Finite-size scaling Casimir force function: Exact spherical-model results

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The excess free energy due to the finite-size contributions to the free energy of the system in a film geometry characterizes a fluctuation-mediated interaction that is termed the Casimir force, or, in the case of a fluid confined between two parallel walls, the solvation force (or the disjoining pressure). The analog of these forces within the three-dimensional mean-spherical model with periodic boundary conditions and geometry $L\times\infty^2$ is investigated in the presence of an external magnetic field. The corresponding analytical expressions for the finite-size scaling functions of the excess free energy and the Casimir (solvation) force and their asymptotic behavior in the vicinity, below and above the critical temperature T_c, are derived and evaluated numerically. In contrast to the Ising-like case the scaling functions of the excess free energy and of the Casimir force below T_c in zero magnetic field do not tend exponentially fast with L to zero, but, tend to some universal constant. The last is supposed to be true for all $O(n)$, $n\geq 2$ models.

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I. INTRODUCTION

A simple $O(n)$-symmetric ($n\geq 1$) system with a geometry $L\times\infty^2$ (and under given boundary conditions τ imposed across the direction L) is a standard statistical mechanical model for describing a magnet, or a fluid, confined between two parallel plates of infinite area. One important quantity which arises naturally in the thermodynamics of these confined systems is

$$f_{(\ldots)}(T,L)=-\frac{\partial f^{\mathrm{ex}}(T,L)}{\partial L}\ , \tag{1}$$

where $f^{\mathrm{ex}}(T,L)$ is the excess free energy

$$f^{\mathrm{ex}}(T,L)=f(T,L)-Lf_b(T)\ . \tag{2}$$

Here $f(T,L)$ is the full free energy per unit area (and per k_BT) of such a system and f_b is the bulk free energy density.

In the case of a fluid (then one actually has to consider the excess grand potential per unit area and the derivative is performed at constant chemical potential μ and temperature T) $f_{(\ldots)}$ is termed the solvation force [1,2] f_{solv}, (sometimes called the disjoining pressure) whereas in the case of a magnet one speaks instead about the Casimir force [3,4] f_{Casimir} (and the derivative is performed at constant temperature and magnetic field h).

Depending on the boundary condition $f^{\mathrm{ex}}(T,L)$ may or may not contain contributions independent of L. For the Ising-like systems these can be the surface free energies $f_{s,1}(T)$ and $f_{s,2}(T)$, and the interface free energy $f_i(T)$ (for brevity we consider the dependence on the temperature T only). For the $O(n)$ models these will only be the contributions stemming from the surface free energies because the analog of the interface free energy is the helicity modulus $\Upsilon(T)$ and the corresponding contribution is of the order $\Upsilon(T)/L$.

The solvation force in two-dimensional Ising strips has been investigated recently by Evans and Stecki [2], whereas the Casimir force in $O(n)$ systems has been considered for $T\geq T_c$ by Krech and Dietrich [4] by means of the field-theoretical renormalization group theory in $4-\epsilon$ dimensions. (For the Ising-like case they have also derived some results for $T<T_c$.) The most results available at the moment are for two-dimensional systems at $T=T_c$ (which can be treated by conformal-field theory methods) where the Casimir force reduces to the so-called Casimir amplitudes Δ_τ. For $d=3$ the results for the Casimir amplitudes available in the Ising-like case have been obtained by Migdal-Kadanoff renormalization-group calculations [5] and by some interpolation of the exact values for $d=2$ and $d=4$ [4]. (For a general review on the Casimir forces see [3]). For $n\geq 2$ the only existing results are obtained by the ϵ-expansion technique, where the calculations are performed up to the first order in ϵ and in the final expressions ϵ is set equal to 1 [4]. For the periodic boundary conditions

$$\Delta_{\mathrm{per}}=\Delta_{\mathrm{per}}^{(1)}\left[1-\frac{5}{4}\epsilon\frac{n+2}{n+8}\right]\ , \tag{3}$$

where $\Delta_{\mathrm{per}}^{(1)}$ is the one-loop calculation result. As is well known, the infinite translational invariant spherical model is equivalent to the $n\to\infty$ limit of the corresponding n-component system [6]. So, the direct investigations of the spherical model will provide independent additional information to these field-theoretical results. [As is clear from Eq. (3) at $\epsilon=1$ for n large enough ($n>22$) Δ_{per} and $\Delta_{\mathrm{per}}^{(1)}$ will be even of different sign; therefore it makes no sense simply to take the limit $n\to\infty$ there.]

In the present article we investigate the analog of the Casimir (solvation) force within the three-dimensional mean-spherical model. We focus only on the case of periodic boundary conditions (then simply $f_{s,1}=f_{s,2}\equiv 0$) in the presence of an external magnetic field h. We recall that within the standard finite-size scaling theory (see,

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e.g., [7] for a general review) near the critical temperature T_c (of the corresponding bulk, i.e., $L=\infty$, system) the behavior of f^{ex} for periodic boundary conditions is given by

$$f^{\text{ex}}(t,L)=L^{-(d-1)}X^{\text{ex}}(atL^{1/\nu},bhL^{\Delta/\nu}) , \tag{4}$$

where $t=(T-T_c)/T_c$ is the reduced temperature, a and b are nonuniversal scaling factors, X^{ex} is the *universal* (usually geometry dependent) scaling function, and ν and Δ are the corresponding (universal) scaling exponents. This slow algebraic decay of f^{ex} (and, therefore, of f_{Casimir}) is, of course, associated with the bulk critical fluctuations, i.e., with the divergence of the bulk correlation length $\xi_b[\xi_b(t)\sim t^{-\nu}, t\geq 0]$. In the Ising-like case $(n=1)$ it is a well established fact that f^{ex} tends to zero exponentially fast with L, when $L\to\infty$ at fixed t and h. The same is true both above $(t>0)$ and below $(t<0)$ T_c. The fundamental fact connected with this behavior is that away from the critical point the bulk correlation length is finite (usually a few lattice spacings). But for the $O(n)$, $n\geq 2$, models at $h=0$ this is true only above T_c. Below T_c, due to the existence of soft modes in the system (spin waves) ξ_b is identically infinite. So, it seems conceivable that in such models f^{ex} below T_c will not tend exponentially fast with L to zero, which, in turn, will lead to a much greater (in comparison with the Ising-like case) Casimir (solvation) force. Of course, close to the critical point the behavior of f_{Casimir} will be given by [see Eqs. (4) and (1)]

$$f_{\text{Casimir}}(t,h)=L^{-d}X_{\text{Casimir}}(x_1,x_2) , \tag{5}$$

where $x_1=atL^{1/\nu}$, $x_2=bhL^{\Delta/\nu}$, and

$$X_{\text{Casimir}}(x_1,x_2)=(d-1)X^{\text{ex}}(x_1,x_2)-\frac{1}{\nu}x_1\frac{\partial}{\partial x_1}X^{\text{ex}}(x_1,x_2) -\frac{\Delta}{\nu}x_2\frac{\partial}{\partial x_2}X^{\text{ex}}(x_1,x_2) . \tag{6}$$

Note, that X_{Casimir} will again be a *universal* function of x_1 and x_2. We note that for finite-size systems this means that X_{Casimir} will be the same for all systems of the same universality class *and* geometry and boundary conditions. It is believed that if the boundary conditions τ are the same at both surfaces X_{Casimir} will be negative. In the case of a fluid confined between identical walls this implies that then the net force between the plates will be attractive for large separations. One of the goals of the present article is to check this general expectation on the example of one exactly solvable model.

The article is organized as follows. In Sec. II we briefly describe the model and present convenient starting expressions for investigation of the excess free energy and the Casimir force. Section III contains the results showing the finite-size behavior of these quantities in the finite-size critical region, as well as above and below the critical point. The paper closes with concluding remarks given in Sec. IV.

II. THE MODEL

We consider the ferromagnetic mean-spherical model on a fully finite d-dimensional hypercubic lattice Λ_d of $|\Lambda|$ sites and with block geometry $L_1\times L_2\times\ldots\times L_d$, where L_i, $i=1,\ldots,d$ are measured in units of the lattice spacing. The Hamiltonian has the form

$$\beta\mathcal{H}_\Lambda^\tau(\{\sigma_i\}_{i\in\Lambda})=-\tfrac{1}{2}K\sum_{i,j\in\Lambda}J_{ij}^\tau\sigma_i\sigma_j+s\sum_{i\in\Lambda}\sigma_i^2-h\sum_{i\in\Lambda}\sigma_i . \tag{7}$$

Here $\sigma_i\in\mathbb{R}$, $i\in\Lambda_d[\sigma_i\equiv\sigma(\mathbf{r}_i)]$ is a variable, describing the spin on lattice site i (at $\mathbf{r}_i$), s is the spherical field, K is a dimensionless coupling, J_{ij}^τ is a matrix with dimensionless elements, so that $(K/\beta)J_{ij}^\tau$ is the exchange energy between the spins at sites i and j (of course, $J_{ij}^\tau=J_{ji}^\tau$), and h is the external magnetic field. The dependence on the boundary condition is denoted by a superscript τ.

In the mean-spherical ensemble the partition function is given by

$$Z_d^{(\tau)}(K,h,s;\mathbf{L}) =\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\prod_{i\in\Lambda}d\sigma_i\exp[-\beta\mathcal{H}_\Lambda^\tau(\{\sigma_i\}_{i\in\Lambda})] , \tag{8}$$

where $\mathbf{L}\equiv(L_1,\ldots,L_d)$. Then the canonical free energy $F_d^{(\tau)}(K,h;\mathbf{L})$ is defined [in units of $(k_BT)^{-1}$] by the Legendre transformation

$$F_d^{(\tau)}(K,h;\mathbf{L})=\sup_s[-\ln Z_d^{(\tau)}(K,h,s;\mathbf{L})-s|\Lambda|] . \tag{9}$$

Let us now suppose that the periodic boundary conditions are applied. Then, for the nearest neighbor interactions ($J_{ij}^p=1$, if i and j are the nearest neighbors under the applied set of boundary conditions, and zero otherwise) the eigenvalues $\tilde{J}^{(p)}(\mathbf{k})$ of the matrix J_{ij}^p are

$$\tilde{J}^{(p)}(\mathbf{k})=2\sum_{i=1}^{d}\cos\left[\frac{2\pi k_i}{L_i}\right] , \tag{10}$$

where $\mathbf{k}\equiv(k_1,\ldots,k_d)$; $k_i=0,\ldots,L_i-1$; $i=1,\ldots,d$.

It is convenient to replace the spherical-field s by another field λ, defined as

$$\lambda=2s/K-2d . \tag{11}$$

Performing now the integration in (9) and taking the limits $L_i\to\infty$, for $i=2,\ldots,d$, we obtain $(L_1\equiv L)$

$$f_L(K,h;d)=\sup_\lambda\left\{\frac{1}{2}\int_0^\infty\frac{dx}{x}\left[\exp(-x)-\frac{1}{L}\exp(-\lambda x)S_L^p(x)\right][\exp(-2x)I_0(2x)]^{d-1}-\frac{1}{2}\frac{h^2}{\lambda K}-\frac{1}{2}\lambda K\right\} +\frac{1}{2}\left[\ln\frac{K}{2\pi}-2dK\right] , \tag{12}$$

where $f_L(K,h;d)$ is the free energy density and

$$S_L^p(x)=\sum_{k=0}^{L-1}\exp\left[-2x\left[1-\cos\frac{2\pi k}{L}\right]\right] . \tag{13}$$

The supremum on the right-hand side of Eq. (12) is attained at a value λ_L that is determined by

$$\frac{1}{L}\int_0^\infty dx\,\exp(-\lambda_L x)S_L^p(x)[\exp(-2x)I_0(2x)]^{d-1}+\frac{h^2}{\lambda_L^2 K}=K . \tag{14}$$

From Eqs. (12) and (14), taking into account that

$$\lim_{L\to\infty}\frac{1}{L}S_L^p(x)=\exp(-2x)I_0(2x) , \tag{15}$$

we obtain for the excess free energy

$$f^{\rm ex}(K,h;L,d)/L=\tfrac{1}{2}\int_0^\infty\frac{dx}{x}[\exp(-2x)I_0(2x)]^{d-1}\times[\exp(-\lambda_\infty x)\exp(-2x)I_0(2x)-\frac{1}{L}\exp(-\lambda_L x)S_L^p(x)]+\frac{1}{2}h^2\left[\frac{1}{\lambda_\infty}-\frac{1}{\lambda_L}\right]+\frac{1}{2}K(\lambda_\infty-\lambda_L) . \tag{16}$$

Here $\lambda_\infty(K,h)$ is the solution of the spherical-field equation for the corresponding bulk system

$$W_d(\lambda_\infty)+\frac{h^2}{\lambda^2 K}=K , \tag{17}$$

where

$$W_d(\lambda)=\int_0^\infty\exp(-\lambda x)[\exp(-2x)I_0(2x)]^d \tag{18}$$

is the d-dimensional Watson function. As is clear from Eqs. (14), (17), and (18) the left-hand side of Eqs. (14) and (17) are monotonously decreasing functions of λ_L and λ_∞, respectively. Therefore, if its exists, the solution of Eqs. (14) and (17) for a given K and h will be unique.

Equations (12)–(18) provide the basis of our further analysis.

III. FINITE-SIZE BEHAVIOR OF THE EXCESS FREE ENERGY AND THE CASIMIR FORCE

Up to now we have not specified the value of the space dimension d. Therefore, the above expressions are quite general. Since our main interest is concentrated on three-dimensional systems, in the remainder of the paper we will only consider the case $d=3$. As is well known, the critical temperature of the system is then given by [8]

$$K_c=\int_0^\infty dx[\exp(-2x)I_0(2x)]^3=0.252\,73 . \tag{19}$$

We will be interested in the region of thermodynamic parameters where $\lambda_L\ll 1$ (and therefore $\lambda_\infty\ll 1$, too), i.e., in the behavior of the system close to the critical point $K=K_c$, $h=0$ and when $K>K_c$ and $|h|\ll 1$. Using the technique developed in [9] and having in mind that for the three-dimensional spherical model $\nu=1$ and $\Delta=5/2$, it can be shown that in the vicinity of the critical point $f^{\rm ex}$ is

$$f^{\rm ex}(K,h;L)=L^{-2}X^{\rm ex}(x_1,x_2) , \tag{20}$$

with

$$X^{\rm ex}(x_1,x_2)=\tfrac{1}{2}(4\pi)^{-3/2}\left[\sum_{k=0}^\infty\frac{(-1)^k(y_L^{k+1}-y_\infty^{k+1})}{(k+1)!(k-1/2)}-\sqrt{4\pi}\int_1^\infty dx\,x^{-2}[1+2R(4\pi^2 x)]\exp[-y_L x]-2\int_0^1 dx\,x^{-5/2}R(1/4x)\exp[-y_L x]+\int_1^\infty dx\,x^{-5/2}\exp(-y_\infty x)\right]+\frac{1}{2}x_2^2\left[\frac{1}{y_\infty}-\frac{1}{y_L}\right]+\frac{1}{2}x_1(y_\infty-y_L) , \tag{21}$$

where

$$R(x)=\sum_{q=1}^\infty\exp[-xq^2] , \tag{22}$$

and

$$y_L=\lambda_L L^2,\quad y_\infty=\lambda_\infty L^2 ,$$
$$x_1=(K-K_c)L,\quad x_2=K_c^{-1/2}hL^{5/2} . \tag{23}$$

Here y_L is the solution of the equation of the spherical field for the finite system [see Eq. (14)] which can be written in the form

$$(4\pi)^{3/2}\left[\frac{x_2^2}{y_L^2}-x_1\right]+\mathcal{F}(y_L)=0 , \tag{24}$$

where

$$\mathcal{F}(y)=2\int_0^1 x^{-3/2}R\left[\frac{1}{4x}\right]\exp(-yx)dx$$
$$+\sqrt{4\pi}\int_1^\infty x^{-1}[1+2R(4\pi^2 x)]\exp(-yx)dx$$
$$-2-2\sqrt{\pi y}-\sqrt{y}\int_y^\infty x^{-3/2}[\exp(-x)-1]dx\ . \tag{25}$$

In order to obtain the above expression use has been made of the identity

$$\sum_{k=0}^{\infty}\frac{(-1)^k y^{k+1}}{(k+1)!(k-1/2)}=-\frac{4\sqrt{\pi}}{3}y^{3/2}$$
$$-\frac{2}{3}y^{3/2}\int_y^\infty x^{-3/2}\exp(-x)dx$$
$$-\tfrac{2}{3}[1-\exp(-y)]\ . \tag{26}$$

For the infinite system, by using the asymptotic expansion of the Watson function (see, e.g., Ref. [8]; $|\lambda_\infty|\ll 1$) the corresponding equation of the spherical field can be recast in the very simple form

$$x_1=\frac{x_2^2}{y_\infty^2}-\frac{1}{4\pi}\sqrt{y_\infty}\ . \tag{27}$$

From Eqs. (1) and (20) it immediately follows that

$$f_{\text{Casimir}}(K,h;L)=L^{-3}X_{\text{Casimir}}(x_1,x_2)\ , \tag{28}$$

where

$$X_{\text{Casimir}}(x_1,x_2)=2X^{\text{ex}}(x_1,x_2)-\frac{5}{2}x_2^2\left[\frac{1}{y_\infty}-\frac{1}{y_L}\right]$$
$$-\frac{1}{2}x_1(y_\infty-y_L)\ . \tag{29}$$

As we see, the results for the spherical model [Eqs. (20), (21), (28), and (29)] are in full agreement with the predictions of the finite-size scaling theory [Eqs. (4) and (5)]. Now we will examine the behavior of the scaling functions X^{ex} and X_{Casimir} in different regimes of K, h, and L. First, let us mention that if $x_1=O(1)$ and $x_2=O(1)$ (i.e., in the "critical region" of the finite system) the solutions of the spherical-field equations will be also $y_\infty=O(1)$ and $y_L=O(1)$ and, therefore, $f_{\text{Casimir}}=O(L^{-3})$. To proceed with the other cases we need the asymptotics of the function $\mathcal{F}$:

$$\mathcal{F}(y)=\begin{cases}-2\sqrt{\pi y}+O[\exp(-\sqrt{y})], & y\to\infty\\ -\sqrt{4\pi}\ln(y)+O(y), & y\to 0\ .\end{cases} \tag{30}$$

Let us first consider the case of zero magnetic field. Then, when $x_1\to-\infty$ (i.e., T is fixed *close* above T_c and $L\to\infty$) it can be easily shown that the solutions of the corresponding spherical-field equations are $y_L\gg 1$ and $y_\infty\gg 1$ and that both equations become identical up to corrections exponentially small in $\sqrt{y}$ (where y is either y_L or y_∞). Next, according to Eqs. (21) and (26) this means that (up to functions which are exponentially small in $\sqrt{y}$) X^{ex} is a function symmetric in y_L and y_∞. Therefore in this case f^{ex} (and, of course f_{Casimir} too) will tend exponentially fast with L to zero. In the opposite case when $x_1\to\infty$ (i.e., when T is fixed *below* T_c and $L\to\infty$) one obtains $y_\infty=0$ and $y_L\ll 1[y_L\approx\exp(-4\pi x_1)]$. Then, as follows from Eqs. (29) and (21):

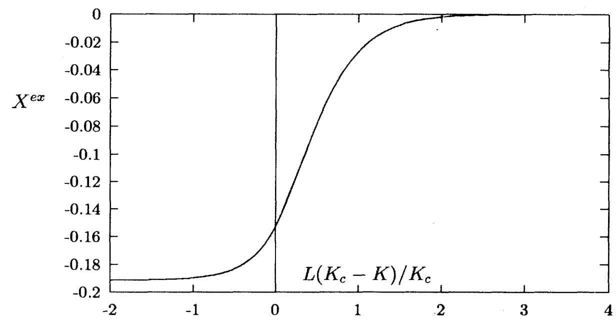

FIG. 1. The finite-size scaling function of the excess free energy is given as a function of $L(K_c-K)/K_c$ for $h=0$. The function is universal.

$$\lim_{x_1\to\infty}X_{\text{Casimir}}(x_1,0)=-\frac{1}{\pi}\zeta(3)\ , \tag{31}$$

where ζ is the Riemann's zeta function. For the intermediate region $x_1=O(1)$ numerical calculations are unavoidable. The corresponding results for the excess free energy [Eq. (21)] are given in Fig. 1 and for the Casimir (solvation) force [Eq. (29)] in Fig. 2. We see that the Casimir force is negative in the whole temperature region, as is to be expected "generally" for systems with "identical boundaries" [2]. At $T=T_c$ the relation $f_{\text{Casimir}}\equiv(d-1)\Delta_{\text{per}}L^{-d}$ defines the universal finite-size amplitude Δ_{per}, commonly denoted as the Casimir amplitude. The numerical estimation of $X^{\text{ex}}(0,0)$ gives

$$X^{\text{ex}}(0,0)\equiv\Delta_{\text{per}}=-0.153 \tag{32}$$

The corresponding value of the spherical field is $y_c=0.926$. Having in mind that according to Ref. [10] in the vicinity of T_c the finite-size correlation length ξ_L in the considered model system under periodic boundary

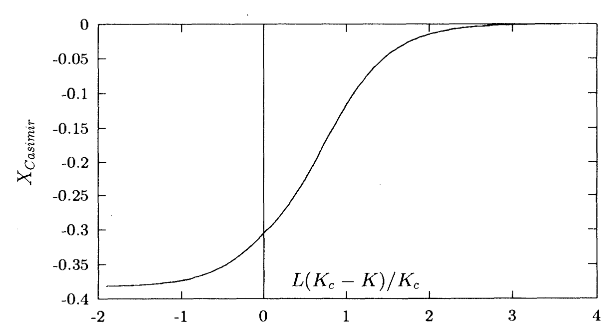

FIG. 2. The universal finite-size scaling function of the Casimir (solvation) force X_{Casimir} as a function of $L(K_c-K)/K_c$ for $h=0$.

conditions is related to the spherical field via

$$\xi_L = L y_L^{-1/2} , \tag{33}$$

we immediately obtain another universal critical constant of interest [11], namely, the correlation length critical amplitude

$$A_c = 1.039 , \tag{34}$$

$[\xi_L(T_c) = A_c L]$. The numerical value of A_c for the spherical model is known also for geometry $L^2 \times \infty$ and it is [12] $A_c = 0.6614$. We see here the clear geometry dependence of these critical amplitudes.

Let us now consider the behavior of the Casimir force in the presence of an external field.

(a) $x_2^2 \gg x_1$. In this regime from Eqs. (24), (27), and (30) it follows that the spherical-field equations have solutions $y_L \gg 1, y_\infty \gg 1$; these equations again become identical up to functions exponentially small in $\sqrt{y}$ and similarly to the case $x_2 = 0$, $x_1 \to -\infty$ we obtain that the Casimir force (and the excess free energy) tends to zero exponentially fast in L. The most interesting case here is the critical isotherm (i.e., $x_1 = 0$). The corresponding numerical results for the scaling function of the Casimir force are given in Fig. 3. One sees the exponential decay of X_{Casimir} when $|x_2| \gg 1$.

(b) $x_2^2 \ll x_1$. In this regime the only solutions of the spherical-field equations are $y_L \ll 1$ and $y_\infty \ll 1$. The behavior of the Casimir force is similar to that when $x_2 = 0$ and $x_1 \to \infty$.

(c) $x_2^2 \sim x_1$. In this regime, which includes also the critical region $x_1 = O(1)$ and $x_2 = O(1)$, one has $y_L = O(1)$ and $y_\infty = O(1)$. the numerical calculations here are unavoidable. The corresponding results for the surface of the finite-size scaling function of the Casimir force in the plane of the scaling parameters are given in Fig. 4. We see that the Casimir force is again always negative. Furthermore, taking into account that from Eqs. (24) and (27) x_1 and x_2 can be expressed in terms of y_L and y_∞ and then replaced in X_{Casimir}, we conclude that everywhere in this region of parameters $f_{\text{Casimir}} = O(L^{-3})$. Finally, it is worthwhile to note that $x_1 \gg 1$ (i.e., $T < T_c$, $L \to \infty$) and $x_2^2 \sim x_1$ implies that $h = O(L^{-2})$ which, according to the general theory [13], is the region where the spin-wave excitation becomes important. The effect of the existence of spin waves therefore is that, when they are essential, the Casimir force is of the same order of magnitude as in the finite-size critical region. When the field is strong enough to suppress the spin-wave excitations, i.e., when $hL^2 \gg 1$, $T < T_c$ [then $x_2^2 \gg x_1 \gg 1$, see case (a) above], the system becomes Ising-like and the excess free energy and the Casimir force tend to zero exponentially fast in L.

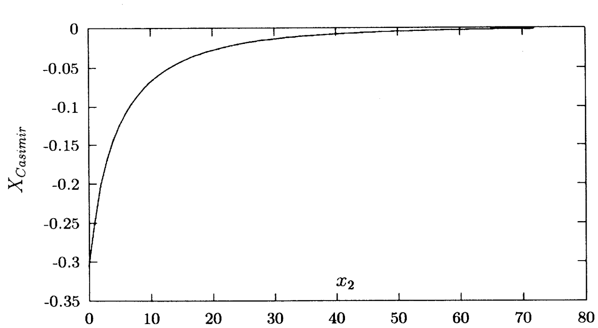

FIG. 3. The universal finite-size scaling function of the Casimir (solvation) force X_{Casimir} as a function of $x_2 = K_c^{-1/2} h L^{5/2}$ for $K = K_c$.

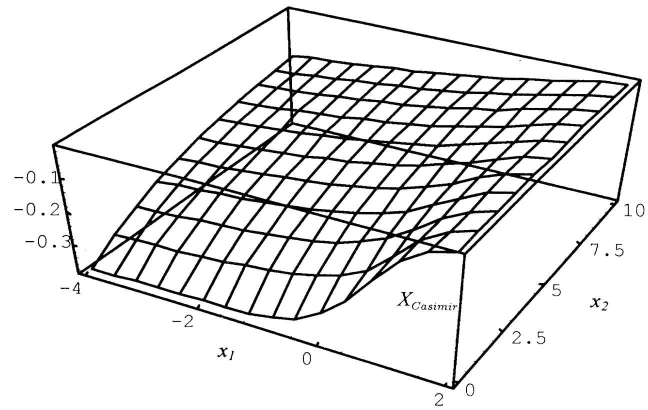

FIG. 4. The universal finite-size scaling function of the Casimir (solvation) force X_{Casimir} as a function of $x_1 = L(K_c - K)/K_c$ and $x_2 = K_c^{-1/2} h L^{5/2}$.

IV. CONCLUDING REMARKS

In the present paper the behavior of the Casimir (solvation) force has been investigated in the framework of the three-dimensional mean-spherical model under periodic boundary conditions and with film geometry, in the presence of an external field. (As is well known, the infinite translational invariant spherical model is equivalent to the $n \to \infty$ limit of the corresponding n-component system.) The obtained results are in full agreement with the predictions of the finite-size scaling theory. First, Eqs. (29) and (21)–(27) give the *universal* finite-size scaling function of the Casimir force. The force is negative in the whole region of the thermodynamic parameters. Next, it should be emphasized that in contrast to the Ising-like case the excess free energy, and, therefore, the Casimir force in the absence of an external field tend to zero below T_c not in an exponential in L way. For example, the finite-size scaling function of the excess free energy and of the Casimir force tend to a constant below T_c [see Eq. (31)]. The explanation of this behavior, which, we believe, is common for all $O(n)$, $n \geq 2$ models, is based on the fact that due to the existence of soft modes in the system (spin waves) below T_c and in the absence of an external field ($h = 0$) ξ_b is identically infinite. If an external field is applied ($h \neq 0$) then $\xi_b < \infty$, and, of course, we expect that $f^{\text{ex}} \to 0$ again exponentially fast in L. For the considered system this is really so, as is demonstrated in the article. Strictly speaking this happens if $hL^2 \gg 1$

[$x_2^2 \gg x_1$; see case (a) above], when the magnetic field is strong enough in order to suppress the effect of the spin-wave excitations. If the spin waves are essential [i.e., when $hL^2 = O(1)$ and $T < T_c$] the Casimir force is of the same order as in the finite-size critical region. Further, we have evaluated numerically the universal Casimir amplitude $\Delta_{\rm per} = -0.153$ [see Eq. (32)]. For comparison we give the corresponding result for the Ising universality class $\Delta_{\rm per} = -0.11$ obtained by the ϵ-expansion (up to the first order in ϵ) technique. In some sense the best available alternative estimation of this amplitude is obtained by some interpolation procedure [4], using the data from the ε expansion, but also the exact values for $d = 2$ and $d = 4$. In this way one obtains $\Delta_{\rm per} = -0.15$, which is surprisingly close enough to the value reported above for the spherical model. Finally, we would like to mention that it is worthwhile to obtain within the spherical model the finite-size scaling functions of the Casimir force (and the corresponding Casimir amplitudes) under other, e.g., antiperiodic, boundary conditions also. We note that for boundary conditions that are not identical at both confining the system surface planes the Casimir force is expected to be *positive* in the whole region of the thermodynamic parameters. We hope to return to this problem later.

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