Decay of classical chaotic systems: The case of the Bunimovich stadium

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The escape of an ensemble of particles from the Bunimovich stadium via a small hole has been studied numerically. The decay probability starts out exponentially but has an algebraic tail. The weight of the algebraic decay tends to zero for vanishing hole size. This behavior is explained by the slow transport of the particles close to the marginally stable bouncing ball orbits. It is contrasted with the decay function of the corresponding quantum system.

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I. INTRODUCTION

The decay of a quantum mechanical system whose states may be treated statistically is — on the average over the states — not always exponential. It is algebraic, if only a few decay channels are open. This has been demonstrated in [1]. There, the quantum mechanical system was simulated experimentally by a quasi-two-dimensional microwave cavity. The classical analog to this is the motion of a particle in a billiard shaped like a stadium, which was shown by Bunimovich [2] to be fully chaotic and especially ergodic. By the present paper, we want to show that the algebraic decay observed in [1] is a quantum mechanical feature that has no classical counterpart. To this end we have studied the escape probability $P(t)$ via a small hole out of the classical Bunimovich stadium and find $P(t)$ "almost exponential." This means that every $P(t)$ has an algebraic tail for large t, but the decay functions approach exponential behavior with decreasing size of the hole.

The algebraic asymptotics of the decay function $P(t)$ can be viewed as an example of the "third type of decay" mentioned in the introduction of [3], where "particles initialized in a chaotic region can stick a long time to the vicinity of the boundary of a regular region." Although there is no region of regular motion in the phase space of the billiard, there exist marginally stable orbits that cause this "sticking." A schematic model will let us — semiquantitatively — understand why the decay is nevertheless almost exponential.

The results of the numerical simulation are presented in Sec. II. They are contrasted with the decay of the corresponding quantum system in Sec. III. The simple model illustrating the interplay between exponential and algebraic decay of the classical system is described in Sec. IV. Its predictions are compared with the present results in Sec. V.

II. NUMERICAL EXPERIMENT

As in [1], a quarter of the Bunimovich stadium has been considered in order to remove symmetry. Its shape is sketched in the inset of Fig. 1. All lengths are given in units of the radius of the circular part of the boundary. The shape parameter γ (see Fig. 1) was $\gamma = 1.8$. An escape hole of size Δ has been assumed, as indicated in Fig. 1, in the upper half of the small straight piece of the boundary. For an ensemble of 10^6 particles, initial conditions were chosen at random. The distribution was a constant times $dxdyd\phi$, where x and y are Cartesian coordinates of the position inside the stadium and ϕ is an angle characterising the direction of motion. The orbit of every particle was followed numerically until it escaped via the hole out of the stadium. The orbits were, however, not followed beyond 10^5 collisions between the particle and the boundary. For a hole of size $\Delta = 0.05$ the result of this numerical experiment is given in Fig. 1 by the histogram of the probability density $P(L)$ for the particle to escape after an orbit of length L. This will be called data in the sequel. Since the velocity of the particle has constant modulus, we identify L with time.

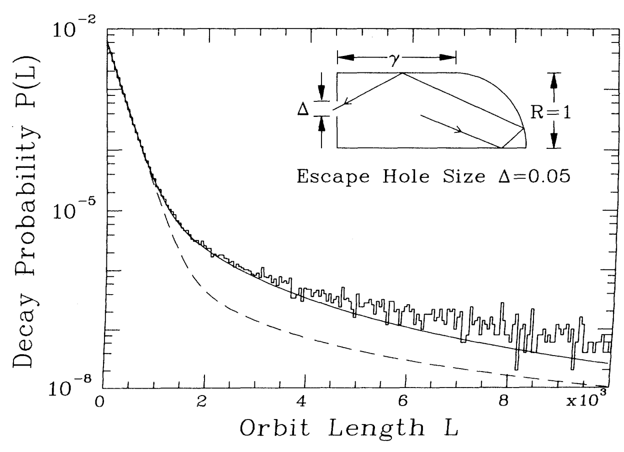

FIG. 1. Decay probability of the stadium with $\gamma = 1.8$ and an escape hole size of $\Delta = 0.05$. The histogram is the result of the numerical experiment. The full line is its parametrization via Eq. (1). The parameters are given in Table I. The dashed line is produced by the model of Sec. IV.

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One finds $P(L)$ exponential for sufficiently small L. Later $P(L)$ turns into a more slowly decaying function. This behavior is typical and it occurs wherever the hole is positioned. The probability density $P(L)$ can be represented by the combination of exponential and algebraic functions

$$P(L) = E\lambda \exp(-\lambda L) + A\alpha(\beta - 1)(1 + \alpha L)^{-\beta}, \quad (1)$$

where $E + A = 1$ for a normalized P. Then A measures the weight of the algebraic term. The full curve in Fig. 1 is a fit to the data. The parameters α, β, λ, and A were searched for. This procedure was followed in 15 "experiments" with holes in the range of $0.25 \geq \Delta \geq 0.0025$. The results are partly reproduced in Table I together with the values of the normalized χ^2. Although only the $\gamma = 1.8$ stadium is discussed and analyzed below, we also display in Table I some results pertaining to the $\gamma = 1$ stadium in order to show that it behaves quite similarly.

Table I and Fig. 2 show that the weight A of the algebraic decay approaches zero for $\Delta \to 0$, i.e., the decay of the stadium becomes exponential in the limit of a vanishing size of the hole. We call this behavior "almost exponential" decay.

In the limit of $\Delta \to 0$ the decay constant approaches — see Table I — the value

$$\lambda_0 = \frac{\Delta}{\pi A_c} = \frac{4\Delta}{\pi(\pi + 4\gamma)} \quad (2)$$

given in [4]. Here, A_c is the area of the billiard; the momentum of the particle has been set equal to unity. Equation (2) can be derived from ergodicity: Every point in phase space should be visited with equal probability independent of the elapsed time. From this one infers [4] that the decay should be exponential with the decay constant λ_0.

We note that the dynamics of the stadium may also be described as a mapping which generates, from given coordinates of collision with the boundary, the coordinates of the next collision. If one uses the number of mappings applied to the initial distribution as time variable one again infers exponential decay, since the motion on the boundary is ergodic, too. The decay constant ν_0 is then simply the ratio of the size of the hole and the perimeter of the billiard,

$$\nu_0 = \frac{2\Delta}{\pi + 2\gamma}. \quad (3)$$

Here, we consider as boundary of the quarter stadium only the piece that it shares with the original full stadium. We disregard the "lines of cut." Similarly a "collision with the boundary" is defined as a collision with the piece of the original boundary (one could as well include the full boundary of the quarter stadium into these definitions but the present choice is more convenient). The ratio

$$\frac{\nu_0}{\lambda_0} = \frac{\pi(\pi + 4\gamma)}{2(\pi + 2\gamma)} \quad (4)$$

TABLE I. Results of parametrizing the numerical experiments on the stadiums with $\gamma = 1.8$ and $\gamma = 1$. The quantities listed are those of Eqs. (1) and (2) together with the normalized χ^2 values.

Δ	A	$10^3\alpha$	β	$10^3\lambda$	$10^3\lambda_0$	χ^2
			$\gamma = 1.8$			
0.2500	0.301	34.00	2.36	34.93	30.77	0.99
0.0500	0.181	4.32	2.89	6.14	6.16	0.61
0.0100	0.038	4.71	1.97	1.22	1.23	0.50
0.0025	0			0.32	0.31	0.82
			$\gamma = 1$			
0.2500	0.223	48.66	2.42	49.93	44.54	1.38
0.0500	0.128	6.40	2.96	8.88	8.91	1.10
0.0100	0.030	2.09	2.81	1.79	1.78	1.03
0.0025	0			0.47	0.46	1.01

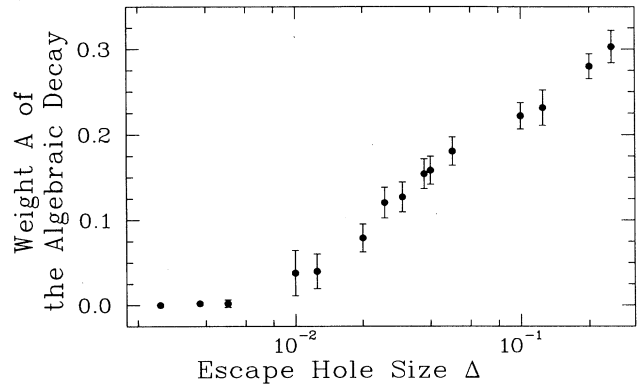

FIG. 2. Weight A of the algebraic decay — see Eq. (1) — vs the size Δ of the escape hole. The errors are statistical ones. They are given if larger than the size of the dots.

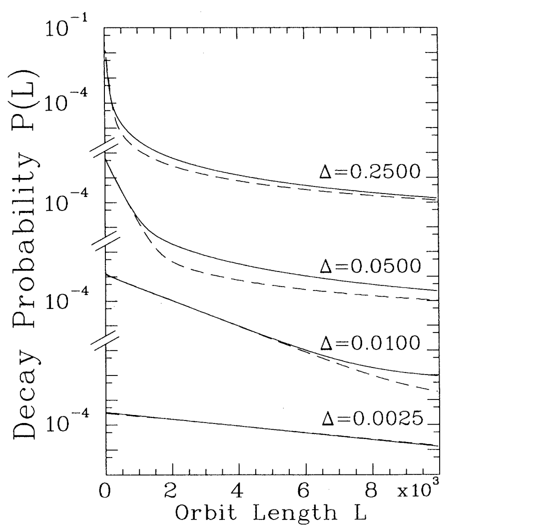

FIG. 3. Decay probability P vs orbit length (or time) L with hole size Δ as parameter. The full lines show the parametrized numerical experiment. The dashed lines result from the schematic model described in the main text.

then is the mean free path of the particle between two collisions with the boundary. We have verified numerically that this relation is fulfilled (again in the limit of $\Delta \to 0$). Equation (4) is a special case of a formula in Appendix B of [5].

The power β of the algebraic part of Eq. (1) is found to lie between 2 and 3. The fitted decay functions for $\gamma = 1.8$ with the parameters of Table I which are useful representations of the data, are given in Fig. 3 as full lines. One notes that with $\Delta \to 0$ the onset of the algebraic decay is shifted to larger L and to decreasing probability level.

III. DECAY OF THE CORRESPONDING QUANTUM SYSTEM

The almost exponential decay found above is in contrast to the behavior of the corresponding quantum system studied with the help of microwaves in [1]. Suppose that the holes through which the system is coupled to the external world (the antennas) are small compared to the wavelengths occurring inside. Every hole may then be identified with one decay channel c. Let $\langle\Gamma_c\rangle$ be the decay width into channel c averaged over the eigenstates of the stadium. We suppose that the decay widths into different channels are uncorrelated. To avoid complications that are not instructive, we take $\langle\Gamma_c\rangle$ to be independent of c. The probability P_q that the quantum system decays at time t after its formation is then

$$P_q(t) \sim (1 + 2\langle\Gamma_c\rangle t)^{-2-\frac{M}{2}}, \tag{5}$$

for t larger than the equilibration time of the system; see Sec. (6.3) of [6] or Eq. (6) of [1]. Here, M is the number of open channels (or holes). The algebraic form of Eq. (5) occurs because P_q is an average over exponentially decaying resonances whose decay amplitudes have a Gaussian distribution (see the introduction of [7]).

The quantum system decays almost exponentially if M is large because Eq. (5) can then be approximated by

$$P_q(t) \sim \exp\left[-(4 + M)\langle\Gamma_c\rangle t\right] \tag{6}$$

and the weight of the algebraic tail becomes negligible. For the channels to be statistically independent, any two antennas must, however, be separated by a typical wavelength or more.

We conclude that the quantum mechanical counterpart of a system with a small hole decays algebraically. If the statistics of the decay amplitudes is not exactly Gaussian it will decay essentially algebraically. Slight deviations from Gaussian distribution have been detected [8] and are due to the presence of bouncing ball orbits between the parallel straight sections of the stadium billiard.

Hence, the essentially algebraic decay of the quantum system has no classical counterpart. The classical system with a small hole decays almost exponentially.

What then is the origin of the algebraic tail in the classical decay functions $P(L)$? It is the fact that the motion in the stadium is ergodic only if it persists indefinitely. If the motion terminates by the escape of the particle, it becomes apparent that ergodicity is not established in any finite time. The reason for this is the existence of marginally stable orbits that can almost trap the particle — the bouncing ball orbits. For details, see the next section.

It is then the same class of orbits, the marginally stable orbits, that add corrections to the essentially different behavior of the classical and the quantum systems.

IV. MODEL OF DELAYED CLASSICAL DECAY

The argument justifying exponential decay with the decay constant λ_0 of Eq. (2) applies if ergodicity is established "sufficiently quickly." A close inspection [9] reveals that the fluctuations of the frequency of the particles' arrival at location of the hole must be small: Let Δt be the difference between two successive times of arrival at the hole. Then $\overline{(\Delta t)^2}/\overline{\Delta t}^2$ must be small compared to unity. Now, when there is a region of phase space into which the particles penetrate very slowly and, by the same token, in which they remain trapped for an exceptionally long time once they are there, then the above fluctuations are large and the release of the trapped particles will eventually dominate the decay process. Such regions of phase space have been described, e.g., in [10–15] and in the first reference of [2]: Close to the family of bouncing ball orbits and the "whispering gallery orbits" there are parts of phase space with volume > 0 in which the particles can be trapped for an arbitrarily long time. (A whispering gallery orbit is the motion of the particle along the boundary.)

The arguments of [10] show that the probability $G(n)$ for the particle to be in an "almost bouncing ball orbit" that will persist for more than n collisions with the boundary is

$$G(n) = \frac{\gamma^2}{(\pi + 2\gamma)n}, \quad n \gg 1. \tag{7}$$

An "almost bouncing ball orbit" allows for the angle between orbit and boundary to be slightly different from $\pi/2$. In this situation, n collisions amount to an orbit of length $L = 2n$ (see the definition of a "collision with the boundary" in Sec. II). Therefore

$$G(n(L)) = \frac{1}{\alpha_0 L}, \quad L \gg 1, \tag{8}$$

is the probability for the particle to be in an orbit that will persist over a length $> L$. Here, we have used

$$\alpha_0 = \frac{\pi + 2\gamma}{2\gamma^2}. \tag{9}$$

The probability to be in a whispering gallery orbit decreases more strongly than L^{-1} and is therefore disregarded in the further discussion.

Hence, the probability that the particle is in an almost bouncing ball orbit that will persist for the length L is asymptotically

$$g_{as}(L) = -\frac{d}{dL}G(n(L)) = \frac{1}{\alpha_0 L^2}. \tag{10}$$

It was therefore anticipated in [9,14] that $P(L)$ should asymptotically be $\sim L^{-2}$. There is some numerical evidence for this in [12]: Note that the function $N(t)$ given there by closed circles in Fig. 2 is the present $G(n(L))$. The present data confirm this asymptotic behavior of $P(L)$ at least qualitatively.

The following schematic model, inspired by the treatment of the Sinai billiard in [13], shows how the algebraically delayed decay comes about. Suppose that the phase space can be split into two parts $\mathcal{C}$ and $\mathcal{L}$ such that the decay happens in $\mathcal{C}$, the delay in $\mathcal{L}$. Once the particle is in $\mathcal{C}$, it shall escape with probability ω or immediately go back to $\mathcal{L}$ with probability $1-\omega$. Consider a particle which may be anywhere at the time $L = 0$. Define $g(L)dL$ as the probability distribution for its next transition from $\mathcal{L}$ into $\mathcal{C}$ to happen at time L. Then the probability density $p_1(L)$ to escape at time L after having made exactly one transition from $\mathcal{L} \to \mathcal{C}$ is

$$p_1(L) = \omega g(L). \tag{11}$$

Obviously g_{as} in Eq. (10) is the asymptotic form of g. Let $f(L)dL$ be the distribution of the time L between two successive transitions $\mathcal{L} \to \mathcal{C}$. Then the probability density $p_2(L)$ for escape after exactly two transitions $\mathcal{L} \to \mathcal{C}$ is

$$p_2(L) = \omega(1-\omega) g \otimes f, \tag{12}$$

where the operator $\otimes$ denotes the convolution. For k transitions $\mathcal{L} \to \mathcal{C}$ one has

$$p_k(L) = \omega(1-\omega)^{k-1} g \otimes (f \otimes \cdots \otimes f), \tag{13}$$

where g is folded with $(k-2)$ convolutions of f with itself. The decay probability $P(L)$ is the sum

$$P(L) = \sum_{k=1}^{\infty} p_k(L). \tag{14}$$

Taking the Laplace transform

$$\hat{p}_k(s) = \int_0^{\infty} dL \exp(-sL) p_k(L) \tag{15}$$

converts the convolution into a product of the Laplace transforms $\hat{g}$ and $\hat{f}$. Therefore the transform $\hat{P}(s)$ of P is a geometric series in $\hat{f}$, which gives

$$\hat{P}(s) = \omega \hat{g}(s)[1-(1-\omega)\hat{f}(s)]^{-1}. \tag{16}$$

The function g is by definition proportional to an integral over f. We note that this is numerically exhibited by [11], where the first-passage-time distribution function P_{fpt} in Ref. [11] corresponds to f and its asymptotic behavior is found to be close to $\sim L^{-3}$. The relation between g and f is also just the relation between "transient chaos" and "chaotic scattering" established in [12]. As the terms are used there, transient chaos corresponds to placing at time zero the particle at random in phase space; chaotic scattering corresponds to the situation in which the particle enters $\mathcal{L}$ at time zero. The results of [12] are compatible with $f \sim dg/dL$ as it should be.

The normalization of f and g requires

$$\frac{dg}{dL} = -g(0) f(L) \tag{17}$$

and the definition of the Laplace transform then leads to

$$\hat{f}(s) = 1 - s\frac{\hat{g}(s)}{g(0)}. \tag{18}$$

Inversion of the Laplace transform [Eq. (16)] then gives $P(L)$ in the form of the Fourier integral

$$\begin{aligned} P(L) &= (2\pi)^{-1} \int_{-\infty}^{\infty} ds' \hat{P}(is') \exp(is'L) \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} ds' \frac{\hat{g}(is')}{1 + \frac{1-\omega}{\omega\alpha_0} is'\hat{g}(is')} \\ &\quad \times \exp(is'L). \end{aligned} \tag{19}$$

A similar formula was used in Eq. (5) of [15] to describe the exponential part only of the decay and to define the "relaxation time" within the closed stadium. It is shown here that Eq. (19) accounts for the decay function altogether if the appropriate probability density $g(L)$ is introduced. It must necessarily allow for large fluctuations of L. There exists, of course, a $g(L)$ such that the observed decay function is exactly reproduced. The present model is schematic by a naive choice of $g(L)$, leaving open the exact definition of the partition of phase space into $\mathcal{C}$ and $\mathcal{L}$. We only know the asymptotic form of G; see Eq. (8). Let us, in the simplest possible way, assume that

$$g(L) = \alpha_0(1+\alpha_0 L)^{-2}. \tag{20}$$

The decay function of Eq. (19) approaches an exponential behavior when $\omega \to 0$. The factor $(1-\omega)/\omega\alpha_0$ in the denominator of the integrand is then very large and this makes the functional form of $\hat{g}$ irrelevant. One can thus approximately put $\hat{g}(is') \approx \hat{g}(0) = 1$ and the pole of $\hat{P}(is')$ at $s' \approx i\omega\alpha_0$ yields $P(L) \sim \exp(-\omega\alpha_0 L)$.

The asymptotic behavior of $P(L)$ is given by setting the denominator of the integrand equal to unity. This is so because the omitted terms are of the type $(is')^n\hat{g}(is')$, leading to derivatives of $g(L)$ that decrease faster than $g(L)$. Hence, $P(L) \to g(L)$ for $L \to \infty$. This means that the asymptotic behavior is independent of the size of the escape hole. This result also implies $P(L) \sim L^{-2}$ for large L, as expected. Hence, the present model incorporates the features of the data that we have termed almost exponential decay.

By equating $\omega\alpha_0$ with the decay constant λ_0, Eqs. (2) and (9) determine the two parameters of the present model. This yields the theoretical decay functions given as dashed lines in Fig. 3 together with the data.

V. DISCUSSION

The numerical experiment yields an almost exponential decay of the classical Bunimovich stadium. This means that the decay function $P(L)$ starts out exponentially and later turns to an algebraic behavior. This contrasts the claim of [4] (see also the discussion in [16]). The weight of the algebraic decay, however, tends to zero with decreasing size of the escape hole. In this modified sense of exponential decay we agree with [4].

Hillermeier *et al.* [3] show how the algebraic decay can be understood formally. The authors of [17,18] have pointed out that the slow transport of particles at the boundary of islands of regular motion can cause algebraic decay. Yet there are no strictly stable orbits in the Bunimovich stadium [2]; in this sense it is a system with "fully developed chaos" [19]. However, there is the family of marginally stable bouncing ball orbits that according to [10] causes the algebraic damping of phase space correlations. It is responsible for the algebraic tail of the decay function $P(L)$, too. More precisely, the bouncing ball orbits are a set of parabolic [20], nonisolated periodic orbits. A similar set of periodic orbits exists in the Sinai billiard and causes similar effects [13].

The algebraic tail of $P(L)$ and its suppression in the limit of vanishing size Δ of the hole — briefly, the almost exponential decay — is semiquantitatively reproduced by the model inspired by [13] and described in Sec. IV. The model reproduces the exponential part of $P(L)$. It should yield a lower limit to the algebraic part because there may be sources of delayed decay other than the bouncing ball orbits. The comparison between data and model in Fig. 3 shows this to be true. Furthermore, the data and the results of the model should converge for $L \to \infty$, since the bouncing ball orbits cause the most pronounced delay. Convergence is indeed indicated by the curves for the largest holes although it is very slow; it takes many inverse decay constants λ_0^{-1} before it is reached. One expects [10] that $P(L)$ tends, for $L \to \infty$, towards a function $g_{as} \sim L^{-2}$, which is independent of the size of the hole and given by the properties of the bouncing ball orbits alone. The model of Sec. IV complies with this expectation. Again it is also compatible with the data: The experimental $P(L)$ asymptotically behaves as $L^{-\beta}$, with β somewhat larger than 2, and thus indicates convergence towards $g_{as}(L)$. For the largest hole ($\Delta = 0.25$) the convergence is essentially achieved within the range of L that we have studied. It is, however, achieved so slowly that for the other hole sizes Δ, even orbit lengths of $L = 10^4$ are insufficient to demonstrate it. The data are, however, fully consistent with convergence towards $g_{as}(L)$.

There are numerous studies of algebraically delayed *decay of correlations* (as, e.g., velocity correlations) in closed chaotic systems, see, e.g., [2,10,21,22]. The authors trace the delay back to the existence of marginally stable periodic orbits. *Anomalous diffusion* is caused by the same type of orbits; see, e.g., [23,24]. Again these orbits are responsible for the algebraically delayed *escape* of particles from fully chaotic systems. This emerges from the arguments of [14], the numerical experiments of [13], and the present paper. Hence, all these phenomena are related to each other. By using the considerations of [10] to define $g_{as}(L)$ we have directly linked the *decay of correlations* to the *escape*. Furthermore, we have pointed out that the algebraic decay of chaotic quantum systems is a consequence of wave mechanics and is not produced by the delayed decay of the classical counterpart.

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