

Concept of complexity in random dynamical systems

Vittorio Loreto,¹ Giovanni Paladin,² and Angelo Vulpiani¹

¹*Dipartimento di Fisica, Università di Roma "La Sapienza," Piazzale Aldo Moro 2, I-00185 Roma, Italy*

²*Dipartimento di Fisica, Università dell'Aquila, Via Vetoio I-67100 Coppito, L'Aquila, Italy*

(Received 23 May 1995)

We introduce a measure of complexity in terms of the average number of bits per time unit necessary to specify the sequence generated by the system. In dynamical systems with small random perturbations, this indicator coincides with the rate K of divergence of nearby trajectories evolving under two different noise realizations. The meaning of K is discussed in the context of the information theory, and it is shown that it can be determined from real experimental data. In the presence of strong dynamical intermittency, the value of K is very different from the standard Lyapunov exponent λ_σ computed considering two nearby trajectories evolving under the same realization of the randomness. However, the former is much more relevant than the latter from a physical point of view, as illustrated by some numerical computations for noisy maps and sandpile models.

PACS number(s): 05.45.+b

I. INTRODUCTION

In deterministic dynamical systems there exist well established ways to define the complexity of a temporal evolution in terms of Lyapunov exponents and Kolmogorov-Sinai entropy. However, the situation becomes much more ambiguous in the presence of a random perturbation, or more in general a randomness, which are always present in physical systems as a consequence of thermal fluctuations or uncontrollable changes of control parameters and in numerical experiments because of the roundoff errors [1].

In the literature, a first rough conclusion is that the presence of a small noise does not change the qualitative behavior of the dynamics [2]. In the case of a regular (stable) system, the random perturbation just changes the very long time behavior by introducing the possibility of jumps among different attractors (stable fixed points, stable limit cycles, or tori). A familiar example is the Langevin equation describing the motion of an overdamped particle in a double well.

Even in the opposite limit of chaotic dissipative systems, the presence of noise is expected not to change the qualitative behavior in a dramatic way. The typical situation is the following: (a) the strange attractor maintains the fractal structure at larger scales, although it is smoothed at small scale $O(\sigma)$, if σ is the strength of the noise; (b) the value of Lyapunov exponents differs from the unperturbed one of a quantity $O(\sigma)$.

However, the combined effects of the noise and of the deterministic part of the evolution law can produce highly nontrivial, and often intriguing, behaviors [3–8]. Let us mention the stochastic resonance where there is a synchronization of the jumps between two stable points [9–12] and the phenomena of the so-called noise-induced order [7] and of the noise-induced stability [5,6].

In our opinion one of the main problems is the lack of a well-defined method to characterize the “complexity” of the trajectories. Usually [2,5,7], the degree of chaoticity

is measured by treating the random term as a usual time-dependent term, and therefore considering the separation of two nearby trajectories with the same realization of the noise. In this way it is possible to compute the maximum Lyapunov exponent λ_σ associated to the separation rate of two nearby trajectories with the same realization of the stochastic term.

Some authors thus argue that there exists a phenomenon of noise-induced order [7], when at increasing the strength σ of the random perturbation, λ_σ passes from positive to negative. Even the opposite phenomenon (noise-induced instability) has been observed: at increasing σ , λ_σ can pass from negative to positive [5,6].

Although the Lyapunov exponent λ_σ is a well-defined quantity, it is neither unique nor the most useful characterization of complexity. In addition, a moment of reflection shows that it is practically impossible to extract λ_σ from experimental data.

In this paper we introduce a more natural indicator of complexity in random dynamical systems computing the separation rate of nearby trajectories evolving in two different realizations of the noise, instead of only one. Let us stress that such a procedure exactly corresponds to what happens when experimental data are analyzed by the algorithm of Wolf *et al.* [13]. Basically, our measure of complexity is related to the average number per time unit of bits necessary to specify the sequence generated by a random evolution law. Our proposal is obviously inspired by the pioneering contribution of Shannon [14].

We stress the fact that even in purely deterministic systems, if many characteristic times are involved, e.g., turbulence, the maximum Lyapunov exponent can have a very limited physical relevance. In particular, the predictability time can be much larger than λ^{-1} [15].

The outline of the paper is the following. In Sec. II we introduce the simplest way to treat the randomness by discussing two specific examples: the Langevin equation describing the motion of an overdamped particle in a double well and the case of the so-called stochastic res-

onance. These two examples provide a clear evidence of the limitations that arise when the Lyapunov exponent is computed by treating the noise term as a usual time-dependent term as well as of the necessity of a better characterization of the complexity of “noisy” systems.

Sections III and IV are devoted to the definition of an appropriate indicator of complexity, respectively, for dynamical systems with noise and random dynamical systems. For this last case, where the randomness is not simply given by an additive noise, we discuss two examples of systems which can be described by random maps: a two block earthquake model [16] and sandpile models [17], an interesting example of self-organized criticality [18].

The basic features of random maps are discussed in a one-dimensional map, which exhibits interesting behaviors, like the so-called *on-off intermittency* [19].

Section V discusses in detail the case of sandpile models with respect to the definition of complexity and to the predictability problem. In Sec. VI we discuss the results and we draw the conclusions.

II. NAIVE APPROACH: NOISE TREATED AS A STANDARD FUNCTION OF TIME

The simplest information about the chaoticity of noisy systems can be obtained by treating the random term as a usual time-dependent term, and therefore considering the separation of two nearby trajectories with the same realization of the noise. Such a characterization can be misleading, as illustrated in the following example.

A. Langevin equation

Let us consider the one-dimensional (1D) Langevin equation

$$\frac{dx}{dt} = -\frac{\partial V(x)}{\partial x} + \sqrt{2\sigma}\eta, \quad (1)$$

where $V(x)$ diverges for $|x| \rightarrow \infty$ and it has more than one minimum, e.g., the usual double well $V = -x^2/2 + x^4/4$, and $\eta(t)$ is a white noise.

The Lyapunov exponent λ_σ associated to the separation rate of two nearby trajectories with the same realization of the stochastic term $\eta(t)$, is

$$\lambda_\sigma = \lim_{t \rightarrow \infty} \frac{1}{t} \ln |z(t)|, \quad (2)$$

where the evolution of the tangent vector [that should be regarded as an infinitesimal perturbation of the trajectory $x(t)$] is

$$\frac{dz}{dt} = -\frac{\partial^2 V(x(t))}{\partial x^2} z(t). \quad (3)$$

Since the system is ergodic with invariant probability distribution $P(x) = Ce^{-V(x)/\sigma}$, one has

$$\begin{aligned} \lambda_\sigma &= \lim_{t \rightarrow \infty} \frac{1}{t} \ln |z(t)| \\ &= -\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \partial_{xx}^2 V(x(t')) dt' \\ &= -C \int \partial_{xx}^2 V(x) e^{-V(x)/\sigma} dx \\ &= -\frac{C}{\sigma} \int (\partial_x V(x))^2 e^{-V(x)/\sigma} dx < 0. \end{aligned} \quad (4)$$

This result is rather intuitive: the trajectory $x(t)$ spends most of the time in one of the “valleys” where $-\partial_{xx}^2 V(x) < 0$ and only short periods on the “hills” where $-\partial_{xx}^2 V(x) > 0$, so that there is a decreasing of the average of the logarithm of the distance between two trajectories evolving in the same noise realization. Let us remark that in Ref. [20], using a wrong argument, an opposite result is claimed.

As matter of fact, $\lambda_\sigma < 0$ implies a fully predictable process *only if* the realization of the noise is known. In the more sensible case of two initially close trajectories evolving in two different noise realizations, after a certain time T_σ , the two trajectories will be very distant since they will be in two different “valleys.” For $\sigma \rightarrow 0$, by the Kramer formula, one has $T_\sigma \sim \exp \Delta V/\sigma$, where ΔV is the difference between the values of V on the top of the hill and on the bottom of the valley.

The result obtained for the one-dimensional Langevin equation can be easily generalized at any dimension for gradient systems if the noise is small enough.

Let us consider the system

$$\dot{x}_i = -\frac{\partial V}{\partial x_i} + \sqrt{2\sigma}n_i, \quad (5)$$

where $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{i,j}\delta(t-t')$. Denoting with $R^2 = \sum |z_i|^2$ one has

$$\begin{aligned} \frac{1}{2} \frac{dR^2(t)}{dt} &= -\sum_{i,j} z_i \frac{\partial^2 V}{\partial x_i \partial x_j} z_j \\ &= -(z(t), \hat{A}(t)z(t)) \leq -l(x(t))R^2(t), \end{aligned} \quad (6)$$

where $l(x)$ is the minimum eigenvalue of the matrix \hat{A} whose elements are

$$A_{i,j} = \frac{\partial^2 V}{\partial x_i \partial x_j}. \quad (7)$$

For the Lyapunov exponent λ_σ one obtains

$$\begin{aligned} \lambda_\sigma &= \lim_{t \rightarrow \infty} \frac{1}{2t} \ln \frac{R^2(t)}{R^2(0)} \leq -\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t l(x(t')) dt' \\ &= -\frac{1}{C} \int l(x) \exp^{-V/\sigma} dx. \end{aligned} \quad (8)$$

Since $l(x)$ is positive around the minimum it follows that $\lambda_\sigma < 0$ for small values of σ . From a more rigorous discussion see [21].

B. Stochastic resonance with and without noise

Let us now discuss a deterministic system close to the onset of chaos when the control parameter varies periodi-

cally in time. We consider the set of differential equations which is a slight modification of the Lorenz model [22],

$$\begin{aligned} dx/dt &= 10(y - x) , \\ dy/dt &= -xz + R(t)x - y , \\ dz/dt &= xy - \frac{8}{3}z , \end{aligned} \quad (9)$$

where the control parameter has a periodic time variation:

$$R(t) = R_0 - A \cos(2\pi t/T) . \quad (10)$$

In our case, the periodic variations of R roughly mimic the seasonal changing on the solar heat inputs.

An interesting situation is when the average Rayleigh number R_0 is assumed to be close to the threshold $R_{cr} = 24.74$ for the transition from stable fixed points to a chaotic attractor in the standard Lorenz model. The value of the amplitude A of the periodic forcing should be such that $R(t)$ oscillates below and above R_{cr} . For very large T , a good approximation of the solution is given by

$$x(t) = y(t) = \pm \sqrt{\frac{8}{3}[R(t) - 1]}z(t) = R(t) - 1 , \quad (11)$$

which is obtained by the fixed points of the standard Lorenz model by replacing R by $R(t)$. The stability of this solution is a rather complicated issue, which depends on the values of R_0 , A , and T .

If R_0 is larger than R_{cr} the solution is unstable. In this case, for A large enough (at least $R_0 - A < R_{cr}$) one observes a mechanism similar to that of the stochastic resonance in bistable systems with random forcing. The value of T is crucial: for large T the system behaves as follows. Let us introduce

$$T_n \simeq nT/2 - T/4 , \quad (12)$$

the times at which $R(t) = R_{cr}$.

For $0 < t < T_1$, the control parameter $R(t)$ is smaller than R_{cr} so that the system is stable and the trajectory is close to one of the two solutions [Eq. (9)]. For $T_1 < t < T_2$, one has $R(t) > R_{cr}$ and both solutions [Eq. (9)] are unstable so that the trajectory in a short time relaxes toward a sort of ‘‘adiabatic’’ chaotic attractor. The chaotic attractor smoothly changes at varying R above the threshold R_{cr} , but if T is large enough, this dependence can be neglected in a first approximation. However, when $R(t)$ becomes again smaller than R_{cr} , the adiabatic attractor disappears and, in general, the system is far from the stable solutions [Eq. (9)]. But, since they are attracting, the system relaxes toward them; see Fig. 1(a).

For a detailed analysis of this behavior see Ref. [23].

It is worth stressing that the system is chaotic, i.e., the first Lyapunov exponent is positive, although the correlation function of the variable z does not decay as a consequence of strong correlation between the regular intervals.

Let us now discuss the effect of a random forcing, of strength σ , in the case where $R(t) - R_{cr}$ changes sign during the time evolution but the solutions [Eq. (9)], in the

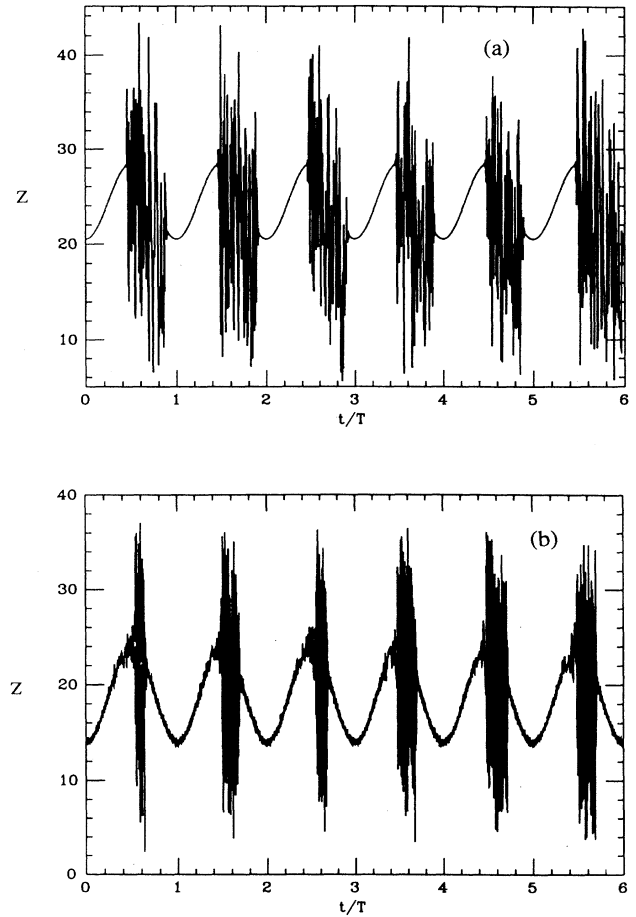


FIG. 1. (a) $z(t)$ vs t/T for the system (9): $R_0 = 25.5$, $A = 4$, and $T = 1600$. (b) $z(t)$ vs t/T for the system (13): $R_0 = 20.0$, $A = 5$, $T = 1600$, and $\sqrt{2\sigma} = 0.15$.

absence of the noise, are stable. In practice, we consider the Langevin equation

$$\begin{aligned} dx/dt &= 10 - (y - x) + \sqrt{2\sigma}\eta_1 , \\ dy/dt &= -xz + R(t)x - y + \sqrt{2\sigma}\eta_2 , \\ dz/dt &= xy - \frac{8}{3}z + \sqrt{2\sigma}\eta_3 , \end{aligned} \quad (13)$$

where $\eta_i(t)$ are uncorrelated white noises, i.e., $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\delta(t - t')$.

The numerical study of the model reveals a phenomenon very close to the original stochastic resonance. For small values of σ one has the same qualitative behavior obtained at $\sigma = 0$, while for σ slightly larger than a critical value σ_{cr} one has an alternation of regular and irregular motions; see Fig. 1(b). Now the Lyapunov exponent, computed treating the noise as an usual time-dependent term, is negative, i.e., two trajectories, initially close, with the same realization of the random forcing do not separate but stick exponentially fast.

Figures 1(a) and 1(b) show in a clear way how, if noise is involved, one can obtain simulations rather close either

in the case of a positive Lyapunov exponent [Fig. 1(a)] or with a negative Lyapunov exponent [Fig. 1(b)].

The two above examples show the limitation of the Lyapunov exponent computed treating the noise term as a usual time-dependent term for the characterization of the “complexity” of noisy systems.

III. COMPLEXITY IN DYNAMICAL SYSTEMS WITH NOISE

The main difficulties to define the notion of complexity in an evolution law with a random perturbation already appear in 1D maps. In fact, the generalization to N -dimensional maps or to coupled ordinary differential equations is straightforward.

Let us therefore consider the model map

$$x(t+1) = f[x(t), t] + \sigma w(t), \quad (14)$$

where t is an integer and $w(t)$ is an uncorrelated random process, e.g., w are independent random variables uniformly distributed in $[-1, 1]$. The maximum Lyapunov exponent λ_σ defined in (2) is given by the map for the tangent vector:

$$z(t+1) = f'[x(t), t]z(t), \quad (15)$$

where $f' = df/dx$. At $\sigma = 0$, λ_0 is the Lyapunov exponent of the unperturbed map.

In order to introduce a more natural indicator of complexity in noisy dynamics it is convenient to follow a quite different approach, where two realizations of the noise, instead of only one, are used [24].

Before discussing our alternative definition of chaos in noisy systems, we must briefly recall what are the characterization of intermittency in deterministic dynamical systems. An effective Lyapunov exponent [25,26] has been introduced to measure the fluctuations of chaoticity

$$\gamma_t(\tau) = \frac{1}{\tau} \ln \frac{|z(t+\tau)|}{|z(t)|}. \quad (16)$$

It gives the local expansion rate in the interval $[t, t+\tau]$. The maximum Lyapunov exponent is thus given by a time average along the trajectory $x(t)$: $\lambda_0 = \langle \gamma_t \rangle$ for $\tau \rightarrow \infty$.

Let us define the new indicator of complexity in the framework of the deterministic map with no random perturbation where it coincides with λ_0 . Let $x(t)$ be the trajectory starting at $x(0)$ and $x'(t)$ be the trajectory starting at $x'(0) = x(0) + \delta x(0)$ with $\delta_0 = |\delta x(0)|$ and indicate by τ_1 the maximum time such that $|x'(t) - x(t)| < \Delta$. Then, we put $x'(\tau_1 + 1) = x(\tau_1 + 1) + \delta x(0)$ and define τ_2 as the maximum τ such that $|x'(\tau_1 + \tau) - x(\tau_1 + \tau)| < \Delta$, and so on. In our context, we can define the effective Lyapunov as

$$\gamma_i = \frac{1}{\tau_i} \ln \frac{\Delta}{\delta_0}. \quad (17)$$

However, we sample the expansion rate in a nonuniform

way, at time intervals τ_i . As a consequence the probability of picking γ_i is $p_i = \tau_i / \sum_i \tau_i$ so that

$$\lambda_0 = \langle \gamma_i \rangle = \frac{\sum_i \tau_i \gamma_i}{\sum_i \tau_i} = \frac{1}{\bar{\tau}} \ln \left(\frac{\Delta}{\delta_0} \right), \quad (18)$$

$$\bar{\tau} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \tau_i.$$

This definition without any modification can be extended to noisy systems by introducing the rate

$$K_\sigma = \frac{1}{\bar{\tau}} \ln \left(\frac{\Delta}{\delta_0} \right), \quad (19)$$

which coincides with λ_0 for a deterministic system ($\sigma = 0$). When $\sigma = 0$ there is no reason to determine the Lyapunov exponent in this apparently odd way, of course. However, the introduction of K_σ is rather natural in the framework of the information theory [27]. Considering again the noiseless situation, if one wants to transmit the sequence $x(t)$ ($t = 1, 2, \dots, T_{\max}$) accepting only errors smaller than a tolerance threshold Δ , one can use the following strategy:

(1) Transmit the rules which specify the dynamical system (1), using a finite number of bits which does not depend on the length T_{\max} .

(2) Specify the initial condition with precision δ_0 using a number of bits $n = \ln_2(\Delta/\delta_0)$ which permits us to arrive up to the time τ_1 where the error equals Δ . Then specify again the new initial condition $x(\tau_1 + 1)$ with a precision δ_0 , and so on. The number of bits necessary to specify the sequence with a tolerance Δ up to $T_{\max} = \sum_{i=1}^N \tau_i$ is $\simeq Nn$ and the mean information for time step is $\simeq Nn/T_{\max} = K_\sigma / \ln 2$ bits.

In the presence of noise, the strategy of the transmission is unchanged but since it is not possible to transmit the realization of the noise $w(t)$, one has to estimate the growth of the error $\delta x(t) = x'(t) - x(t)$, where $x(t)$ and $x'(t)$ evolve in two different noise realizations $w(t)$ and $w'(t)$, and $|\delta x(0)| = \delta_0$.

The resulting equation for the evolution of $\delta x(t)$ is

$$\delta x(t+1) \simeq f'[x(t), t] \delta x(t) + \sigma \tilde{w}(t), \quad (20)$$

$$\tilde{w}(t) = w'(t) - w(t).$$

For the sake of simplicity we discuss the case $|f'[x(i), i]| = \text{const} = \exp \lambda_0$, where (20) gives the bound on the error:

$$|\delta x(t)| < e^{\lambda_0 t} (\delta_0 + \tilde{\sigma}) \quad \text{with} \quad \tilde{\sigma} = \frac{2\sigma}{e^{\lambda_0} - 1}. \quad (21)$$

This formula shows that δ_0 and $\tau = \bar{\tau}$ are not independent variables, but are linked by the relation

$$e^{\lambda_0 \tau} (\delta_0 + \tilde{\sigma}) \simeq \Delta. \quad (22)$$

As a consequence, we have only one free parameter, say τ , to optimize the information entropy K_σ in (19), so that

the complexity of the noisy system can be estimated by

$$G_\sigma = \min_\tau K_\sigma = \lambda_0 + O(\sigma/\Delta), \quad (23)$$

where the minimum is reached at an optimal time $\tau = \tau_{\text{opt}}$ from the transmitter point of view.

In the case of a deterministic system K_σ does not depend on the value of τ (i.e., it is equivalent to use a long $\bar{\tau}$ and to transmit many bits a few times or a short $\bar{\tau}$ and to transmit a few bits many times). On the contrary, in noise systems there exists an optimal time τ_{opt} which minimizes K_σ : using relation (21) one sees that $\Delta = \exp(\lambda_0 \bar{\tau})(\delta_0 + \bar{\sigma})$ and K_σ has a minimum for $\tau_{\text{opt}} \simeq 1/\lambda_\sigma$. Let us stress that this corresponds to transmitting the initial condition with a precision $\delta_0 \simeq (\Delta - \bar{\sigma})/e$. For small noise $\delta_0 \gg \sigma$ and it is convenient to transmit rather often the initial condition with a small accuracy. This result might appear trivial but has a relevant consequence from a theoretical point of view in the presence of noise; even if the value of the entropy G_σ changes only $O(\sigma/\Delta)$, there exists an optimal time for the transmission.

An interesting situation happens for strong intermittency when there is an alternation of positive and negative γ during long time intervals. In this case the existence of an optimal time for the transmission induces a dramatic change for the value of G_σ . This becomes particularly clear when considering the limit case of positive γ_1 in an interval $T_1 \gg 1/|\gamma_1|$ followed by a negative γ_2 in an interval $T_2 \gg 1/|\gamma_2|$, and again a positive effective Lyapunov exponent, and so on. In the expanding intervals, one can transmit the sequence using $\simeq T_1/(\gamma_1 \ln 2)$ bits, while during the contracting interval one can use only a few bits. Since in the expanding intervals the transmission has to be repeated rather often and moreover $|\delta x|$ cannot be lower than the noise amplitude σ , contrary to the noiseless case, it is impossible to use the contracting intervals to compensate the expanding ones. This implies that in the limit of very large T_i only the expanding intervals contribute to the evolution of the error $\delta x(t)$ and the information entropy is given by an average of the positive effective Lyapunov exponents:

$$G_\sigma \simeq \langle \gamma \theta(\gamma) \rangle. \quad (24)$$

For the approximation considered above, $G_\sigma \geq \lambda_\sigma = \langle \gamma \rangle$. Note that by definition $G_\sigma \geq 0$ while λ_σ can be negative. The estimate (24) stems from the fact that δ_0 cannot be smaller than σ so the typical value of τ_i is $O(1/|\gamma_i|)$ if γ_i is positive. We stress again that (24) holds only for strong intermittency, while for uniformly expanding systems or rapid alternations of contracting and expanding behaviors $G_\sigma \simeq \lambda_\sigma$.

It is not difficult to estimate the range of validity of the two limit cases $G_\sigma \simeq \lambda_\sigma$ and $G_\sigma \simeq \langle \gamma \theta(\gamma) \rangle$. Denoting by $\gamma_+ > 0$ and $\gamma_- < 0$ the typical values of the effective Lyapunov exponent in the expanding and contracting time intervals of length T_+ and T_- , respectively, (24) holds if during the expanding intervals there are at least two repetitions of the transmission and the duration of the contracting interval is long enough to allow

the noise to be dominant with respect to the contracting deterministic effects. In practice one should require

$$\exp(\gamma_+ T_+) \gg \frac{\Delta}{\sigma}, \quad \exp(|\gamma_-| T_-) \gg \frac{\Delta}{\sigma}. \quad (25)$$

In a similar way, $K \simeq \lambda_0$ holds if

$$\exp(|\gamma_-| T_-) \ll \frac{\Delta}{\sigma}. \quad (26)$$

We report the results of some numerical simulations in two different systems, which are shown in Figs. 2 and 3, respectively. Let us stress that we have directly computed K_σ , and since $\tau_i = O(1/|\gamma_i|)$, we automatically are very close to the optimal strategy so that $K_\sigma \simeq G_\sigma$, without performing a minimization. The random perturbation $w(t)$ is an independent variable uniformly distributed in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

The first system is given by periodic alternation of two piecewise linear maps of the interval $[0,1]$ into itself:

$$f[x, t] = \begin{cases} ax \bmod 1 & \text{if } (2n-1)T \leq t < 2nT \\ bx & \text{if } 2nT \leq t < (2n+1)T, \end{cases} \quad (27)$$

where $a > 1$ and $b < 1$. Note that in the limit of small T , $G_\sigma \rightarrow \max[\lambda_\sigma, 0]$ since it is a non-negative quantity as shown in Fig. 2, where for $b = \frac{1}{4}$, λ_σ is negative.

The second system is strongly intermittent without an external forcing. It is the Beluzov-Zhabotinsky map [4,7] related to a famous chemical reaction:

$$f(x) = \begin{cases} [(\frac{1}{8} - x)^{1/3} + a]e^{-x} + b & \text{if } 0 \leq x < \frac{1}{8} \\ [(\frac{1}{8} - x)^{1/3} + a]e^{-x} + b & \text{if } \frac{1}{8} \leq x < \frac{3}{10} \\ c(10xe^{-10x/3})^{19} + b & \text{if } \frac{3}{10} \leq x \end{cases} \quad (28)$$

with $a = 0.50607357$, $b = 0.0232885279$, $c = 0.121205692$. The map exhibits a chaotic alternation of expanding and very contracting time intervals. Although the value of T_- is very small because $|\gamma_-| \gg 1$, the first inequality (25) is unsatisfied because the expand-

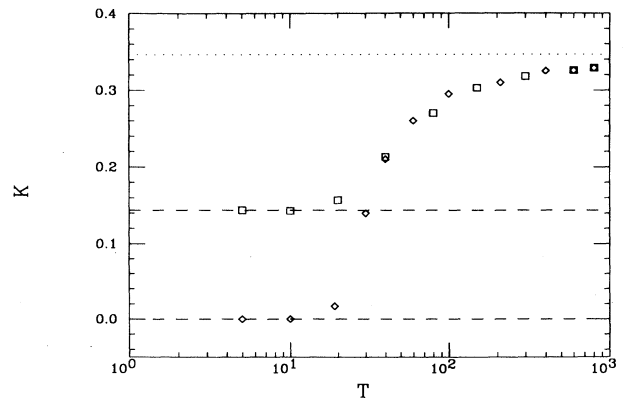


FIG. 2. K_σ versus T with $\sigma = 10^{-7}$ for the map (27). The parameters of map (27) are $a = 2$ and $b = \frac{2}{3}$ (squares) or $b = \frac{1}{4}$ (diamonds). The dotted line indicates the Pesin-like relation (24) while the dashed lines are the noiseless limit of K_σ . Note that for $b = \frac{1}{4}$ the Lyapunov exponent λ_σ is negative.

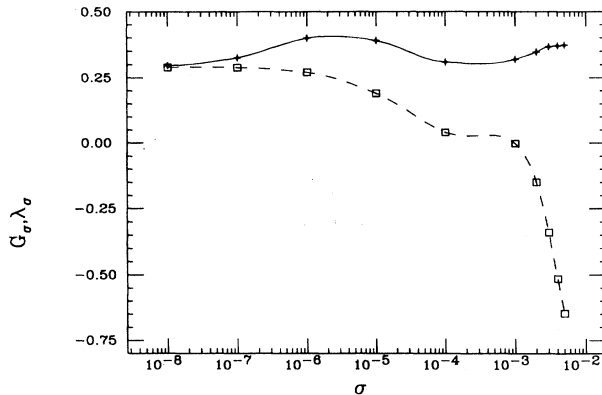


FIG. 3. λ_σ (squares) and G_σ (crosses) versus σ for map (28).

ing time intervals are rather short. As a consequence the asymptotic estimate $G_\sigma \simeq \langle \gamma \theta(\gamma) \rangle$ cannot be observed. In Fig. 3, one sees that while λ passes from negative to positive values at decreasing σ , G_σ is not sensitive to this transition to “order.” Another important remark is that in the usual treatment of the experimental data, if some noise is present, one practically computes G_σ and the result can be completely different from λ_σ . Let us mention, for example, [6], where the author studies a one-dimensional nonlinear time-dependent Langevin equation. A numerical computation shows that λ_σ is negative while the author claims to find, using the Wolf method, a positive “Lyapunov exponent.”

Our results show that the same system can be regarded either as regular (i.e., $\lambda_\sigma < 0$), when the same noise realization is considered for two nearby trajectories, or as chaotic (i.e., $G_\sigma > 0$), when two different noise realizations are considered. The situation is similar to what is observed in fluids with Lagrangian chaos [28]. There, a pair of particles passively advected by a chaotic velocity field might remain closed following together a “complex” trajectory. The Lagrangian Lyapunov exponent is thus zero. However, a data analysis gives a positive Lyapunov exponent because of the “Eulerian” chaos. We can say that λ_σ and G_σ correspond to the Lagrangian Lyapunov exponent and to the exponential rate of separation of a particle pair in two slightly different velocity fields, respectively.

The relation $G_\sigma \simeq \langle \gamma \theta(\gamma) \rangle$ is, in some sense, the time analogue to the Pesin relation $h \simeq \sum_i \lambda_i \theta(\lambda_i)$ between the Kolmogorov-Sinai entropy h and the Lyapunov spectrum [29], where the negative Lyapunov exponents do not decrease the value of h since the contraction along the corresponding directions cannot be observed for any finite space partition. In the same way the contracting time intervals, if long enough, do not decrease G_σ .

It is important to note that the limit $\sigma \rightarrow 0$ is very delicate. Indeed, for small σ , say $\sigma < \sigma_c$, the inequality (26) will be fulfilled and $G_\sigma \simeq \lambda_\sigma \rightarrow \lambda_0$ for $\sigma \rightarrow 0$. However, in strongly intermittent systems T_- can be very long so that the noiseless limit $G_\sigma \rightarrow \lambda_0$ is practically unreachable, as illustrated by Fig. 3.

IV. COMPLEXITY IN RANDOM DYNAMICAL SYSTEMS

In this section we discuss dynamical systems (mainly maps) where the randomness is not simply given by an additive noise, as in Sec. III. This kind of system has been the subject of much interest in the past few years in relation to problems involving disorder [30,31], the characterization of the so-called *on-off intermittency* [19], and the modeling of transport problems in turbulent flows [32]. In these systems, in general, the random part represents an ensemble of hidden variables, that is, unknown observables, believed to be implicated in the dynamics: the turbulent convection in the solar cycle or several economic factors for the stock market prices are just two examples of this situation. The random part can also mimic the effect of a set of variables that vary in a chaotic way or that vary on a time scale very small with respect to the time scale of the phenomenon under investigation. Random maps exhibit very interesting features ranging from stable or quasistable behaviors to chaotic behaviors and intermittency. In particular, on-off intermittency is an aperiodic switching between static, or laminar, behavior and chaotic bursts of oscillation. It can be generated by systems having an unstable invariant manifold, within which is possible to find a suitable attractor (i.e., a fixed point). The intermittency is linked, in the simplest case, to the loss of stability of the fixed point. For further details we refer to [19].

A random map can be defined in the following way. Denoting with $x(t)$ the state of the system at the discrete time t , the evolution law is given by

$$x(t+1) = F(x(t), I(t)), \quad (29)$$

where $I(t)$ is a random variable (RV). If the RV $I(t)$ is discrete with an entropy h_s , according to the general ideas discussed in Sec. III, a measure of the complexity of the evolution can be defined in terms of the mean number of bits that must be specified, at each iteration, in order to have a certain tolerance Δ on the knowledge of the state x .

Of course, it is possible to introduce a Lyapunov exponent λ_I , which is the analogue of λ_σ , computed considering the evolution of the tangent vector of Eq. (29) once given the realization $I(1), I(2), \dots, I(t)$ of the random process.

Therefore, there are two different contributions to the complexity: (a) one has to specify the sequence $I(1), I(2), \dots, I(t)$ which implies $h_s / \ln 2$ bits per iteration; (b) if λ_I is positive, one has to specify the initial condition $x(0)$ with a precision $\Delta \exp^{-\lambda_I T}$, where T is the time length of the evolution; it is necessary to give $\lambda_I \ln 2$ bits per iteration; if λ_I is negative the initial condition can be specified using a number of bits which does not depend on T .

Therefore, the complexity of the dynamics can be measured as

$$\tilde{K} = h_s + \lambda_I \theta(\lambda_I), \quad (30)$$

where θ is the Heaviside step function. Strictly speak-

ing, (30) holds just for the case with only one positive Lyapunov exponent. In general, it is possible to write an analogous expression just replacing λ_I with the Kolmogorov-Sinai entropy computed with the same realization of the randomness.

We stress again that a negative value of λ_I does *not* imply predictability.

A. Two examples of random maps

As a specific example, we discuss a random map obtained from the deterministic chaotic evolution of a system of two sliding blocks [33] (see also [34]) on a rough surface. Such a model provides a good description of the dynamics of two coupled large segments of a fault.

The equations of the motion for the position of the two blocks during a slip can be written as

$$\frac{d^2 Y_1}{dt} + Y_1 + \alpha(Y_1 - Y_2) = 1 / \left(1 + \gamma \left| \frac{dY_1}{dt} - \nu \right| \right), \quad (31)$$

$$\frac{d^2 Y_2}{dt} + Y_2 + \alpha(Y_2 - Y_1) = \beta / \left(1 + \gamma \left| \frac{dY_2}{dt} - \nu \right| \right),$$

while when one of the two blocks sticks, one has

$$\frac{d^2 Y_1}{dt} = 0, \quad \frac{dY_1}{dt} = \nu \quad (32)$$

or

$$\frac{d^2 Y_2}{dt} = 0, \quad \frac{dY_2}{dt} = \nu, \quad (33)$$

respectively, if $|Y_1 + \alpha(Y_1 - Y_2)| < 1$ or $|Y_2 + \alpha(Y_2 - Y_1)| < \beta$, where the Y_i are the rescaled displacements from the equilibrium position, and α and γ are related to the coupling constants and the friction dynamical coefficient. The quantity $T_\nu = \nu^{-1}$ is the natural (adimensional) time unit of the system. For details on the model see Ref. [16].

Although there is no randomness in the starting model one can obtain a random map in a new set of physically relevant variables. In the model (31), the seismic moment (proportional to the energy released) is the sum of the sliding runs during a single seismic event, that is

$$M_n = \sum_{i=1}^2 |Y_i^{(n+1)} - Y_i^{(n)}|, \quad (34)$$

where $Y_i(n)$ is the position of the i th block before the n th slip. As shown in Fig. 4, the map M_{n+1} versus M_n of the seismic moment computed at subsequent events is multivalued on the definition domain. This is a general feature which must be taken into account when analyzing realistic signals generated from dynamical systems exhibiting low-dimensional chaos.

Since some points have more than one image, an appropriate description of the dynamics is through a random map where a weight is assigned to each possible option.

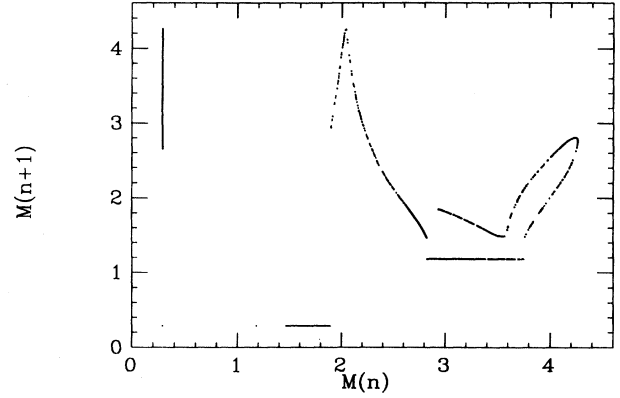


FIG. 4. Multivalued map of the seismic moments: M_{n+1} versus M_n generated by Eqs. (31) where $\beta = 2.0$, $\alpha = 1.2$, and $\gamma = 3.0$.

A good approximation of the deterministic evolution is obtained even considering the same weights for the two options.

Another example of a system which can be treated in the framework of random maps is represented by the so-called sandpile models [17]. These models represent an interesting example of self-organized criticality (SOC) [18,35–38]. This term refers to the tendency of large dynamical systems to evolve *spontaneously* toward a critical state characterized by spatial and temporal self-similarity. The original sandpile models are cellular automata inspired to the dynamics of avalanches in a pile of sand. Dropping sand slowly, grain by grain on a limited base, one reaches a situation where the pile is critical, i.e., it has a critical slope. That means that a further addition of sand will produce slidings of sand (avalanches) that can be small or cover the entire size of the system. In this case the critical state is characterized by scale-invariant distributions for the size and the lifetime and it is reached without the fine tuning of any critical parameter.

We will refer in particular to the so-called Zhang model [39], a continuous version of the original sandpile model (the BTW model) [17], defined on a d -dimensional lattice. The variable on each site E_i (interpretable as energy, sand, heat, mechanical stress, etc.) can vary continuously in the range $[0,1]$ with the threshold fixed to $E_c = 1$. The dynamics is the following:

(a) We choose a site in random way and we add to this site an energy δ (rational or irrational).

(b) If at a certain time t a site, say i , exceeds the threshold E_c , a relaxation process is triggered, defined as

$$E_{i+nn} \rightarrow E_{i+nn} + \frac{E_i}{2d}, \quad (35)$$

$$E_i \rightarrow 0,$$

where nn indicates the $2d$ nearest neighbors of the site i .

(c) We repeat point (b) until all the sites are relaxed.

(d) We go back to point (a).

We can also define a deterministic version of this model in which, at each addition time, we increase the variable of every site of a quantity δ and then follow the same rules as above, updating all the sites over threshold in a parallel way.

The dynamics of this model can be seen as described by a piecewise linear map [40]. In fact, indicating with $x \equiv \{x_i\}_{i \in D}$ the configuration of the system at a certain time, where $D \subset Z^d$ is the bounded domain whose cardinality is $|D| = N^d$ with N being the linear dimension of the lattice, the operator Δ_i corresponding to a toppling at site i is given by

$$(\Delta_i x)_j = x_j - \delta_{i,j} x_i + \frac{1}{2d} \sum_{i'}^* \delta_{i',j} x_{i'} , \quad (36)$$

where \sum^* means the sum over the nearest neighbors site of i .

Equation (36) shows that the single toppling is a linear operator and acts as a local Laplacian. The evolution of a configuration up to the time t can be written as [40]:

$$x(t) = T^t x(0) = L_{x,t} x(0) + \delta \sum_{s=1}^t L_{x,t-s+1} 1_{k(s)} , \quad (37)$$

where $L_{x,t}$ is a linear operator defined as a suitable product of linear operator Δ , $x(0)$ is the initial configuration and 1_i is a vector in R^D whose component i is 1 and all the others are 0. $k(s)$ defines the sequence of site over which there will be the random addition of energy at the time s . Equation (37) shows as the evolution of the Zhang model can be seen as the sequential application of maps, chosen, time by time, in a random way. Sandpile models, thus, belong to the wide class of the random maps.

B. Toy model: One-dimensional random maps

Let us discuss a random map which, in spite of its simplicity, captures some basic features of this kind of system:

$$x(t + 1) = a_t x(t) [1 - x(t)] , \quad (38)$$

where a_t is a random dichotomic variable given by

$$a_t = \begin{cases} 4 & \text{with probability } p \\ \frac{1}{2} & \text{with probability } 1 - p . \end{cases} \quad (39)$$

It is easy to understand the behavior for $x(t)$ close to zero. The solution of (38), keeping the linear part is

$$x(t) = \prod_{j=0}^{t-1} a_j x(0) . \quad (40)$$

The long-time behavior of $x(t)$ is given by the product $\prod_{j=0}^{t-1} a_j$. Using the law of large numbers one has that the typical behavior is

$$x(t) \sim x(0) e^{(\ln a)^t} . \quad (41)$$

Since $\langle \ln a \rangle = p \ln 4 + (1 - p) \ln \frac{1}{2} = (3p - 1) \ln 2$ one has that, for $p < p_c = \frac{1}{3}$, $x(t) \rightarrow 0$ for $t \rightarrow \infty$. On the contrary for $p > p_c$ after a certain time $x(t)$ is far from the fixed point zero and the nonlinear terms are relevant. Figure 5 shows a typical on-off intermittency behavior for p slightly larger than p_c .

Let us note that, in spite of this irregular behavior, numerical computations show that the Lyapunov exponent λ_I is negative for $p < \tilde{p}_c \simeq 0.5$: this is essentially due to the nonlinear terms. For such a system with on-off inter-

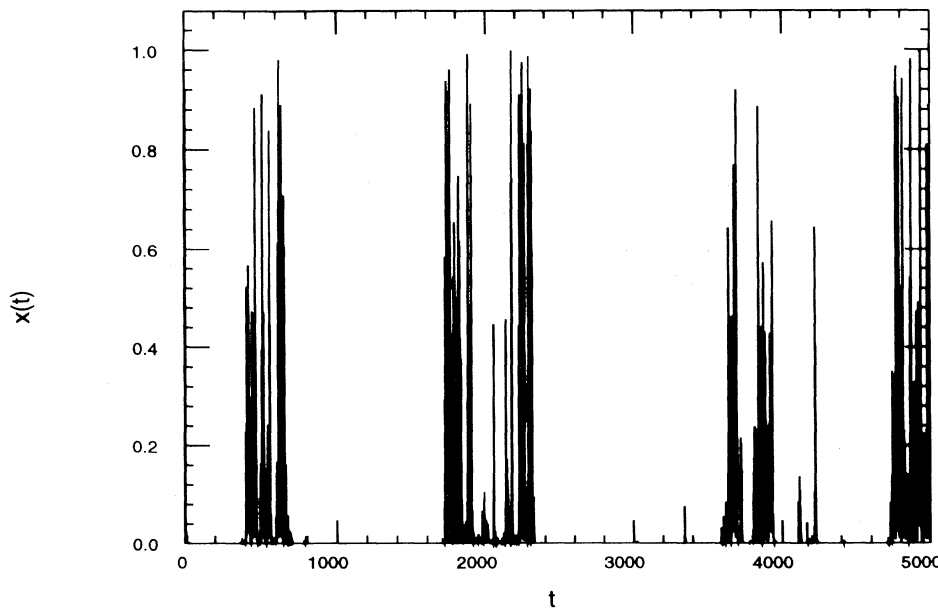


FIG. 5. $x(t)$ vs t for the random map (38) and (39) with $p = 0.35$.

mittency it is possible, in practice, to define a complexity of the sequence which turns out to be much smaller than the value given by the general formula (30).

Let us denote with l_L and l_I the average lifetimes, respectively, of the laminar and of the intermittent phases for p close to p_c ($l_I \ll l_L$). It is easy to realize that the mean number of bits per iteration one has to specify in order to transmit the sequence is

$$\tilde{K} \simeq \frac{l_I h_s}{(l_I + l_L) \ln 2} \simeq \frac{l_I}{l_L} \frac{h_s}{\ln 2}. \quad (42)$$

The previous formula is obtained noting that on an interval T one has $\simeq T/(l_I + l_L)$ intermittent bursts. Since during the intermittent bursts, i.e., $x(t)$ far from zero, there is not an exponential growth of the distance between two trajectories initially close and computed with the same sequence of a_t . So, one has just to give the sequence of a_t on the intermittent bursts. Equation (42) has an intuitive interpretation: in systems with a sort of “catastrophic” events, the most important feature is the mean time between two subsequent events.

V. CASE OF SANDPILE MODELS

In this section we discuss the problem of the predictability in sandpile models [17]. Different authors [41] suggested that self-organized critical systems occupy a particular position among the dynamical systems, which has been named weak chaos. This is because it has been argued that the maximum Lyapunov exponent of these systems is zero. From this it is deduced that two initially close trajectories in the phase space will diverge just algebraically in time and not in an exponential way, as do the chaotic systems. From this point of view these systems would seem more predictable than chaotic systems in that a better knowledge of the initial conditions would considerably improve the predictability time T_p ,

$$T_p \simeq (\Delta_{\max}/\delta_0)^\alpha, \quad (43)$$

where δ_0 is the error on the determination of the initial conditions, Δ_{\max} is the maximum tolerance between the real evolution and the simulation that makes any prediction, and α is just the exponent of the algebraic divergence of the error. We recall that for chaotic systems the predictability time is given by

$$T_p \simeq 1/\lambda \ln \left(\frac{\Delta_{\max}}{\delta_0} \right). \quad (44)$$

In this case an improvement in δ_0 would increase T_p only in a logarithmic way.

In this context we would like to discuss this problem on the basis of some recent rigorous results [40] in order to clarify the role of the Lyapunov exponents for this class of systems and to address the problem of the predictability. We will refer to the Zhang model defined in Sec. IV A.

The evolution of a configuration up to the time t is given by (37), which shows that the evolution of the

Zhang model can be seen as the sequential application of maps, chosen, time by time, in a random way.

The Lyapunov exponent corresponding to a given trajectory $x(t) = T^t x$ can be defined, linearizing the dynamics in the neighborhood of $x(t)$, as

$$\lambda \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{|z(t)|}{|z(0)|}, \quad (45)$$

where $z(t)$ represents the distance between two different configurations x and y at the time t . For example, in the L^1 norm $z(t) = \sum_i |y_i(t) - x_i(t)|$, with $i = 1, N^d$. If the two trajectories $x(t)$ and $y(t)$ make the same sequence of toppling, Eq. (45) holds with the substitution $y - x \rightarrow z$. In fact, in this case, $T^t y - T^t x = L_t z(0) = z(t)$ holds. Therefore the definition (45) for the Lyapunov exponent fails when the two configurations begin to follow different sequences of toppling. It is easy to see that such a situation occurs when, for one configuration, it holds that $x_i(t) = 1$ for some i and t . In this case a small difference in the second configuration $y_i(t) = x_i(t) + \epsilon$ will produce a toppling just in the y configuration. From this point onwards the two configurations will follow different sequences and the definition (45) fails definitely.

It is easy to see that the Lyapunov exponent is not positive. In fact, the dynamics in the tangent space, the dynamics of a small difference between two configurations, follows the same rules of the usual dynamics and the “error” is redistributed to the nearest neighbors site.

It is then clear that the distance between two configurations, being conserved in the toppling from the boundaries, can only decrease when a site of the boundary topples. We can conclude that $\lambda \leq 0$.

In [40] it has been obtained rigorously that, for the maximum Lyapunov exponent λ , as defined in (45), it holds that

$$\lambda \leq -\frac{1}{N^d [R(D) + 1]^2 (1/\delta + 1) (\ln N^d + 1)}, \quad (46)$$

where we indicate with $R(D)$ the diameter of the domain D , that is, that the Lyapunov exponent is strictly lower than 0.

An immediate consequence of the above result is that the dynamics, up to the time t (for t sufficiently large) is given by a *piecewise linear contractive map*.

At first, one could think that the existence of a negative Lyapunov exponent should assure a perfect predictability. That is not true. What makes the situation complex is the existence of a splitting mechanism in the configuration space which affects the so-called snapshot attractor. A snapshot attractor is obtained by considering a cloud of initial conditions and letting it evolve forward in time under a given realization of the noisy dynamics. We can identify two different mechanisms which concur to the formation of the snapshot attractor: (a) a volume contraction mechanism due to the effect of the negative Lyapunov exponent; (b) a splitting mechanism which tends, by virtue of the piecewise structure of the map, to map single sets of configurations in two or more distinct sets also far apart in phase space. The splitting mechanism

(b) tends to create a partition of the configuration space in regions which follow the same sequence of toppling, whereas mechanism (a) tends to contract the volumes of the elements of the partition.

It is worth it to stress how, in some cases, it happens that the evolution of all the possible configurations shrink to the evolution of a single configuration (a point in the configuration space) whose evolution corresponds, at each time, to a snapshot attractor given by just one point.

This happens, for example, in the case of a one-dimensional (linear) chain of N sites driven with an arbitrary δ . In [40] the case in which $\delta = \frac{1}{2}$ has been studied and it has been shown that in this case the partition is time independent. Let us discuss, for the sake of simplicity and without loss of generality, this last case. A certain cloud of configurations, i.e., belonging to the same element of the partition, will evolve in a cloud of configurations, in principle smaller than the initial one due to the contractivity of the map, belonging entirely to another element of the partition; in its turn this cloud will evolve in a smaller cloud of configurations belonging to another element of the partition, and so on. This process continues until all the configurations shrink to just one that continues to evolve jumping between different elements of the partition and evolving according to the map corresponding to each element of the partition. The Lyapunov exponent, in this case, gives information about the rate of shrinking of the different clouds of configurations, i.e., it gives the typical exponential contracting rate of the radius of the snapshot attractor.

The rigorous study of the properties of the snapshot attractors is out of the purposes of present work and it will be treated elsewhere. Here we just want to note how this situation does not change the problem of the predictability in that, in order to forecast the system, one should be able to know the random sequence which drives it.

This puts the problem of the definition of predictability in a wider perspective in which the Lyapunov exponent is not the only relevant quantity. Since the Lyapunov exponent gives information only at very large time and for infinitesimal perturbations the dynamical balance of the two effects (a) and (b) represents a basis for the definition of a predictability for such a system.

Let us consider initially the situation in which two different configurations are driven with the same realization of the noise: that means that at each time the sand (energy) is added to the same sites for the two configurations.

Up to the time in which two different configurations make the same sequence of toppling, the error ϵ (the distance between the two configurations) will decrease. When the configurations begin to follow a different sequence of toppling the error ϵ becomes of order 1 in a single time step, whatever ϵ was before this time. From this point onwards the evolution of the distance between the two configurations seems far from being linked to the Lyapunov exponent. The threshold mechanism, and then the splitting mechanism, plays therefore a crucial role in determining the predictability of such a system.

The system remains definitely predictable up to the time in which two different configurations make the same sequence of toppling. This time can be defined as the predictability time. For a more complete treatment of this point we refer to [42]. In particular, it is possible to show how a predictability for such a class of models can be related to a threshold mechanism, in which the Lyapunov exponent is not the only relevant quantity. The smaller is the initial distance between the two configurations, the smaller will be the probability of different toppling. The threshold in the initial distance between the two configurations, say ϵ_T , has, in this case, a probabilistic value. If $\epsilon(t=0) < \epsilon_T$ we cannot exclude the possibility that the configurations will follow different sequences of toppling, but the probability associated with this event becomes exponentially small as time goes on because, due to the negative Lyapunov exponent, two different configurations tend to converge to each other.

In order to confirm these predictions we simulated the parallel evolution of two different configurations in the random case (for a system with $N = 30$) with different starting error ϵ and we plotted the distance (in the L^1 norm) between the two orbits. Figures 6(a) and 6(b) show the results, respectively, for $\epsilon = 10^{-2}$ and 10^{-3} . These results seem to confirm the existence of a proba-

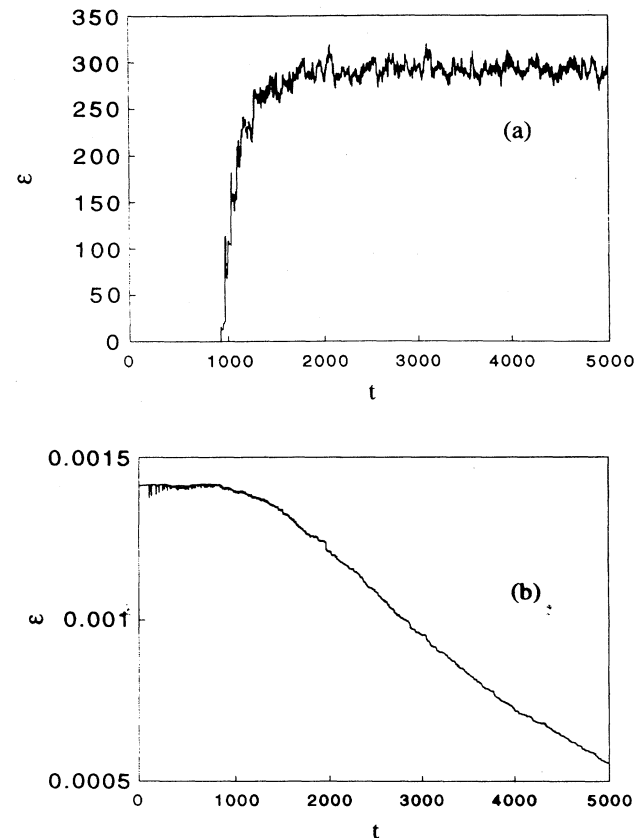


FIG. 6. Evolution of the distance ϵ between two configurations driven with the same realization of the randomness and with a starting distance of (a) 10^{-2} and (b) 10^{-3} . $N = 30$.

bilistic threshold in ϵ which determines the divergence or the asymptotic convergence of two orbits.

A. Predictability with different realizations of the randomness

It is very interesting to investigate what happens when one considers the case, more relevant from the point of view of the predictability, in which two configurations are driven by different randomness. That means that at each time the sand is added in different sites for the two realizations. Obviously we can imagine a situation in which the uncertainty in the knowledge of the noise can be varied. In fact, choosing a site for one configuration, we can drive the other configuration by putting the sand in a site which can be one of the nearest neighbors sites, or one of the second nearest neighbors sites, etc., of the site of the first configuration. In this way, due to the discrete structure of the system, the uncertainty cannot be reduced at will. The minimum uncertainty is obtained by putting the sand in one of the nearest neighbors of the site chosen for the first configuration. In our simulations we considered this last situation. The results are shown in Fig. 7. As it is possible to see, the situation in this case is much more involved and the threshold mechanism, described above for the case of the same realizations of the randomness, does not hold anymore. That is because, in this case, the two configurations can start to follow different sequences of toppling at the first toppling. From this point onwards we return to the situation in which, with the same realization of the noise, the two configurations start to follow different sequences of toppling; the system becomes unpredictable. Also in this case, the Lyapunov exponent does not play the role of the only relevant quantity.

In order to better explain this point it is possible to define the complexity K for such a kind of system. In this case we can use Eq. (30), where b_s defines the complexity relative to the choice of the random sequence of addition of energy. In sandpile models, for example, since

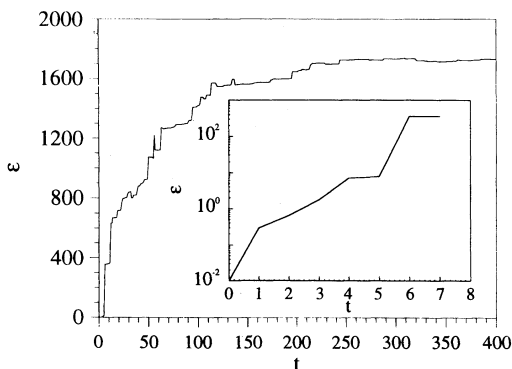


FIG. 7. Evolution of the distance ϵ between two initially identical configurations driven with different realizations of the randomness. $N = 30$.

each site has the same probability to be selected, one has $h_s = \ln N$, where N is the number of sites of the system and the second term does not exist in that the Lyapunov exponent is negative. We then obtain the result that the complexity for sandpile models is just determined by the randomness in the choice of the sequence of addition of energy; nevertheless, once this sequence is known, the system could be, all the same, unpredictable, at least for some initial conditions and noninfinitesimal perturbations, due to the splitting mechanism cited above. Once more, let us emphasize that a negative Lyapunov exponent implies predictability only if the sequence which drives the system is exactly known.

VI. CONCLUSIONS

In this paper we focus on the problem of an appropriate definition of the concept of complexity in random dynamical systems. At first, one could follow a naive approach where the randomness is considered as a standard time-dependent time. In this way, the Lyapunov exponent λ_σ is given by the rate of divergence of two initially close trajectories evolving under the same realization of the randomness.

Although well defined from a mathematical point of view, such an approach leads to paradoxical situations. For instance, in a system driven by the one-dimensional Lyapunov equation, the existence of a negative Lyapunov exponent does not imply the possibility to forecast the future state of the system unless one exactly knows the realization of the noise. Another paradox is represented by the situation discussed in Sec. III B, where two different systems, one with positive Lyapunov exponent, and the other with a negative one, appear practically indistinguishable. Last but not least, it is practically impossible to extract λ_σ from an analysis of experimental data.

The main result of the paper is the definition of a measure of complexity K in terms of the mean number of bits per time unit necessary to specify the sequence generated by the random evolution law. We have also shown that from a practical point of view, this definition corresponds to considering the divergence of nearby trajectories evolving in different noise realizations. The great advantage is that K can be extracted from experimental data [13]. The two indicators K and λ_σ have close values and are practically equivalent in systems with weak dynamical intermittency. However, in the presence of strong intermittency (say, irregular alternations of long regular periods with sudden chaotic bursts) K and λ_σ become very different and in extreme situations it may happen that K is positive while λ_σ negative. It is thus questionable whether such a system is chaotic or regular and to speak of noise induced order.

A special class of systems well described by our characterization is random maps, where at each time step different possible evolution laws are chosen according to a given probabilistic rule. Sandpile models form an important group of systems that can be described in terms of random maps. The existence of a negative Lyapunov exponent does not allow one to capture the basic features

of these spatially extended systems, while our measure of complexity is able to describe in an appropriate way the dynamical behavior.

It seems to us that the study of the complexity and of the predictability is completely understood only in the case of deterministic dynamical systems with few degrees of freedom. We have just mentioned [15] the case of systems with many characteristic times where the predictability time is not trivially related to the Lyapunov exponent. Our work may be a step toward a deeper comprehension of these issues in systems with many degrees of freedom or in interactions with many degrees of free-

dom represented by a noise, problems that are still open and sometimes controversial.

ACKNOWLEDGMENTS

It is a pleasure to thank E. Caglioti for useful suggestions and discussions. We also give thanks to R. Kapral and T. Tel for the useful correspondence. G.P. is grateful to NORDITA (Nordic Institute for Theoretical Physics) of Copenhagen for their kind hospitality during the completion of this paper.

-
- [1] F. Moss, P. W. E. McClintock, *Noise in Nonlinear Dynamical Systems* (Cambridge University Press, Cambridge, 1989), Vols. 1-3.
 - [2] J. P. Crutchfield, D. J. Farmer, and B. A. Huberman, *Phys. Rep.* **92**, 45 (1982).
 - [3] H. Herzel, W. Ebeling, and Th. Schulmeister, *Z. Naturforsch. A* **41**, 801 (1986).
 - [4] H. Herzel and B. Pompe, *Phys. Lett. A* **122**, 121 (1987).
 - [5] A. R. Bulsara, E. W. Jacobs, and W. C. Schieve, *Phys. Rev. A* **42**, 4614 (1990).
 - [6] Z. Y. Chen, *Phys. Rev. A* **42**, 5837 (1990).
 - [7] K. Matsumoto and I. Tsuda, *J. Stat. Phys.* **31**, 87 (1983).
 - [8] R. J. Deissler and D. J. Farmer, *Physica D* **55**, 155 (1992).
 - [9] R. Benzi, A. Sutera, and A. Vulpiani, *J. Phys. A* **14**, L453 (1981).
 - [10] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, *Tellus* **34**, 10 (1982).
 - [11] C. Nicolis, *Tellus* **34**, 1 (1982).
 - [12] For a general review on stochastic resonance see *J. Stat. Phys.* **70**, No. 1/2 (1993).
 - [13] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, *Physica D* **16**, 285 (1985).
 - [14] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communications* (University of Illinois Press, Urbana, 1949).
 - [15] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin, and A. Vulpiani, *Phys. Rev. E* **53**, 2337 (1996).
 - [16] G. Lacorata and G. Paladin, *J. Phys. A* **26**, 3463 (1993).
 - [17] P. Bak, C. Tang, and K. Wiesenfeld, *Phys. Rev. A* **38**, 364 (1988).
 - [18] P. Bak, C. Tang, and K. Wiesenfeld, *Phys. Rev. Lett.* **59**, 381 (1987).
 - [19] N. Platt, E. A. Spiegel, and C. Tresser, *Phys. Rev. Lett.* **70**, 279 (1993); J. F. Heagy, N. Platt, and S. M. Hammel, *Phys. Rev. E* **49**, 1140 (1994).
 - [20] C. van der Broeck and G. Nicolis, *Phys. Rev. E* **48**, 4845 (1993).
 - [21] F. Martinelli and E. Scoppola, *Commun. Math. Phys.* **120**, 25 (1988).
 - [22] E. D. Lorenz, *J. Atmos. Sci.* **20**, 130 (1963).
 - [23] A. Crisanti, M. Falcioni, G. Paladin, and A. Vulpiani, *J. Phys.* **27A**, L597 (1994).
 - [24] G. Paladin, M. Serva, and A. Vulpiani, *Phys. Rev. Lett.* **74**, 66 (1995).
 - [25] J. S. Nicolis, G. Mayer-Kress, and G. Haubs, *Z. Naturforsch. Teil A* **38**, 1157 (1983); B. Pompe, J. Kruscha, and R. W. Leven, *ibid.* **41**, 801 (1986); J. S. Nicolis, *Dynamics of Hierarchical Systems: An Evolutionary Approach* (Springer-Verlag, Berlin, 1986); R. Feistel and W. Eberling, *Evolution of Complex Systems: Self-Organization, Entropy and Development* (Kluwer Academic Press, Dordrecht, 1989).
 - [26] G. Paladin and A. Vulpiani, *Phys. Rep.* **156**, 147 (1987).
 - [27] For a physical discussion see J. Ford, *Phys. Today* **36**, (4), 40 (1983).
 - [28] A. Crisanti, M. Falcioni, G. Paladin, and A. Vulpiani, *Riv. Nuovo Cimento* **14**, 1 (1991).
 - [29] J. P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **57**, 617 (1985).
 - [30] C. Van Der Broeck and T. Tel, in *From Phase Transitions to Chaos: Topics in Modern Statistical Physics*, edited by G. Gyorgyi, I. Kondor, L. Sasvari, and T. Tel (World Scientific, Singapore, 1992), p. 227.
 - [31] R. Kapral and S. J. Fraser, *J. Stat. Phys.* **70**, 61 (1993).
 - [32] L. Yu, E. Ott, and Q. Chen, *Phys. Rev. Lett.* **65**, 2935 (1990); *Physica D* **53**, 102 (1993); S. Galluccio and A. Vulpiani, *Physica A* **212**, 75 (1994).
 - [33] J. Huang and D. L. Turcotte, *Geophys. Res. Lett.* **17**, 223 (1990); *Nature* **348**, 234 (1990); R. Burridge and L. Knopoff, *Bull. Seismol. Soc. Am.* **57**, 341 (1967).
 - [34] G. Narkounskaia, J. Huang, and D. L. Turcotte, *J. Stat. Phys.* **68**, 1151 (1992).
 - [35] T. A. Witten and L. M. Sander, *Phys. Rev. Lett.* **47**, 1400 (1981).
 - [36] L. Pietronero and H. J. Wiesmann, *J. Stat. Phys.* **36**, 811 (1984).
 - [37] P. Bak and K. Sneppen, *Phys. Rev. Lett.* **71**, 4083 (1993).
 - [38] D. Wilkinson and J. F. Willemsen, *Physica A* **16**, 3365 (1983).
 - [39] Y. C. Zhang, *Phys. Rev. Lett.* **63**, 470 (1987).
 - [40] E. Caglioti and V. Loreto (unpublished).
 - [41] K. Chen, P. Bak, and M. H. Jensen, *Phys. Lett. A* **149**, 207 (1990).
 - [42] E. Caglioti and V. Loreto, *Phys. Rev. E* **53** (to be published).