Stochastic Hopf bifurcation: The effect of colored noise on the bifurcation interval

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We consider a general nonlinear dynamical system undergoing a supercritical Hopf bifurcation, and driven by a Gaussian colored noise. Using normal form theory and an approximate effective Fokker-Planck equation, valid for weak intensity and small correlation time of the noise, we obtain an explicit expression for the stationary probability distribution of the full general system close to the bifurcation, and analyze the changes in the shape of the distribution to discuss the effect of the noise on the bifurcation interval.

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There are many physical situations in which the stochastic nature of some of the relevant parameters involved in the problem play a fundamental role in the evolution of the system [1]. Of particular significance is the analysis of the behavior of systems perturbed by colored noise, i.e., with a nonzero correlation time.

Among the many different aspects analyzed in the literature, the study of the influence of fluctuations on the appearance of a Hopf bifurcation has a prominent place [2–11]. In the absence of fluctuations, center manifold and normal form theories provide a powerful tool for the study of dynamical systems. The idea of introducing successive coordinate transformations to simplify the analytic expression of a general problem has been extended, with different success, to stochastic problems. Some of the early works on this subject started with the reduced dynamical system and then introduced the fluctuations [4,5], or made the reduction of variables in the associated Fokker-Planck equation [6]. However, those descriptions do not contain all the terms describing the system-noise coupling as they arise when the parameter fluctuations are included in the original system and all the nonlinear variable changes to arrive at the normal form are performed [3,7,9,11]. It is clear that this more complete analysis is necessary to fully account for the effect of noise on the bifurcation.

When the reduction to the normal form is done in a two-dimensional system of equations including a stochastic term, only the deterministic part of the equations retain the characteristic radial symmetry. This makes it necessary to work with the two-dimensional probability distribution. In Ref. [3] Hoffmann considered the effect of white noise of weak intensity on the Hopf bifurcation of a general two-dimensional system. After performing the necessary transformations to obtain the normal form and making an expansion on powers of the noise intensity he was able to identify a noise-dependent bifurcation interval wherein the two-dimensional approximate stationary distribution of the system changed its shape from a focus type to a craterlike (limit cycle) structure. The existence of a whole bifurcation interval instead of a single bifurcation point is a feature of not making any kind of phase averaging in order to obtain an oversimplified description of the bifurcation in terms of a single variable, and has been further discussed theoretically [8,12], and observed in analog experiments [10]. However, to our knowledge, the influence of the correlation time of the noise on the position and width of the interval has not been previously considered in the literature. In this paper we fill that gap by extending the method used by Hoffmann to the case of colored noise for a general model that includes most of the physical significant systems that undergo a supercritical Hopf bifurcation. In order to do that, we use a recently proposed geometrical method [13] to obtain an effective Fokker-Planck equation, the so-called best Fokker-Planck equation (BFPE) [14], and look for the changes in the shape of the probability distribution as indicators of the transitions. As in the white noise case the transition from a single peak (focuslike distribution) to a crater (limit cycle behavior) occurs in a whole interval instead of at a single point, but now the characteristics of the interval depend not only on the intensity but also on the correlation time of the noise.

In a practical situation, we normally begin with a physical model perturbed by noise (it may be multiplicative, additive, or both). To analyze the effect of the fluctuations close to a Hopf bifurcation point, it is convenient to look for the corresponding (stochastic) normal form. Performing the adequate nonlinear variable changes in the original model, and being consistent with the noise in the different transformations, we may write the general complex stochastic differential equation

$$\dot{z} = f(z,\bar{z}) + \xi(t)g(z,\bar{z}) \ , \tag{1}$$

which we will take as our starting point, and in which an overbar denotes complex conjugation. We will also assume the following conditions:

(a) The function $f(z,\bar{z})$ admits an expansion of the form [16]

$$f(z,\bar{z}) = \lambda z + c_1 z^2 \bar{z} + \cdots + c_k z^{k+1}\bar{z}^k + O(|z|^{2k+3}) \ , \tag{2}$$

where we also have $\mathrm{Re}(c_1) < 0$ to recover the standard deterministic normal form of a supercritical Hopf bifurcation, and the complex eigenvalue $\lambda = \mu + i\omega$ is chosen such that $\omega > 0$.

(b) The noisy term $\xi(t)$ is a real-valued Ornstein-Uhlenbeck process with zero mean and correlation function given by

$$\langle \xi(t)\xi(s)\rangle = \sigma^2/2\tau_n \; \exp\left(-|t-s|/\tau_n\right) \ , \tag{3}$$

with σ and τ_n the intensity and correlation time of the noise, respectively.

(c) The function $g(z,\bar{z})$ is an analytic function of z that accounts for the coupling between the noise and the system. Due to the complexity of the nonlinear coordinate changes involved in the passage to the normal form, $g(z,\bar{z})$ appears as a power series in z, $\bar{z}$, even in the simplest case when one started with an additive stochastic force. Moreover we assume that $\lim_{|z|\to 0} g(z,\bar{z}) \neq 0$,and we can write

$$g(z,\bar{z}) = g_0 e^{i\varphi_0} + O(|z|) \ , \tag{4}$$

with $\varphi_0 \in [0, 2\pi]$, and $g_0 \neq 0$.

From the deterministic point of view, the proximity to the origin can be invoked to truncate the normal form expansion (2). To be more precise, the amplitude of the oscillations appearing at the bifurcation point grows as $|z| \sim \mu^{1/2}$, remaining small near the critical value $\mu_c = 0$. When a stochastic term is added, as in Eq. (1), the situation becomes more delicate. If the noise is strong enough or if it is highly correlated, a trajectory could leave the neighborhood of the origin, and the higher-order terms in (2) and (4) would become important. It is, therefore, appropriate to restrict the analysis to the case of small intensity and correlation time, and we shall calculate the stationary probability density of the process z as an expansion around zero noise.

For this purpose we rescale appropriately our variables and introduce polar coordinates by means of the coordinate change

$$z = \sqrt{-\,\omega/\mathrm{Re}(c_1)}\;\eta\, r \exp i[\theta + \varphi_0] \tag{5}$$

with $\eta = [g_0\sqrt{-\mathrm{Re}(c_1)}\omega^{-1}\,\sigma]^{1/2}$ as the parameter that we shall assume small. The deterministic behavior near the bifurcation point suggests the advantage of a change to a new order parameter δ^2 defined by [3]

$$\delta^2 = \mu\,\left(\omega\eta^2\right)^{-1} \ . \tag{6}$$

With these new variables, defining the parameters $\Gamma = -\mathrm{Im}(c_1)/\mathrm{Re}(c_1)$, $\gamma = -\omega\,\mathrm{Re}(c_2)/\left[\mathrm{Re}(c_1)\right]^2$, $\nu = -\omega\,\mathrm{Im}(c_2)/\left[\mathrm{Re}(c_1)\right]^2$, and using ω^{-1} as the time unit, Eq. (1) reads

$$\frac{d}{dt}\begin{bmatrix} r \\ \theta \end{bmatrix} = \mathbf{F}\begin{bmatrix} r \\ \theta \end{bmatrix} + \tilde{\xi}(t)\mathbf{G}\begin{bmatrix} r \\ \theta \end{bmatrix} \ , \tag{7}$$

where $\tilde{\xi}(t)$ is a new Ornstein-Uhlenbeck process with correlation function $\langle\tilde{\xi}(t)\tilde{\xi}(s)\rangle = 1/2\tau\exp\left(-|t-s|/\tau\right)$ where $\tau = \omega\tau_n$, and the new functions $\mathbf{F}$ and $\mathbf{G}$ are vectors given by (note that in the expression for $\mathbf{G}$, the second- and third-order terms in η are general [3]. In a particular application, the explicit calculation of the model-dependent coefficients α_i and β_i will indicate which are the angular terms actually present)

$$\mathbf{F}\begin{bmatrix} r \\ \theta \end{bmatrix} = \sum_{k=0}^{\infty}\eta^{2k}\mathbf{F}_{2k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \eta^2\begin{bmatrix} \delta^2 r - r^3 \\ -\Gamma r^2 \end{bmatrix} + \eta^4\begin{bmatrix} \gamma r^5 \\ \nu r^4 \end{bmatrix} + O(\eta^6) \tag{8}$$

and

$$\begin{aligned}\mathbf{G}\begin{bmatrix} r \\ \theta \end{bmatrix} &= \sum_{k\geq 1}^{\infty}\eta^{k}\mathbf{G}_{k} = \eta\begin{bmatrix} \cos\theta \\ -\frac{1}{r}\sin\theta \end{bmatrix} + \eta^2\begin{bmatrix} r\left[\alpha_1\cos^2\theta + (\alpha_2+\beta_1)\cos\theta\sin\theta + \beta_2\sin^2\theta\right] \\ \beta_1\cos^2\theta + (\beta_2-\alpha_1)\cos\theta\sin\theta - \alpha_2\sin^2\theta \end{bmatrix} \\ &+\eta^3\begin{bmatrix} r^2\left[\alpha_3\cos^3\theta + (\alpha_4+\beta_3)\cos^2\theta\sin\theta + (\alpha_5+\beta_4)\cos\theta\sin^2\theta + \beta_5\sin^3\theta\right] \\ r\left[\beta_3\cos^3\theta + (\beta_4-\alpha_3)\cos^2\theta\sin\theta + (\beta_5-\alpha_4)\cos\theta\sin^2\theta - \alpha_5\sin^3\theta\right] \end{bmatrix} + O(\eta^4) \ .\end{aligned} \tag{9}$$

It is important to notice that the different time scales governing the dynamics of the system are now related through the parameters η and τ. Thus, η gives the separation between time scales of the angular and radial variables, and τ measures whether the noise evolves faster than the phase ($\tau < 1$) or vice versa.

The two-dimensional process solution of (7) is non-Markovian, and to obtain an approximation for the stationary probability density $P(r,\theta)$ we use the geometrical treatment of Ref. [13] to write the BFPE [14], valid for short correlation time and small intensity [17].

In the BFPE approximation the stationary probability density $P(r,\theta)$ satisfies the equation (see Ref. [13] for details)

$$\left[-L_{\mathbf{F}} + L_{\mathbf{G}}L_{\mathbf{D}}\right]P_s(r,\theta) = 0 \ , \tag{10}$$

where $L_{\mathbf{F}}$ is the Lie derivative in the direction of the field $\mathbf{F}$ [18], and the field $\mathbf{D}$ is given by

$$\mathbf{D} = \int_0^{\infty} ds\frac{1}{2\tau}e^{-\frac{|s|}{\tau}}\left[\phi_0^{-s}\right]^{*}\mathbf{G} \ , \tag{11}$$

where ϕ_0^t is the flow defined by the deterministic part of Eq. (7).

To obtain an expression for the stationary distribution we use a perturbative method similar to that of Ref. [3] and some properties of the Lie derivatives. Skipping the technical details, and after some tedious calculations we finally obtain the stationary probability density, up to second order in η, as

$$P_s(r,\theta) = N\,r\left\{1+\eta^2 r^2\left[ar^4+b\delta^2 r^2+c-\frac{1}{6}\tau(1-\tau^2)[\delta^2-r^2]^2\right.\right.$$
$$\left.\left.+(\sin 2\theta-\tau\cos 2\theta)\left[2(1+\tau^2)(\delta^2-r^2)^2-1\right]\right]\right\}\exp\left[(1+\tau^2)(2\delta^2 r^2-r^4)\right]+O(\eta^3)\ , \qquad (12)$$

where

$$a = -(1+\tau^2)\left[\frac{2}{3}\gamma+\frac{1}{6}b\right]\ , \qquad (13)$$

$$b = (1+\tau^2)\left[6\alpha_3+2\alpha_5+2\beta_4+2(\alpha_1+\beta_2)^2+8\Gamma\tau^2+\frac{1+\tau^2}{1+4\tau^2}\left[(\alpha_1-\beta_2)^2+(\alpha_2+\beta_1)^2\right]\right]\ , \qquad (14)$$

$$c = -\frac{1}{2}\left[2\alpha_3+\beta_4+(1+\tau^2)(\alpha_1+\beta_2)^2-\tau(\alpha_4+2\beta_5)-8\tau\frac{1-\tau^2}{1+\tau^2}\right]\ . \qquad (15)$$

N is the normalization constant, and the multiplicative factor r is a consequence of the use of polar coordinates.

We now take as an indicator of the transitions the changes in the shape of $r^{-1}P_s(r,\theta)$ [1,3,11]. It is not difficult to see that $r=0$ is always an extremum, and the others, if any, are given by

$$r_m^2(\theta) = \delta^2+\frac{\eta^2}{2\,(1+\tau^2)}\,\left[c-(\sin 2\theta-\tau\cos 2\theta)\right]\ , \qquad (16)$$

where terms of $O(\eta^4,\delta^4)$ have been neglected.

The angular term in (16) oscillates between the values $\pm\sqrt{1+\tau^2}$, and, therefore, three intervals of the parameter δ^2 can be distinguished in which $r^{-1}P_s(r,\theta)$ shows different structures.

(a) $\delta^2 \le \delta^2_{c,1} = -\frac{\eta^2}{2(1+\tau^2)}\left[c+(1+\tau^2)^{1/2}\right]$. The distribution exhibits a single peak centered at the origin. We can think of this shape as representative of a stable focus.

(b) $\delta^2_{c,1} < \delta^2 \le \delta^2_{c,2} = -\frac{\eta^2}{2(1+\tau^2)}\left[c-(1+\tau^2)^{1/2}\right]$. For certain angles one maximum at $r\neq 0$ and a local minimum at $r=0$ appear, while for others there is a unique maximum located at the origin. This is a typical situation of stochastically perturbed systems that has no equivalence with any deterministic case. The existence of this *bifurcation interval* has been established theoretically for the white noise case [3,8], and also observed experimentally [10]. The width of the interval is $\Delta = \eta^2/(1+\tau^2)^{1/2}$, and it is centered at $-c\eta^2/2(1+\tau^2)$. The nonzero correlation time tends to reduce the width of the bifurcation interval.

(c) $\delta^2 > \delta^2_{c,2}$. For any value of the angle we always have a maximum at $r\neq 0$ and a minimum at $r=0$. Therefore, all vertical cross sections are bimodal, and the probability density undergoes a transition to a crater shape with a closed rim at $\delta^2=\delta^2_{c,2}$. A limit cycle behavior appears.

The presence of region (b) makes it difficult to define the bifurcation position in the stochastic case. Instead of a point we find a whole interval separating stable focus and limit cycle behavior. Following the arguments in [2,10], we adopt $\delta^2_{c,2}$ as the bifurcation value because it fixes the point at which stochastic fluctuations are not able to destroy the limit cycle oscillations.

Now we apply the previous results to the model known as the Brusselator [15], defined by the equations

$$\dot{x} = A-(1+B)x+x^2y\ , \qquad (17a)$$
$$\dot{y} = Bx-x^2y\ . \qquad (17b)$$

Without loss of generality, we take $A=1$, and consider B as the control parameter. As is well known the system undergoes a supercritical Hopf bifurcation at the critical value $B_c = 1+A^2 = 2$.

We are interested in the effect of fluctuations of the parameter B on this transition. Taking $B=B_c(1+\beta)+\xi(t)$, where β is a new bifurcation parameter ($\beta\sim 0$) and $\xi(t)$ is an Ornstein-Uhlenbeck process as defined in (3), we arrive, neglecting terms $O(\eta^4)$, at the stochastic normal form Eqs. (7)–(9) with $\tau=\tau_n$, $\delta^2=\beta/\eta^2$, $\Gamma=1/9$, and $\alpha_1=-2\sqrt{2}/3$, $\alpha_3=16/9$, $\alpha_5=-16/9$ as the only coefficients different from zero in (9). Therefore, the coefficient c for this model is

$$c_B = -\frac{4}{9(1+\tau^2)}\left[\tau^4+9\tau^3+6\tau^2-9\tau+5\right]\ . \qquad (18)$$

Note that $c_B<0$, $\forall\tau$, and, therefore, the bifurcation is always postponed ($\delta^2_{2,c}>0$) with respect to its deterministic position , as observed in the analog experiments [10]. For very short correlation times, for which our theory is valid, this postponement is smaller than that obtained in the white noise situation, since $c_B-c_{B,\tau=0}>0$ for $\tau\ll 1$, in agreement with the results in [10,11]. Finally, in terms of the original parameter B, the position of the second transition point is

$$B_{c,2} = 2+2\sigma^2\,[1.2-4\tau]+O(\tau^2)\ . \qquad (19)$$

To complement our analysis we compare this theoretical result with a numerical simulation of the equations (17) when the noise is present. Figure 1 depicts the pre-

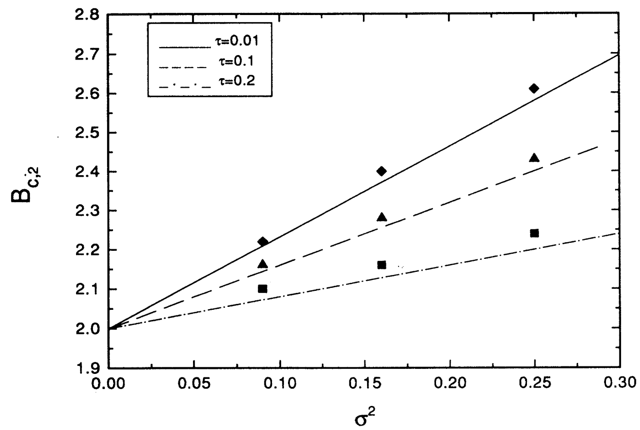

FIG. 1. Critical parameter $B_{c,2}$ of the Brusselator as a function of σ^2, and for different values of τ. The theoretical results are obtained from (19), and the points from the numerical simulations are indicated by diamonds ($\tau = 0.01$), triangles ($\tau = 0.1$), and squares ($\tau = 0.2$).

dicted variation of $B_{c,2}$ with respect to σ^2 for a fixed value of τ ($\tau = 0.01$, $\tau = 0.1$, and $\tau = 0.2$, respectively) compared with the critical values obtained from the simulations. Although the theory always underestimates the values (with an error of 0.5% for the best case of small σ^2 and τ, and of 2% for the worst situation of high σ^2 and τ), the qualitative and quantitative coincidence between theory and numerical simulations is quite good. Notice also how the postponement is reduced by increasing the correlation time of the noise.

To summarize, in this paper we have calculated an approximate stationary probability distribution for a general system showing a supercritical Hopf bifurcation and perturbed by a colored noise. Our results are valid close to the bifurcation point, and with a noise of small intensity and short correlation time.

Because of its explicit angular dependence, the stationary probability distribution is no longer symmetric, and as a fundamental consequence we do not have a single bifurcation point but an interval, and inside that interval, and for the same values of the parameters, bimodal vertical cross sections for some angles coexist with monomodal ones. We wish to stress that this angular dependence is an important ingredient in the behavior of the system, and therefore, any study in which the angular variable is integrated out can only capture a limited amount of information, and in particular cannot predict the existence of the bifurcation interval [9,11].

The different transition values of the new control parameter δ^2 depend not only on the deterministic order parameter, but also on the characteristic values of the noise, and in particular the correlation time of the noise makes the width of the bifurcation interval smaller. Therefore, noise-induced changes in the bifurcation values, as the advancements and postponements predicted in [9], can in general occur, depending on the values of c, although the actual appearance of these changes will be model dependent.

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