# Experimental visualization of acoustic resonances within a stadium-shaped cavity

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Acoustic resonances of an insonified water-filled stadium-shaped cavity are located and visualized in a noninvasive manner using a schlieren technique. The chaotic nature of the geometry is seen to affect the form of the resonance patterns observed. Individual eigenstates of the cavity can be resolved at low frequencies; in particular, the "bouncing ball" modes. In the high-frequency (overlapping resonance) regime, nodal patterns are characterized by a network of ridges similar in form to those produced by a random superposition of plane waves.

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The solutions of the Helmholtz equation are of great importance in acoustic, electromagnetic, water-wave, and quantum contexts. Recent interest in the spectra and eigenfunctions of two-dimensional cavities (or "billiards") arose out of semiclassical studies of quantum systems whose classical mechanics are chaotic, such as the stadium-shaped cavity of Bunimovich [1–4]. It is known that the *shape* of the boundary is of great importance in determining the behavior of the system: integrable or nonintegrable, regular or chaotic motion. The equivalence of the time independent Schrödinger equation and the Helmholtz equation would seem to suggest that phenomena discovered in the quantum context can be sought in other fields.

The study of chaotic systems is important in all fields since such systems are ubiquitous in realistic scattering situations; integrable systems are the exceptions, not the rule. The spectra of nonintegrable chaotic systems can be described by Gaussian orthogonal ensemble (GOE) random matrix statistics [5,3], unlike the spectra of integrable systems (circular, elliptical, rectangular geometries, etc.), which obey Poisson statistics. Also, unlike integrable systems, the eigenfunctions of chaotic systems are characterized by nodal lines that meander around with few crossings; eigenfunctions of integrable systems have families of nodal lines with many perpendicular crossings (Ref. [4], Chap. 15).

The classical trajectories of a completely chaotic system, such as the stadium-shaped cavity, access every region of phase space, and it was conjectured [6] that the eigenstates of such systems are governed by the random superposition of plane waves, all having the same wave-vector magnitude but differing amplitude, phase, and direction. Such a superposition was shown [7] to result not in a "speckle pattern," but in a wave field characterized by a network of ridges which have since become known as "scarlets" [8]. In the stadium-shaped cavity these ridges appear to be localized along classical periodic orbits, manifesting themselves as enhanced (or diminished) amplitude; they are known as "scars" of the periodic orbit. The importance of the periodic orbits was shown by Gutzwiller [9], who expressed the density of states as a sum over the classical periodic orbits. More recently Doron and Smilansky [10] have extended the summation approach to *scattering* systems. Many theoretical predictions of scarred eigenstates have been presented in the literature (see, for example, Ref. [2], and references therein). Scars have even been identified in quite low-energy states [11]; scars and scarlets are thought to be general wave phenomena. Scars have also been observed in the high-frequency vibrations of stadium-shaped plates [12].

The little published experimental work concerning itself with nonintegrable systems and chaotic scattering has mainly involved microwave cavities [13,11], although the manifestation of scarlets and scars have been demonstrated for water surface waves by Blümel [14]. Stöckman and Stein have used microwave resonators to obtain spectra of both the stadium and the Sinai billiard [15] and have obtained wave functions for the stadium-shaped cavity [16] which give good agreement with Gutzwiller's semiclassical representation, convincingly demonstrating the influence of the periodic orbits on the wave functions.

In this paper we describe an experimental arrangement which permits the location and visualization of the wave functions of two-dimensional acoustic cavities. By using a schlieren technique we are able to study the fluid-column resonances within insonified cylindrical cavities of "stadium" cross section, and compare them with the resonances of a circular cavity. This technique has previously been used successfully to locate and image the resonances within submerged circular and elliptical cylindrical shells [17,18] and to study the scattering of pulses by various objects and geometries [19]. The extension of this technique to the study of nonintegrable systems seems a natural one. To our knowledge this is the first application of acoustics to the determination of the wave functions of two-dimensional cavities having nonintegrable topology.

The pressure ($\psi$) within the fluid column of a cylindrical shell satisfies the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$ with impedance boundary conditions. For a shell presenting an *infinite* impedance to the cavity, this is just the requirement that the normal derivative of the pressure field vanishes at the boundary (Neuman boundary conditions), and the resonance frequencies constitute the spectrum of eigenvalues of the cavity. However, an *insonified* shell having *finite* impedance constitutes a true scattering problem and yields a *resonance* spectrum characterized by resonances of finite width which may overlap at high frequencies. Of particular interest in the circular geometry is the hybridization of eigenstates in the overlapping resonance regime, and the effect this has on the resonance patterns.

One advantage of this technique is that it is noninvasive and permits wave fields to be recorded instantly on film or video (with a resolution limited only by that of the film), or analyzed by computer using an image-grabbing system. This allows resonance modes to be rapidly identified as the frequency is swept. A disadvantage of the present experimental arrangement is the low $Q$ factor of the acoustic system, which prevents individual resonances from being resolved at high frequency (although the semiclassical regime *is* one of overlapping resonances [20]).

The experimental arrangement is shown in Fig. 1 and has been described in detail elsewhere [17,19]. The principle upon which schlieren operates is that light is diffracted during its passage through an acoustic field in a fluid; the diffracted light contains information about the spatial distribution of the acoustic field. Light from a high-power light emitting diode is focused onto a pinhole, or an array of pinholes having a random basis. The resultant light beam is collimated by the first parabolic mirror and passed through a glass walled tank containing the acoustic field. In the present application a transducer placed in the tank, with its axis perpendicular to the light beam, is used to insonify a cylinder suspended with its axis parallel to the light beam. The light emerging from the tank is brought to a focus by the second parabolic mirror; the resultant diffraction pattern contains the light which has passed straight through the acoustic field undiffracted (zeroth-order) and the diffracted light (higher orders). By removing a part of the light in the diffraction pattern and allowing the remainder to recombine in a still camera or video, an "image" of the acoustic field is obtained. The optical distribution in the image has, in general, a complicated dependence on the spatial filtering arrangement and the pressure amplitude in the acoustic field. However, it has been shown [21] that for *low* acoustic pressures and zeroth-order filtering, the optical distribution approximates the *square* of the acoustic pressure distribution. As any variations in the acoustic field *along* the light path are integrated out, the system is restricted to the visualization of resonances of bodies having translational symmetry.

The cylinder used in these experiments was a stadium-shaped cavity in an aluminum block; the semicircular ends of the cavity had radii ($a$) 12.7 mm and the straight sections of the cavity were of length $2a$. The cylinder was 100 mm long and could be insonified at various angles to the minor axis of the stadium (see Fig. 1). The transducer was driven in continuous mode, and by sweeping the frequency and altering the angle of insonification, resonances within the cavity were easily located and recorded on film. The current experimental arrangement incorporates parabolic mirrors of focal length 1.8 m, has an aperture of approximately 25 cm, and can be used to visualize acoustic fields down to about 100 kHz. The upper-frequency limit of the visualization system is unlimited, although individual wave fronts cannot be resolved at very high frequencies (above several megahertz).

The ability to resolve individual modes of the cavity depends upon the density of resonances and the $Q$ factor of the scattering system; in this case influenced by the losses from the cavity due to the finite impedance of the shell, and the method of excitation. For a truly bound system (discrete eigenvalues) the average density of states of a cavity of area $A$ is given by

$$\overline{\rho}(\nu) = \frac{2\pi A\nu}{c^2}\ , \tag{1}$$

where $c$ is the speed of sound in water and $\nu$ is the frequency. The average separation of states is $1/\overline{\rho}(\nu)$ Hz and it follows that $Q$ factors of the order

$$Q = \frac{\nu}{\Delta\nu} = \overline{\rho}(\nu)\nu = \frac{2\pi A}{c^2}\nu^2 \tag{2}$$

are necessary for individual states to be resolved at a frequency $\nu$. The average number of states with a frequency *less* than $\nu$ is given by

$$\overline{N}(\nu) = \frac{\pi A}{c^2}\nu^2 \approx \frac{Q}{2}. \tag{3}$$

For our stadium $[A = (4+\pi)a^2,\ c = 1480\ \mathrm{ms}^{-1}]$ the 1000th eigenstate exists around 800 kHz and a $Q$ of about 2000 would be required to resolve these 1000 states; the $Q$ of our present system is not that high. The overlapping resonance regime begins at around several hundred kilohertz, in the megahertz region we have *many* over-

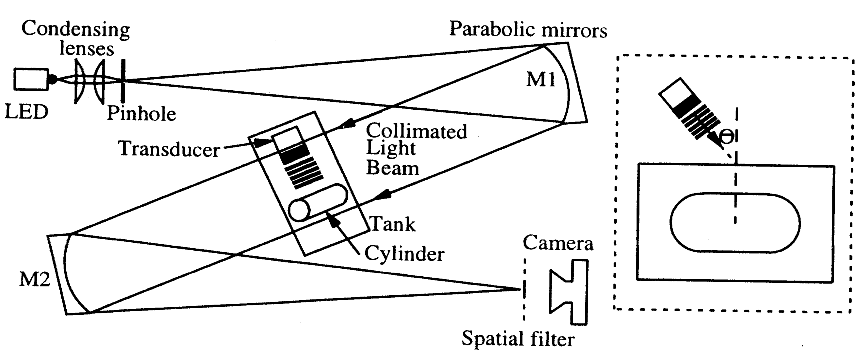


FIG. 1. The schlieren visualization system and scattering geometry (inset).

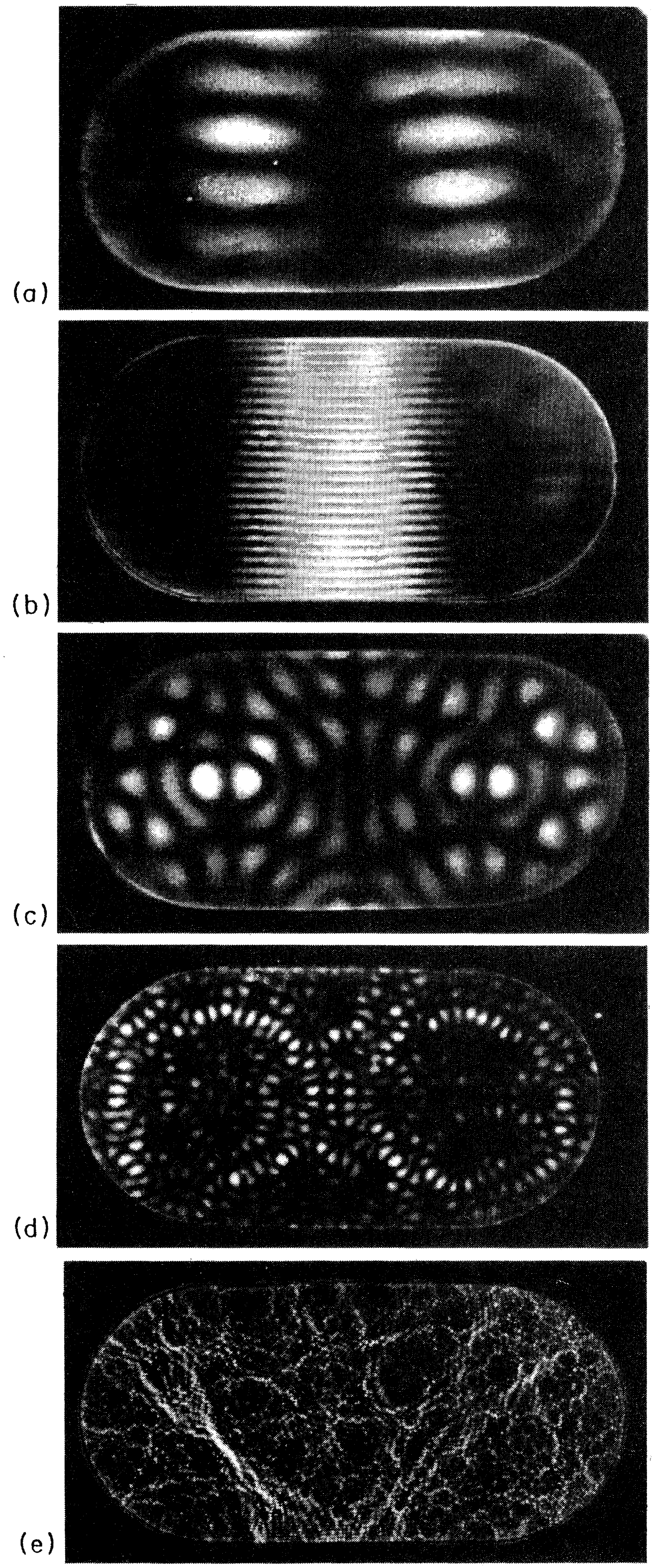


FIG. 2. Acoustic wave fields within the stadium-shaped cavity. Frequencies are (a) 150.9 kHz, (b) 720.2 kHz, (c) 261.1 kHz, (d) 664.8 kHz, and (e) 2.5 MHz. Different angles of the incident beam (not shown) were used to excite each resonance [see Fig. 1 (inset)].

lapping states and the consequences of their interference must be considered when interpreting the resonance patterns obtained with the system.

At the lowest frequencies analyzed with this system the cavity resonances are clearly resolved in frequency. Figure 2(a) shows a typical low-frequency resonance found at 150.9 kHz. The "bouncing ball" (BB) periodic orbit can be associated with this mode which is confined to the portion of the cavity between the flat sides, avoiding the semicircular ends. Such resonance modes are the most easily excited and identified; by insonifying the stadium at "normal incidence," members of the family of BB resonances can be isolated and visualized over a wide frequency range; a mode found at 720.2 kHz is shown in Fig. 2(b). The resonance shown in Fig. 2(c) was found at 261.1 kHz and has an altogether different appearance. Unlike the BB modes, a significant portion of the disturbance is present in the semicircular ends, and the focusing effect of these ends is clearly seen. At 664.8 kHz a wave function is shown [Fig. 2(d)] that can be associated with the "double-diamond" periodic orbit. The resonance widths have become significant at this frequency; modes are thought to overlap and the appearance of the wave field alters as the incident angle is varied.

Figure 2(e) shows a typical wave field observed at very

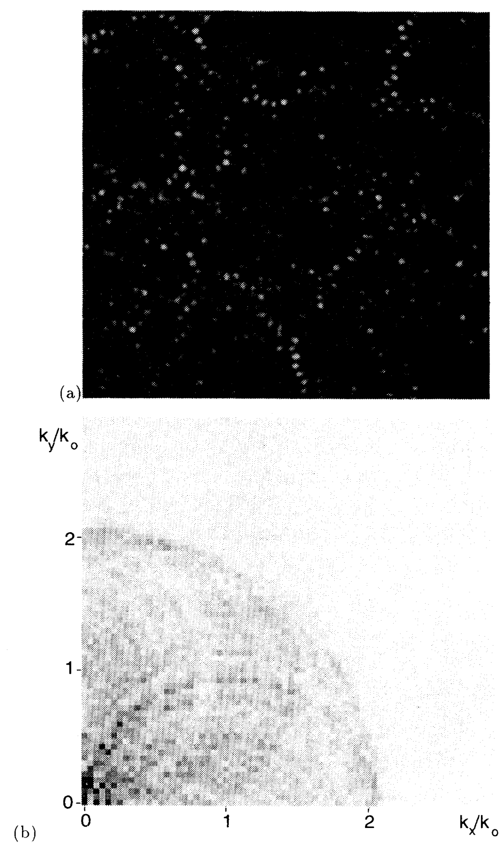


FIG. 3. (a) A portion of the schlieren image shown in Fig. 2(e); $500\times500$ pixels. (b) Two-dimensional Fourier transform of (a); higher Fourier amplitudes are represented by darker pixels. $k_0$ is the wave number of the incident (driving) field.

high frequency (2.5 MHz). The network of ridges is reminiscent of the "scarlets" predicted in [7] and observed in water surface waves [14]. In the acoustic case presented here there are many overlapping resonances contributing to the wave function at this frequency and varying the angle of the incident beam causes the pattern to change, although the ridge structure remains. There is some evidence in this image that the scarlets are aligning themselves along the trajectory of a periodic orbit.

In an effort to obtain more quantitative information from the schlieren images the photograph shown in Fig. 2(e) was digitized and a square portion selected for further analysis [Fig. 3(a)]. The discrete Fourier transform of this portion of the field was calculated and the resultant Fourier amplitudes are shown in Fig. 3(b). The observed circular symmetry of the Fourier transform indicates that there is no preferred direction. This is consistant with a wide distribution of plane waves. As the optical field is (approximately) proportional to the square of the pressure distribution, each plane wave component interacts with all the others to produce a a series of circles in the Fourier transform plane, each passing through the origin, and all bounded by a limiting circle at twice the frequency of the incident wave field ($k = 2k_0$).

Figure 4(a) shows the square of the pressure in a portion of wave field obtained by adding together 1000 cosine waves with random phases and directions, and Gaussian random amplitudes. The Fourier transform of this field is shown in Fig. 4(b). The resulting distribution of Fourier amplitudes is similar in form to the experimental results [Fig. 3(b)]. The intersecting circles can be seen more clearly here and this is thought to reflect the fact that there are more plane wave components present here than in the experimental result.

For comparison we present results for an insonified *circular* shell, a geometry that is integrable. The shell is made of brass, and has inner and outer radii of 14.25 mm and 15.85 mm. In the low-frequency regime we again see isolated resonance modes $(n, m)$ whose wave functions are of the form $J_n(k_{n,m})\cos(n\phi)$; the separable nature of this geometry reflects itself in the presence of nodal line families which intersect perpendicularly. Figure 5(a) shows the (2,4) fluid column resonance. At a higher frequency of 2.5 MHz where resonances overlap, the nodal crossings are, in places, destroyed and new symmetries appear [Fig. 5(b)]; in this case a five-pointed star is clearly evident. Similar effects are to be expected in the overlapping resonance regime of the stadium-shaped cavity.

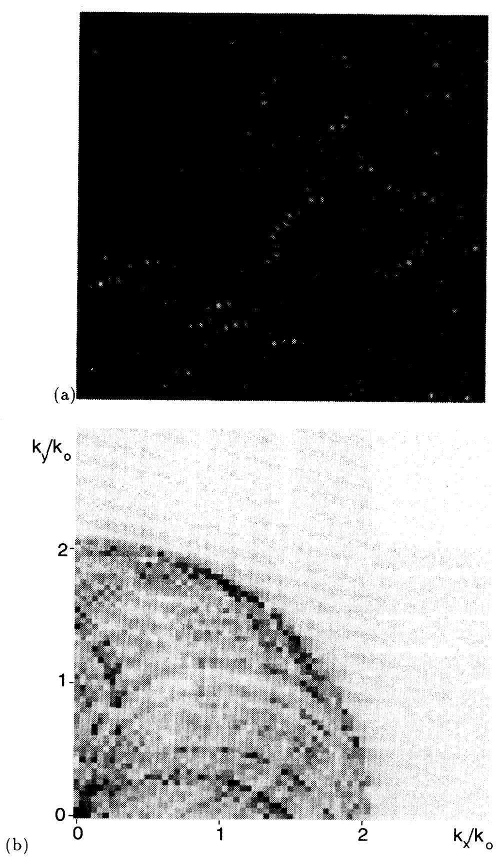


FIG. 4. (a) Superposition of 1000 cosine waves with random direction, random phase, and Gaussian random amplitudes. The square of the pressure amplitude is plotted, white representing higher values. (b) Two-dimensional Fourier transform of (a).

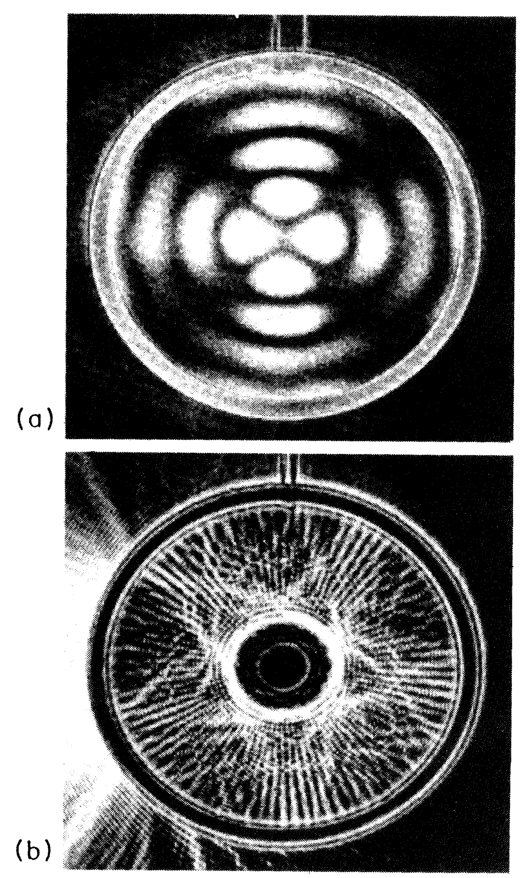


FIG. 5. Wave fields within the circular cavity (a) 218 kHz and (b) 2.5 MHz.

In conclusion, we note the ease with which the resonance modes can be located and visualized over a wide range of frequencies makes the schlieren system a valuable tool for studying the eigenfunctions, and to a limited extent the spectra, of two-dimensional cavities. The main disadvantage of the current experimental setup is the low $Q$ factor of the acoustic system, preventing spectra from being obtained at high frequencies. It is proposed that alternative acoustic arrangements, perhaps involving excitation of water filled shells *in air*, would yield significantly higher $Q$ factors, permitting a greater number of resonance modes to be resolved and spectra obtained. A preliminary study undertaken by the authors has demonstrated that such a technique could, with care, be incorporated into the schlieren visualization system. The association of periodic orbits with the acoustic wave functions of the stadium-shaped cavity has been demonstrated, and the appearance of scarlets at higher frequencies noted. The importance of considering the effects of overlapping resonance states in realistic scattering systems has also been noted.

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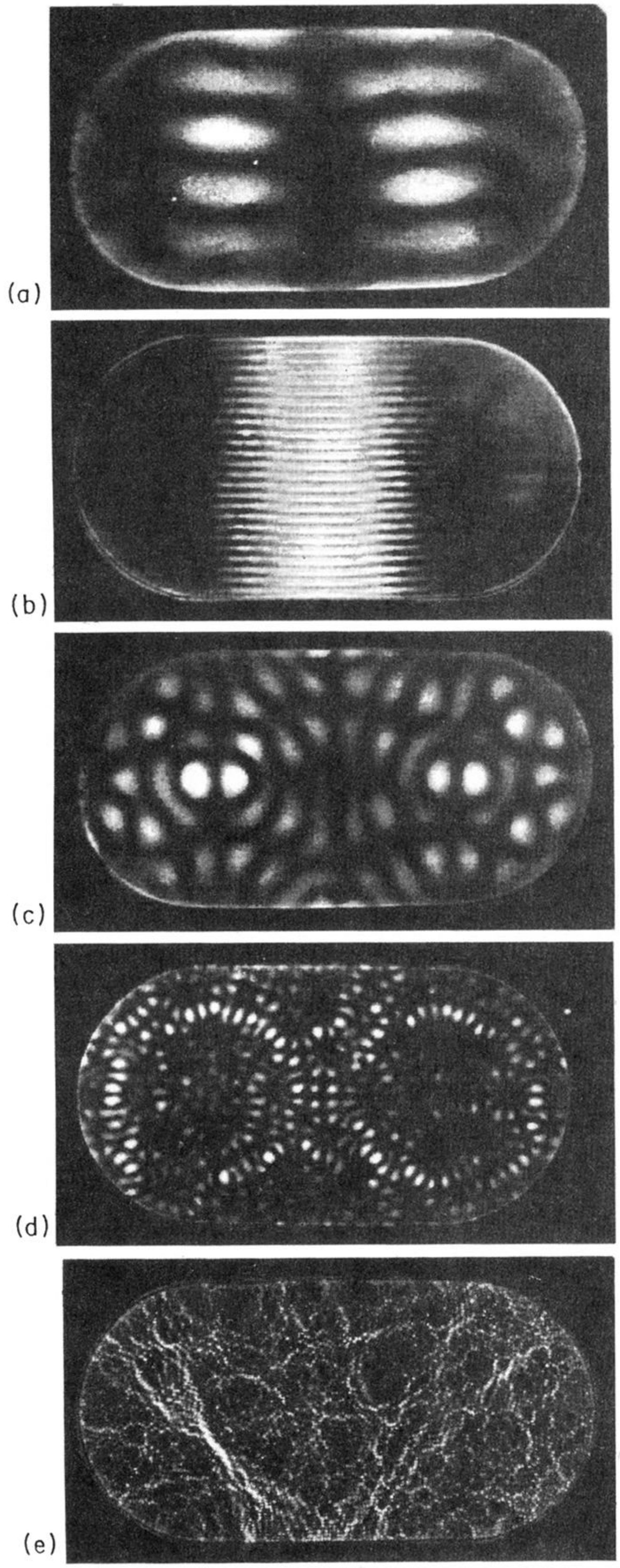

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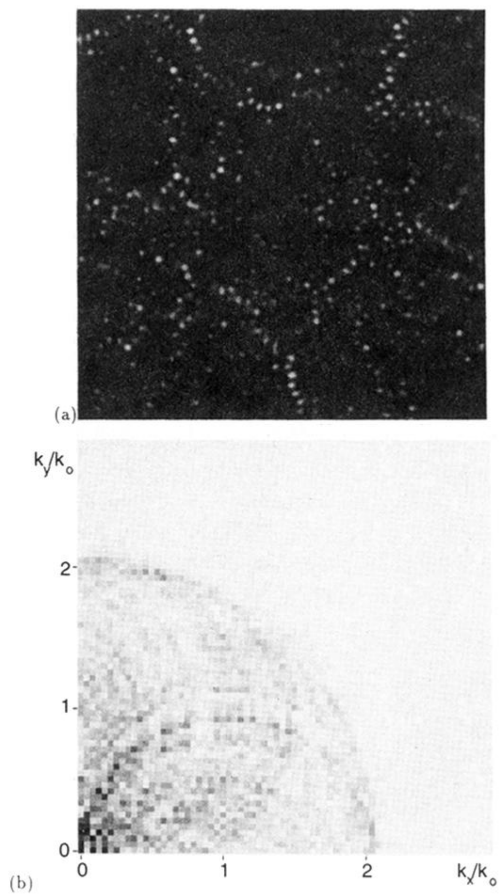


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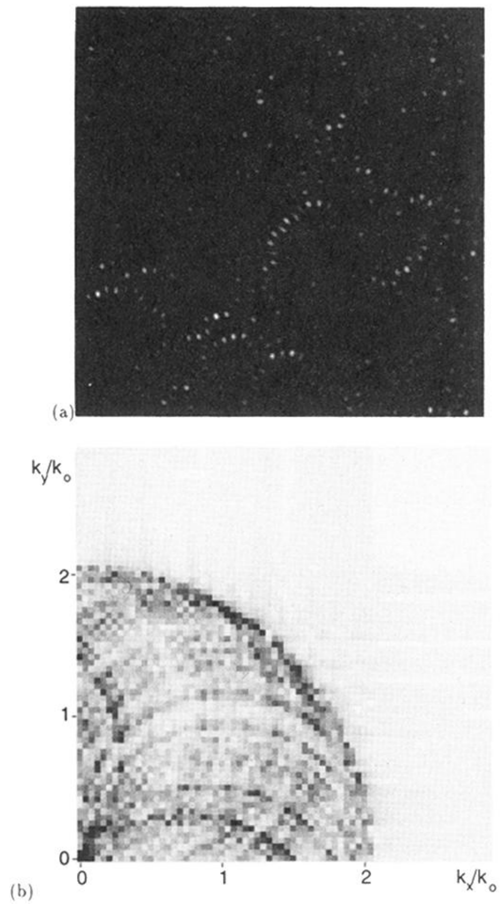


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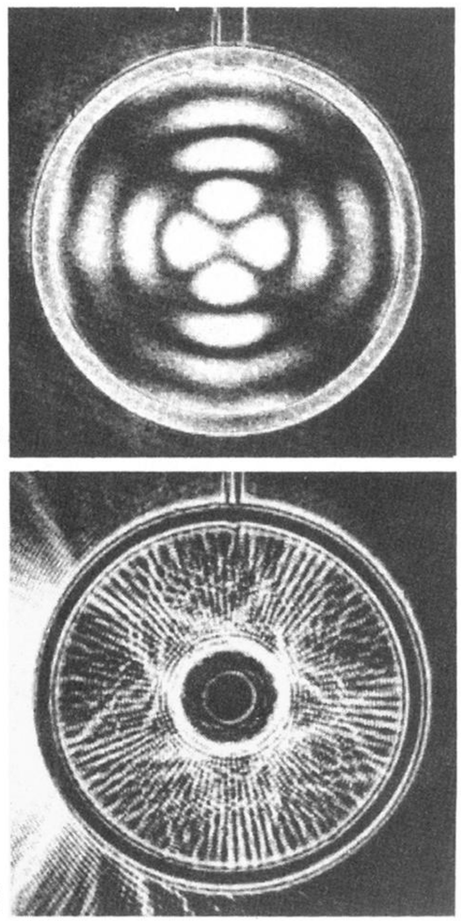


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