parallel}$ , and  $\mu$  where  $v_{\parallel} (\partial f_{g\nu}^{(0)} / \partial v_{\parallel}) > 0$ . Outside this region  $G_{\nu}^{(1)}$  vanishes. All the other  $G_{\lambda}^{(1)}$ , i.e., with  $\lambda \neq \nu$ , are set equal to zero. The sign of  $F^{(2)}$  is then determined only by the sign of the integrand in the region of localization. This result agrees with those obtained by Correa-Restrepo and Pfirsch for several Vlasov-Maxwell equilibria [7–10].

##### B. Oblique modes ( $k_{\parallel} \neq 0$ and $k_{\perp} \neq 0$ )

With the definitions

$$C = v_{\parallel} \frac{k_{\parallel}}{k_{\perp}} - \frac{\mu c}{e_{\nu} B_{\nu\parallel}^{*(0)}} \frac{dB^{(0)}}{dr} + \frac{v_{\parallel}}{\omega_{\nu}^{*(0)}} \frac{(b_{\theta}^{(0)})^2}{r} \quad (40)$$

and

$$D = \frac{k_{\parallel}}{k_{\perp}} \frac{\partial f_{g\nu}^{(0)}}{\partial v_{\parallel}} - \frac{1}{\omega_{\nu}^{*(0)}} \left[ \frac{\partial f_{g\nu}^{(0)}}{\partial r} - v_{\parallel} \frac{(b_{\theta}^{(0)})^2}{r} \frac{\partial f_{g\nu}^{(0)}}{\partial v_{\parallel}} \right], \quad (41)$$

Eq. (37) yields  $F^{(2)} < 0$  if

$$C > 0 \quad \text{and} \quad D > 0 \quad (42)$$

or

$$C < 0 \quad \text{and} \quad D < 0. \quad (43)$$

The following two cases are now considered separately.

(a) Let us first assume that

$$v_{\parallel} \frac{\partial f_{g\nu}^{(0)}}{\partial v_{\parallel}} > 0 \quad (44)$$

again holds locally in  $r$ ,  $v_{\parallel}$ , and  $\mu$  for any particle species  $\nu$ . It then follows from inequalities (42) and (43) that

$$\frac{k_{\parallel}}{k_{\perp}} < \min(\Lambda_{\nu}, M_{\nu}) \quad \text{or} \quad \frac{k_{\parallel}}{k_{\perp}} > \max(\Lambda_{\nu}, M_{\nu}), \quad (45)$$

with

$$\Lambda_\nu \equiv \frac{1}{v_\parallel} \frac{\mu c}{e_\nu B_\nu^{*(0)}} \frac{dB^{(0)}}{dr} - \frac{1}{\omega_\nu^{*(0)} v_\parallel} \frac{(b_\theta^{(0)})^2}{r}$$

and

$$M_\nu \equiv -\frac{v_\parallel}{\omega_\nu^{*(0)}} \frac{(b_\theta^{(0)})^2}{r} + \frac{1}{\omega_\nu^{*(0)}} \frac{\partial f_{g\nu}^{(0)}}{\partial r} \left( \frac{\partial f_{g\nu}^{(0)}}{\partial q^4} \right)^{-1}.$$

The perturbations  $G_\nu^{(1)}$  are localized as in the previous case of parallel propagation. The orders of magnitude of  $\Lambda_\nu$  and  $M_\nu$  depend on the particle energy. For thermal particles, these being the most representative particles, it holds that  $|\Lambda_\nu| \approx |M_\nu| \approx (r_{L\nu})_{th}/r_0 \ll 1$ , [ $(r_{L\nu})_{th}/r_0$  is the thermal Larmor radius], and consequently condition (45) imposes no essential restriction on the magnitude or orientation of  $\mathbf{k}_{\theta z}$  associated with negative-energy perturbations.

(b) If one has

$$v_\parallel \frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel} < 0, \quad (46)$$

at some  $r$ ,  $v_\parallel$ , and  $\mu$  for any  $\nu$ , a condition which is more frequently satisfied (e.g., in the case of Maxwellian distribution functions), it follows from inequalities (42) and (43) that negative-energy perturbations exist if, in addition to (46),

$$\min(\Lambda_\nu, M_\nu) < \frac{k_\parallel}{k_\perp} < \max(\Lambda_\nu, M_\nu) \quad (47)$$

holds. For thermal particles the latter condition implies that

$$\frac{k_\parallel}{k_\perp} \approx \frac{(r_{L\nu})_{th}}{r_0} \ll 1. \quad (48)$$

Therefore, the most important negative-energy perturbations, in the sense that the less restrictive condition (46) is involved, concern nearly perpendicular modes.

### C. Perpendicular modes ( $k_\parallel = 0$ )

In this case, with the aid of the equilibrium condition (16), Eq. (37) reduces to

$$F^{(2)} = 4\pi S \sum_\nu \int r dr dv_\parallel d\mu |G_\nu^{(1)}|^2 \frac{B_\nu^{*(0)}}{m_\nu^2} \times \frac{W_{\nu\perp}}{(B^{(0)})^2} \left( \frac{k_\perp}{\omega_\nu^{*(0)}} \right)^2 R_\nu Q_\nu \quad (49)$$

with

$$R_\nu = \frac{dP^{(0)}}{dr} + \frac{(B_\theta^{(0)})^2}{4\pi r} \left( 1 + \frac{2W_{\nu\parallel}}{W_{\nu\perp}} \right) + \Pi(r) \quad (50)$$

and

$$Q_\nu = \left( \frac{\partial f_{g\nu}^{(0)}}{\partial r} - \frac{(b_\theta^{(0)})^2}{r} \frac{\partial f_{g\nu}^{(0)}}{\partial v_\parallel} \right). \quad (51)$$

Here,  $W_{\nu\parallel}$  and  $W_{\nu\perp}$  are the parallel and perpendicular particle energies. Negative-energy perturbations exist whenever the condition

$$R_\nu < 0 \text{ and } Q_\nu > 0 \quad (52)$$

or

$$R_\nu > 0 \text{ and } Q_\nu < 0 \quad (53)$$

holds. Condition (53), which cannot be satisfied by plane equilibria with singly peaked pressure profiles for which  $R_\nu = dP^{(0)}/dr \leq 0$ , determines a new regime of negative-energy perturbations. The consequences of (52) and (53) for straight tokamak and reversed-field pinch equilibria are examined in Sec. V. To simplify the notation, the superscript (0) will be suppressed on the understanding that all quantities pertain to equilibrium.

## V. PERPENDICULAR NEGATIVE-ENERGY PERTURBATION IN EQUILIBRIA OF MAGNETIC CONFINEMENT SYSTEMS

### A. Straight tokamak equilibria

Straight tokamak plasmas which are close to thermal equilibrium can be described by shifted Maxwellian distribution functions

$$f_{g\nu} = \left( \frac{m_\nu}{2\pi} \right)^{\frac{1}{2}} \frac{N_\nu(r)}{T_\nu^{3/2}(r)} \times \exp \left\{ -\frac{\mu B(r) + \frac{1}{2m_\nu} [v_\parallel - V_\nu(r)]^2}{T_\nu(r)} \right\}. \quad (54)$$

$N_\nu$  and  $T_\nu$  are, respectively, the number density and temperature (in energy units) for particles of species  $\nu$ .  $V_\nu$  leads to a net "toroidal" current and satisfies

$$\frac{V_\nu}{(v_\nu)_{th}} \approx \frac{(r_{L\nu})_{th}}{r_0} \ll 1. \quad (55)$$

In the remainder of the paper the analysis will be carried out up to zeroth order in  $(r_{L\nu})_{th}/r_0$ , i.e., small terms of the order of  $[(r_{L\nu})_{th}/r]^n$  (with  $n \geq 1$ ) will be dropped. In this context, from (18) one obtains  $\Pi_\nu \approx 0$ , and Eqs. (16) and (50) reduce to, respectively,

$$\frac{d}{dr} \left( P + \frac{B^2}{8\pi} \right) + \frac{B_\theta^2}{4\pi r} = 0 \quad (56)$$

and

$$R_\nu = \frac{dP}{dr} + \frac{B_\theta^2}{4\pi r} \left( 1 + 2 \frac{W_{\nu\parallel}}{W_{\nu\perp}} \right). \quad (57)$$

For distribution functions (54), negative-energy perturbations exist if the relation

$$R_\nu Q_\nu = R_\nu \left( \frac{N'_\nu}{N_\nu} \right) U_\nu f_{g\nu} < 0 \quad (58)$$

is satisfied. Here,

$$U_\nu \equiv 1 - \frac{3}{2}\eta_\nu + \eta_\nu \frac{W_{\nu\perp}}{T_\nu} \left( 1 + \frac{W_{\nu\parallel}}{W_{\nu\perp}} \right) + \frac{4\pi}{B^2} \frac{W_{\nu\perp}}{T_\nu} \frac{N_\nu}{N'_\nu} R_\nu, \quad (59)$$

with

$$\eta_\nu \equiv \frac{\partial \ln T_\nu}{\partial \ln N_\nu}. \quad (60)$$

It is now assumed that both the density and temperature profiles are singly peaked and therefore  $\eta_\nu \geq 0$  for all  $\nu$ . Negative-energy perturbations thus exist in the following two regimes.

(a)  $R_\nu < 0$ . This implies that  $R_\nu (N'_\nu/N_\nu) > 0$  and, consequently, condition (58) is satisfied if  $U_\nu < 0$ . Since the last two terms of  $U_\nu$  are non-negative and vanish for  $W_{\nu\parallel} = W_{\nu\perp} = 0$ , the condition  $U_\nu < 0$  can be satisfied if

$$\eta_\nu > \frac{2}{3} \quad (61)$$

holds for some particle species  $\nu$ . The existence of perpendicular negative-energy perturbations for any perpendicular wave number is therefore related to the threshold value of  $2/3$  of the quantity  $\eta_\nu$ . As discussed in Ref. [14], this threshold value is subcritical in the sense that it is lower than the critical value  $\eta_\nu^c \approx 1$  for linear stability of temperature-gradient-driven modes.

(b)  $R_\nu > 0$ . Condition (58) is now satisfied if  $U_\nu > 0$ . In this case negative-energy perturbations exist for any  $k_\perp$  without restriction on the values of  $\eta_\nu$ .

We now find the part of the velocity space occupied by particles associated with negative-energy perturbations (active particles). The particular particles with energy components  $W_{\nu\parallel} = T_\nu/2$  and  $W_{\nu\perp} = T_\nu$ , and consequently with velocities equal to the root mean square velocity  $(v_\nu)_{rms} = \sqrt{3T_\nu/m_\nu}$ , are first examined. For these particles, henceforth called *representative particles*, the quantity  $U_\nu$  becomes *independent of  $\eta_\nu$* . Condition  $R_\nu < 0$ ,  $U_\nu < 0$  is then impossible and condition  $R_\nu > 0$ ,  $U_\nu > 0$ , concerning the new regime, takes the simpler form

$$-1 < \frac{4\pi}{B^2} \frac{N_\nu}{N'_\nu} \left( P' + \frac{B_\theta^2}{2\pi r} \right) < 0. \quad (62)$$

Condition (62) guarantees that the representative particles are active particles.

For particles with arbitrary velocities the part of the velocity space occupied by active particles is determined on the basis of analytic solutions constructed in the following way.

Inserting the distribution function (54) into the equilibrium equation (15) and carrying out the integrations with respect to  $v_\parallel$  and  $\mu$ , one obtains

$$J_\theta = b_\theta \sum_\nu e_\nu N_\nu V_\nu + \frac{cb_z}{B} P' = -\frac{c}{4\pi} B'_z \quad (63)$$

and

$$-J_z = -b_z \sum_\nu e_\nu N_\nu V_\nu + \frac{cb_\theta}{B} P' = -\frac{c}{4\pi} \frac{1}{r} (rB_\theta)'. \quad (64)$$

To get some simple kind of insight, we now restrict discussion to  $T_i = 0$ , a case often considered in the literature, e.g., [21, 22]. For cold ions Eqs. (63) and (64) yield

$$c \frac{b_z}{B} P' - eb_\theta N_e V_e = -\frac{c}{4\pi} B'_z \quad (65)$$

and

$$c \frac{b_\theta}{B} P' + eb_z N_e V_e = -\frac{c}{4\pi} \frac{1}{r} (rB_\theta)' \quad (66)$$

with  $e_e = -e$  and

$$P = N_e T_e. \quad (67)$$

To obtain analytic straight tokamak equilibria, it is convenient to use, instead of Eqs. (65) and (66), Eq. (65) and

$$\nabla^2 \psi = -4\pi \frac{d}{d\psi} \left( P(\psi) + \frac{B_z^2(\psi)}{8\pi} \right), \quad (68)$$

which is equivalent to the equilibrium condition (56). Here,  $\psi(r)$  is the usual poloidal flux function. Assigning the  $\psi$  dependence of the functionals  $P(\psi)$  and  $B_z(\psi)$  and the  $r$  dependence of  $V_e(r)$ , one obtains from the solution of Eq. (68) the poloidal magnetic field  $\mathbf{B}_\theta = \nabla\psi \times \mathbf{e}_z = -(d\psi/dr)\mathbf{e}_\theta$ , the electron density from Eq. (65), and the electron temperature from Eq. (67).

We have considered two classes of equilibria: (i)  $B_z^2$  and  $P$  are linear in  $\psi$  and (ii)  $B_z = \text{const}$  and  $P = \text{quadratic}$  in  $\psi$ . For both classes we chose  $\eta_e = 0$ ,  $\eta_e = 1$ ,  $\eta_e \rightarrow \infty$ , and  $\eta_e < 0$ , the latter with singly peaked density and hollow temperature profiles or with singly peaked temperature and hollow density profiles. From these equilibria the following results are deduced (an example is discussed in Appendix B).

(1) A substantial fraction of the thermal electrons are active, e.g., (a) for linearly (marginally) stable equilibria of a strongly diamagnetic plasma with  $\eta_e = 1$ , more than one-third of the thermal electrons are active; (b) for linearly stable equilibria of a paramagnetic plasma with flat electron temperature profiles, the entire velocity space is occupied by active electrons.

(2) The fraction of active particles increases from the center to the plasma edge.

(3) The fraction of active particles in a paramagnetic plasma is higher than in a diamagnetic plasma with the same pressure profile.

## B. Reversed-field pinch equilibria

The same distribution function (54) is employed to derive force-free equilibria. Linearizing Eq. (68) by means of the *Ansatzes*  $P' = 0$  and  $B_z \propto \psi$  and then solving the resulting equation, one obtains  $B_z = B_z(0)J_0(\rho)$  and  $B_\theta = B_z(0)J_1(\rho)$ . These profiles satisfactorily describe the central region of the relaxed state of a reversed-field pinch [23]. We note that perpendicular negative-energy perturbations do not exist in force-free plane equilibria with sheared magnetic field, which were studied in Ref. [14], because for this case the second-order perturbation energy vanishes. For cold ions and by appropriately as-

signing the mean electron velocity profile, one can derive equilibria with various density and temperature profiles having nonpositive values of  $\eta_e$  for which negative-energy perturbations exist and a substantial fraction of active, thermal electrons are involved.

As an example we consider an equilibrium with peaked density and hollow temperature electron profiles:

$$V_e = \text{const}, \quad N_e = N_e(0) \frac{B}{B(0)}, \quad T_e = T_e(0) \frac{B(0)}{B}. \quad (69)$$

Condition (53) then yields

$$2 \frac{W_{e\perp}}{T_e} + 3 \frac{W_{e\parallel}}{T_e} < \frac{5}{2},$$

for any  $\rho$ , which implies that more than half of the thermal electrons throughout the poloidal cross section are active.

## VI. CONCLUSIONS

The general expression for the second-order perturbation energy, derived by Pfirsch and Morrison in the framework of linearized collisionless Maxwell-drift kinetic theory, was evaluated for the case of circularly cylindrical equilibria and vanishing initial field perturbations. From this expression we obtained the following conditions for the existence of negative-energy perturbations, which need only be satisfied locally in  $r$ ,  $v_{\parallel}$ , and  $\mu$  and are valid in the reference frame of minimum equilibrium energy. (i) If the equilibrium guiding center distribution function  $f_{g\nu}^{(0)}$  of any species  $\nu$  has the property  $v_{\parallel}(\partial f_{g\nu}^{(0)}/v_{\parallel}) > 0$ , parallel and oblique negative-energy perturbations ( $k_{\parallel} \neq 0$ ) exist with no essential restriction on  $\mathbf{k}$ . (ii) If  $v_{\parallel}(\partial f_{g\nu}^{(0)}/v_{\parallel}) < 0$ , the oblique negative-energy perturbations possible are nearly perpendicular. With the quantities  $R_{\nu}$  and  $Q_{\nu}$  defined by (50) and (51), the condition for perpendicular perturbations is  $R_{\nu}Q_{\nu} < 0$ . From this condition it follows that the curvature, which is associated with  $B_{\theta}^{(0)}$ , modifies the plane-equilibrium condition  $dP^{(0)}/dr \partial f_{g\nu}^{(0)}/\partial r < 0$ .

For the case of tokamak equilibria there are two regimes. (i) If  $R_{\nu} < 0$ , the existence of negative-energy perturbations is related to the threshold value of  $2/3$  of the quantity  $\eta_{\nu} \equiv \partial \ln T_{\nu} / \partial \ln N_{\nu}$ . (ii) If  $R_{\nu} > 0$ , a new regime appears, not present in plane equilibria, in which negative-energy perturbations exist for any value of  $\eta_{\nu}$ .

For various tokamak cold-ion equilibria with non-negative and negative values of  $\eta_e$ , a substantial fraction

of the thermal electrons are associated with negative-energy perturbations (active particles). In particular, (i) for linearly (marginally) stable equilibria of a strongly diamagnetic plasma with  $\eta_e = 1$ , more than one-third of the thermal electrons are active; (ii) for linearly stable equilibria of a paramagnetic plasma with flat electron temperature profiles, the entire velocity space is occupied by active electrons.

The part of velocity space occupied by active particles increases from the center to the plasma edge region and is larger in a paramagnetic plasma than in a diamagnetic plasma with the same density and temperature profiles.

It is also shown that, unlike in plane equilibria, negative-energy perturbations exist in force-free, reversed-field pinch equilibria with a substantial fraction of active particles. The present results, in particular the fact that a threshold value of  $\eta_{\nu}$  is not necessary for the existence of negative-energy perturbations, enhance even more the relevance of these modes.

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## APPENDIX A: CALCULATION OF THE EXPRESSION $\mathcal{A}$ APPEARING IN THE SECOND-ORDER PERTURBATION ENERGY

The two terms of this expression, Eq. (33), are calculated separately.

We start with the term

$$\frac{\partial S_{\nu}^{(1)}}{\partial \tilde{P}_i} \frac{\partial}{\partial \tilde{q}^i} \left( \frac{\partial H_{\nu}^{(0)}}{\partial \tilde{q}^j} \frac{\partial S_{\nu}^{(1)}}{\partial \tilde{P}_j} \right).$$

To make treatment of the constraints easier, we first introduce the vector

$$\mathbf{V} \equiv \frac{1}{m_{\nu}} \left[ \mathbf{P} - \frac{e_{\nu}}{c} \mathbf{A}_{\nu}^{*(0)}(\mathbf{x}, q^A) \right]. \quad (A1)$$

Using the relation  $\partial H_{\nu}^{(0)}/\partial \tilde{q}^j|_{\mathbf{V}=\mathbf{0}} = 0$ , one has

$$\begin{aligned} \frac{\partial S_{\nu}^{(1)}}{\partial \tilde{P}_i} \frac{\partial}{\partial \tilde{q}^i} \left( \frac{\partial H_{\nu}^{(0)}}{\partial \tilde{q}^j} \frac{\partial S_{\nu}^{(1)}}{\partial \tilde{P}_j} \right) &= \frac{\partial^2 H_{\nu}^{(0)}}{\partial q^i \partial q^j} \frac{\partial S_{\nu}^{(1)}}{\partial P_i} \frac{\partial S_{\nu}^{(1)}}{\partial P_j} \\ &= \frac{\partial^2 H_{\nu}^{(0)}}{\partial q^k \partial q^l} \frac{\partial S_{\nu}^{(1)}}{\partial P_k} \frac{\partial S_{\nu}^{(1)}}{\partial P_l} + 2 \frac{\partial}{\partial q^4} \left( \frac{\partial H_{\nu}^{(0)}}{\partial P^l} \right) \frac{\partial S_{\nu}^{(1)}}{\partial q^l} \frac{\partial S_{\nu}^{(1)}}{\partial P_4} + \frac{\partial^2 H_{\nu}^{(0)}}{\partial (q^4)^2} \left( \frac{\partial S_{\nu}^{(1)}}{\partial P_4} \right)^2 \end{aligned} \quad (A2)$$



with  $i, j = 1, \dots, 4$  and  $k, l = 1, \dots, 3$ . We note that the constraint  $P_4 = 0$  is not involved here, because  $P_4$  does not appear in  $H_\nu^{(0)}$ . Since the equilibrium quantities depend on just  $q^1$ , the only nonvanishing components of  $\partial^2 H_\nu^{(0)}/\partial q^k \partial q^l$  and  $\partial/\partial q^4(\partial H_\nu^{(0)}/\partial q^l)$ , according to Hamiltonians (31), are

$$\left. \frac{\partial^2 H_\nu^{(0)}}{\partial (q^1)^2} \right|_{\mathbf{v}=0} = -\frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^1} \quad (\text{A3})$$

and

$$\left. \frac{\partial}{\partial q^4} \left( \frac{\partial H_\nu^{(0)}}{\partial q^1} \right) \right|_{\mathbf{v}=0} = -\frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^4}. \quad (\text{A4})$$

On the basis of relations (A3) and (A4) and

$$\left. \frac{\partial^2 H_\nu^{(0)}}{(\partial q^4)^2} \right|_{\mathbf{v}=0} = -\frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^4} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^4}, \quad (\text{A5})$$

one obtains

$$\begin{aligned} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i} \frac{\partial}{\partial \tilde{q}^i} \left( \frac{\partial S_\nu^{(1)}}{\partial \tilde{q}^j} \frac{\partial H_\nu^{(0)}}{\partial \tilde{P}_j} \right) &= -\frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^1} \\ &\quad - 2 \frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^4} \\ &\quad - \frac{e_\nu}{c} \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^4} \frac{\partial (A_\nu^{*(0)})_l}{\partial q^4}. \end{aligned} \quad (\text{A6})$$

We now calculate the second term of Eq. (33), i.e.,

$$\frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i} \frac{\partial}{\partial \tilde{q}^i} \left( \frac{\partial S_\nu^{(1)}}{\partial \tilde{q}^j} \frac{\partial H_\nu^{(0)}}{\partial \tilde{P}_j} \right).$$

By virtue of  $\partial H_\nu^{(0)}/\partial P_4 = 0$ , the second term on the right-hand side of

$$\begin{aligned} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i} \frac{\partial}{\partial \tilde{q}^i} \left( \frac{\partial S_\nu^{(1)}}{\partial \tilde{q}^j} \frac{\partial H_\nu^{(0)}}{\partial \tilde{P}_j} \right) &= \frac{\partial S_\nu^{(1)}}{\partial P_i} \frac{\partial}{\partial q^i} \left( \frac{\partial S_\nu^{(1)}}{\partial q^l} \frac{\partial H_\nu^{(0)}}{\partial P_l} \right) \\ &\quad + \frac{\partial S_\nu^{(1)}}{\partial P_i} \frac{\partial}{\partial q^i} \left( \frac{\partial S_\nu^{(1)}}{\partial q^4} \frac{\partial H_\nu^{(0)}}{\partial P_4} \right) \end{aligned} \quad (\text{A7})$$

( $i, j = 1, \dots, 4, l = 1, \dots, 3$ ) vanishes. We note here that, whereas Eq. (27) for  $f_\nu$  is sufficient in the nonlinear theory to pick out the correct solutions, this is not so with the linearized theory. In this case, since the constraints are imposed along the perturbed orbits, a displacement vector  $(\xi, \xi_4)$  in  $\mathbf{x}, q^4$  space, similar to that in macroscopic theory, is introduced [6]. That is, since the zeroth-order distribution function always selects  $\mathbf{V} = \mathbf{0}$  and  $P_4 = 0$ , with  $\mathbf{V}$  as defined by Eq. (A1), it is reasonable to expand  $S_\nu^{(1)}$  in powers of  $\mathbf{V}$  and  $P_4$ :

$$\begin{aligned} S_\nu^{(1)} &= \hat{S}_\nu^{(1)}(\mathbf{x}, q_4) - \xi \cdot m_\nu \mathbf{V} - \xi^4 P_4 \\ &\quad + (\text{higher-order terms}), \end{aligned} \quad (\text{A8})$$

so that

$$\left. \frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}} \right|_{\mathbf{v}=0, P_4=0} = -\xi, \quad \left. \frac{\partial S_\nu^{(1)}}{\partial P_4} \right|_{\mathbf{v}=0, P_4=0} = -\xi^4. \quad (\text{A9})$$

Using Eq. (A8), one has

$$\begin{aligned} \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{P}} &= \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} - \left. \frac{\partial P_k}{\partial q^l} \right|_{\mathbf{v}} \left. \frac{\partial S_\nu^{(1)}}{\partial P_k} \right|_{\mathbf{x}} \\ &= \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} - \frac{e_\nu}{c} \frac{\partial (A_\nu^{*(0)})_k}{\partial q^l} \left. \frac{\partial S_\nu^{(1)}}{\partial P_k} \right|_{\mathbf{x}} \end{aligned} \quad (\text{A10})$$

and, therefore,

$$\begin{aligned} \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{P}} \frac{\partial H_\nu^{(0)}}{\partial P_l} &= \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{P}} (v_{g\nu}^{(0)})^l \\ &= \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} (v_{g\nu}^{(0)})^l - \frac{e_\nu}{c} (v_{g\nu}^{(0)})^l \frac{\partial}{\partial q^l} - \frac{e_\nu}{c} (v_{g\nu}^{(0)})^l \frac{\partial (A_\nu^{*(0)})_k}{\partial q^l} \left. \frac{\partial S_\nu^{(1)}}{\partial P_k} \right|_{\mathbf{x}}. \end{aligned} \quad (\text{A11})$$

Since  $A_\nu^{*(0)}$  depends only on  $q^1$  and  $\mathbf{v}_{g\nu}^{(0)}$  is perpendicular to  $\mathbf{e}_r$ , the last term on the right-hand side of Eq. (A11) vanishes. This has the consequence that higher-order terms in expansion (A8), after the constraint  $\mathbf{V} = \mathbf{0}$  is imposed, do not contribute to Eq. (A13) below. Applying the operator  $\partial/\partial q^m|_{\mathbf{P}}$ ,  $m = 1, 4$ , to Eq. (A11), one has

$$\begin{aligned} \frac{\partial}{\partial q^m} \left( \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{P}} \frac{\partial H_\nu^{(0)}}{\partial P_l} \right) \Big|_{\mathbf{P}} &= \frac{\partial}{\partial q^m} \left[ (v_{g\nu}^{(0)})^l \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} \right] \Big|_{\mathbf{v}} - \frac{e_\nu}{c} \frac{\partial (A_\nu^{*(0)})_k}{\partial q^m} \frac{\partial}{\partial P_k} \left[ (v_{g\nu}^{(0)})^l \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} \right] \Big|_{\mathbf{x}} \\ &= \frac{\partial}{\partial q^m} \left[ (v_{g\nu}^{(0)})^l \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} \right] \Big|_{\mathbf{v}} - \frac{e_\nu}{c} \frac{\partial (A_\nu^{*(0)})_k}{\partial q^m} (v_{g\nu}^{(0)})^l \frac{\partial}{\partial P_k} \left( \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} \right) \Big|_{\mathbf{x}} \\ &= \frac{\partial}{\partial q^m} \left[ (v_{g\nu}^{(0)})^l \left. \frac{\partial S_\nu^{(1)}}{\partial q^l} \right|_{\mathbf{v}} \right] \Big|_{\mathbf{v}} - \frac{e_\nu}{c} \frac{\partial (A_\nu^{*(0)})_k}{\partial q^m} (v_{g\nu}^{(0)})^l \frac{\partial}{\partial P_l} \left( \left. \frac{\partial S_\nu^{(1)}}{\partial q^k} \right|_{\mathbf{x}} \right) \Big|_{\mathbf{v}}. \end{aligned} \quad (\text{A12})$$

With the expansion (A8) and (A9), the last equation yields

$$\frac{\partial}{\partial q^m} \left[ \frac{\partial S_\nu^{(1)}}{\partial q^l} \bigg|_{\mathbf{P}} \frac{\partial H_\nu^{(0)}}{\partial P_l} \bigg|_{\mathbf{P}} \bigg|_{\mathbf{V}=0} = \frac{\partial}{\partial q^m} \left[ (v_{g\nu}^{(0)})^l \frac{\partial \hat{S}_\nu^{(1)}}{\partial q^l} \right] + \frac{e_\nu}{c} \frac{\partial (A_\nu^{*(0)})_k}{\partial q^m} (v_{g\nu}^{(0)})^l \frac{\partial \xi^k}{\partial q^l}. \quad (\text{A13})$$

Equation (A7) can then be written in the form

$$\begin{aligned} \frac{\partial S_\nu^{(1)}}{\partial \hat{P}_i} \frac{\partial}{\partial \hat{q}^i} \left( \frac{\partial \hat{S}_\nu^{(1)}}{\partial \hat{q}^j} \frac{\partial H_\nu^{(0)}}{\partial \hat{P}_j} \right) &= \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^1} \frac{\partial \hat{S}_\nu^{(1)}}{\partial q^l} \xi^1 + (v_{g\nu}^{(0)})^l \frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial q^k \partial q^l} \xi^k + \frac{\partial (v_{g\nu}^{(0)})^l}{\partial q^4} \frac{\partial \hat{S}_\nu^{(1)}}{\partial q^l} \xi^4 \\ &+ (v_{g\nu}^{(0)})^l \frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial q^4 \partial q^l} \xi^4 + \frac{e_\nu}{c} (v_{g\nu}^{(0)})^l \frac{\partial (A_\nu^{*(0)})_k}{\partial q^1} \frac{\partial \xi^k}{\partial q^l} \xi^1 + \frac{e_\nu}{c} (v_{g\nu}^{(0)})^l \frac{\partial (A_\nu^{*(0)})_k}{\partial q^4} \frac{\partial \xi^k}{\partial q^l} \xi^4. \end{aligned} \quad (\text{A14})$$

Insertion of (A6) and (A14) into (33) leads to expression (34) for  $\mathcal{A}$ .

### APPENDIX B: ACTIVE PARTICLES FOR AN EQUILIBRIUM WITH $\eta_e = 0$

With the ansatz  $d/d\psi (P + B_z^2) = \text{const}$ , the solution of Eq. (68) is of the form  $\psi \propto \rho^2$ , with  $\rho \equiv r/r_0$ . This yields a class of equilibria with the following characteristics.

Peaked parabolic pressure profile  $P = P(0)(1 - \rho^2)$ ,

$$B_z = [B_z^2(0) + 8\pi P(0)(1 - \alpha^2)\rho^2]^{1/2}; \quad (\text{B1})$$

$\alpha$  is a parameter. The plasma is diamagnetic for  $\alpha^2 < 1$  and paramagnetic for  $\alpha^2 > 1$ ,

$$B_\theta = 2\sqrt{\pi P(0)}\alpha\rho; \quad (\text{B2})$$

constant "toroidal" current density. Choosing the  $V_e$  profile as

$$V_e = V_e(0) \frac{B_z^2}{B_z B} (1 - \rho^2)^{-1} \quad (\text{B3})$$

with

$$B^2 \equiv (B_\theta^2 + B_z^2) = B_z^2(0) + 4\pi P(0)(2 - \alpha^2)\rho^2 \quad (\text{B4})$$

and

$$B_z^2 \equiv [B_z^2(0) + 4\pi P(0)(1 - \alpha^2)\rho^2]^{1/2}, \quad (\text{B5})$$

one obtains

$$N_e = N_e(0)(1 - \rho^2) \quad \text{and} \quad T_e = T_e(0) = \text{const}. \quad (\text{B6})$$

We note here that, owing to the  $(1 - \rho^2)^{-1/2}$  dependence of  $V_e$ , the equilibrium profiles are possible only in the interval  $0 \leq \rho \leq \rho_s < 1$ , with  $\rho_s$  appropriately chosen so that inequality (55) is satisfied (e.g.,  $\rho_s = \frac{3}{4}$ ).

Condition (62) concerning the representative particles leads to

$$0 < \beta(\alpha^2 - 1) < 1 \quad (\text{B7})$$

with

$$\beta \equiv \frac{P(0)}{B^2/8\pi} \approx \frac{P(0)}{B_z^2(0)/8\pi} = \text{const}. \quad (\text{B8})$$

The requirement that the "toroidal" magnetic field, Eq. (B1), must be a real function, sets an upper limit  $1 + \beta^{-1}$  on the values of  $\alpha^2$ . Thus, the right-hand inequality of condition (62),  $\beta(\alpha^2 - 1) < 1$ , is satisfied for all possible values of  $\alpha^2$ . The left-hand inequality,  $0 < \beta(\alpha^2 - 1)$ , is satisfied for  $\alpha^2 > 1$ , and therefore only in a paramagnetic plasma are the representative particles active.

For particles with arbitrary velocities, conditions (52) and (53) yield, respectively,

$$\frac{W_{e\parallel}}{W_{e\perp}} < \frac{1}{2} \left( \frac{2}{\alpha^2} - 1 \right)$$

and

$$\frac{\beta}{4} (2 - \alpha^2) \frac{W_{e\perp}}{T_e} + \frac{\beta}{2} \alpha^2 \frac{W_{e\parallel}}{T_e} < -1 \quad (\text{B9})$$

and

$$\frac{W_{e\parallel}}{W_{e\perp}} > \frac{1}{2} \left( \frac{2}{\alpha^2} - 1 \right)$$

and

$$\frac{\beta}{4} (2 - \alpha^2) \frac{W_{e\perp}}{T_e} + \frac{\beta}{2} \alpha^2 \frac{W_{e\parallel}}{T_e} > -1. \quad (\text{B10})$$

Condition (B9) is impossible for any  $\alpha$ , as is expected because  $\eta_e$  takes its lowest non-negative value well below the subcritical one. For  $\alpha \rightarrow 0$  condition (B10), concerning the new regime of negative-energy perturbations, is also impossible and therefore no negative-energy perturbations exist in a strongly diamagnetic plasma. For  $\alpha^2 = 1$  condition (B10) yields

$$\frac{W_{e\parallel}}{W_{e\perp}} > \frac{1}{2} \quad \text{and} \quad \frac{\beta}{2} \left( \frac{1}{2} \frac{W_{e\perp}}{T_e} + \frac{1}{2} \frac{W_{e\parallel}}{T_e} \right) > -1 \quad (\text{B11})$$

and therefore half of the velocity space is occupied by active electrons for all  $\rho$ . For  $\alpha^2 = 2$  condition (B10) leads to

$$\frac{W_{e\parallel}}{W_{e\perp}} > 0 \quad \text{and} \quad \beta \frac{W_{e\perp}}{T_e} > -1 \quad (\text{B12})$$

and therefore all particles are active. Thus, since the value  $\eta_e = 0$  is far lower than the critical value for linear stability ( $n_e^c \approx 1$ ), negative-energy perturbations involving a large number of thermal electrons exist in a linearly stable regime.

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