Nonlinear theory of oscillating, decaying, and collapsing solitons in the generalized nonlinear Schrödinger equation

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A nonlinear theory describing the long-term dynamics of *unstable solitons* in the generalized nonlinear Schrödinger (NLS) equation is proposed. An analytical model for the instability-induced evolution of the soliton parameters is derived in the framework of the perturbation theory, which is valid near the threshold of the soliton instability. As a particular example, we analyze solitons in the NLS-type equation with two power-law nonlinearities. For weakly subcritical perturbations, the analytical model reduces to a second-order equation with quadratic nonlinearity that can describe, depending on the initial conditions and the model parameters, *three possible scenarios* of the long-term soliton evolution: (i) periodic oscillations of the soliton amplitude near a stable state, (ii) soliton decay into dispersive waves, and (iii) soliton collapse. We also present the results of numerical simulations that confirm excellently the predictions of our analytical theory.

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I. INTRODUCTION

As is well known, propagation of self-guided beams in dielectric optical waveguides [1,2] and short nonlinear pulses in optical fibers [2,3] which exhibit Kerr nonlinearity is governed by a conventional model, the cubic nonlinear Schrödinger (NLS) equation. However, practical materials often display physical effects, such as nonlinearity saturation, which can only be described by generalized models of the nonlinear refractive index. For such *non-Kerr* materials theoretical predictions of new nonlinear effects, e.g., nonlinearity-induced focusing of self-guided beams via wave collapse [4], multistability [5,6], and nonlinear switching and steering due to spatial solitons [7], are very important for practical applications. Therefore models with a more general form of the intensity-dependent refractive index are often used to analyze nonlinear effects in *non-Kerr materials*. In dimensionless units, these models can be described by the following generalized NLS equation:

$$i\frac{\partial\Psi}{\partial t}+\frac{\partial^2\Psi}{\partial x^2}+f(|\Psi|)\Psi=0, \tag{1}$$

with rather general nonlinearity $f(|\Psi|)$. Here Ψ is a slowly varying envelope of the electric field, and t and x have different meanings depending on the physical context. For example, for the problems of stationary wave propagation in a dielectric waveguide (i.e., *spatial solitons*) t and x stand for the longitudinal and transverse coordinates, respectively. In the case of *temporal solitons* in optical fibers, t is the coordinate along the fiber whereas x is the retarded time. In various nonlinear models of solid state physics (see, e.g., [8] and references therein) the variable t is time and x is the propagation coordinate. Below, for simplicity, we call these variables "time" and "coordinate," respectively.

In all cases, the function $f(|\Psi|)$ characterizes nonlinearity, e.g., the nonlinear correction to the refractive index of the optical material. For many physical problems this function can be approximated by a power-law dependence, $f\sim|\Psi|^{2p}$ (see, e.g., [4,9]) and, in the particular case of the Kerr nonlinearity, we have $p=1$. This model has been investigated in detail in the context of the existence and stability of (bright) optical solitons which are stationary solutions of the form $\Psi_s(x,t)=\Phi(x;\omega)e^{i\omega t}$, where $\Phi(x;\omega)$ vanishes for $|x|\to\infty$, and ω is either the nonlinearity-induced shift of the mode propagation constant (e.g., for spatial solitons) or the soliton frequency (see, e.g., the review papers [8,10] and references therein). In particular, it has been proven that solitons can become unstable for stronger nonlinearity ($p\geq 2$) and they either decay or exhibit wave collapse (i.e., singularities of the wave field $|\Psi|$ are formed in a finite time from localized initial perturbations). However, it still remains unclear whether this scenario can be applied to the model (1) with a more general nonlinear function $f(|\Psi|)$, or the generalized NLS models may display other types of the long-term instability-induced soliton dynamics.

The standard approach of the soliton theory is to analyze the *linear stability* of the stationary, spatially localized solutions. For the generalized NLS equation (1) the soliton stability is determined by the well known Vakhitov-Kolokolov criterion [11] (see also [12] for a rigorous mathematical proof) which allows one to predict the parameter region in ω where the soliton amplitude can grow or decay exponentially with a nonzero growth rate. However, the linear stability analysis does not allow one to predict the subsequent evolution of the unstable soliton when the linearized equation describing the initial

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exponential growth of *small perturbations* of the soliton profile becomes invalid.

The main objective of the present paper is to investigate analytically and numerically the dynamics of the soliton instabilities in the model (1) with a rather general form of nonlinearity $f(|\Psi|)$ in the vicinity of the instability threshold. We derive, for the first time to our knowledge, an analytical model which describes *not only linear instabilities* of solitons but also their nonlinear long-term evolution. The approach we use is based on a modification of the soliton perturbation theory (see [8] and references therein) under the condition of slow, almost adiabatic, evolution of solitons near the instability threshold. In particular, we show that our analytical model predicts (at least in the case when there exists only one stable stationary soliton state) *three possible scenarios* of the long-term instability-induced dynamics of solitons, which depend on the initial conditions and the sign of the derivative $N_s''(\omega) \equiv d^2N_s(\omega)/d\omega^2$, where $N_s(\omega)$ is the energy (or "the number of particles") invariant calculated for the soliton solutions,

$$N_s(\omega) = \frac{1}{2}\int_{-\infty}^{+\infty} |\Psi_s(x,t)|^2 dx. \tag{2}$$

Note that the condition $N_s'(\omega) \equiv dN_s(\omega)/d\omega = 0$ determines exactly the instability threshold $\omega = \omega_c$ in the dependence of the soliton energy $N_s(\omega)$ on the parameter ω, and solitons are unstable provided the condition $N_s'(\omega) < 0$ is valid [10–12]. We reveal that in the vicinity of the instability threshold the stable solitons have always a nontrivial "internal mode" which corresponds to *long-lived nonlinear amplitude oscillations* of the stable solitons. As a result, the soliton instability leads, for certain types of nonlinearities, not to wave collapse but to an excitation of the amplitude oscillations around a new (stable) stationary state. Similar oscillating solitons have been recently reported in Ref. [13] for a special case of so-called threshold nonlinearity. This oscillating regime in the soliton dynamics disappears when the soliton parameter ω approaches the threshold curve of soliton stability. The frequency Ω_s of the soliton internal mode vanishes exactly at the instability threshold $\omega = \omega_c$, and for $\Omega_s^2 < 0$ it gives rise to exponentially growing perturbations with the growth rate of the linear instability $\lambda \approx \sqrt{-\Omega_s^2}$. In such a case, for nonlinear systems with only one stable soliton state, our model describes *two other scenarios* of the dynamics of unstable solitons, either *spreading* of an unstable soliton and its *decay into linear diffractive waves*, for $N_s''(\omega) > 0$, or *soliton collapse* and formation of singularities in a finite time, for $N_s''(\omega) < 0$.

To make more specific predictions, we apply our nonlinear theory to a generalization of the NLS equation which includes two power-law nonlinearities

$$f(|\Psi|) = a|\Psi|^p + b|\Psi|^{2p}, \tag{3}$$

where a, b, and p are arbitrary parameters (but $b, p > 0$). This model can be used to describe the effect of saturation (e.g., for $a < 0$) of the nonlinear refractive index [5,6,14], and its exact soliton solutions are known [14–16]. We show that for the case $a < 0$ the solitons of the model (1) and (3) become unstable and decaying near the instability threshold for $1 \le p < 2$, while for $p \ge 2$ they might collapse in a finite time. On the other hand, for $a > 0$ the instability of large-amplitude solitons is observed for $2 \le p < 4$, and it results in a wave collapse. Moreover, all solitons become collapsing provided $p \ge 4$.

The paper is organized as follows. In Sec. II we present the analytical theory describing the long-term *nonlinear dynamics of unstable solitons* in the model (1) with a general nonlinearity $f(|\Psi|)$. Then, in Sec. III we discuss the soliton solutions and their stability for the model (1) with two power-law nonlinearities (3). In particular, we analyze in detail the dynamics of unstable solitons near the instability threshold for two cases: (i) $a < 0$, $1 < p < 2$, and (ii) $a > 0$, $2 < p < 4$. The special degenerate case, $p = 2$, is discussed in Sec. IV where we show that it corresponds to the case of critical collapse [17–19] observed for large-amplitude solitons. Analysis of the model in the other degenerate case, $p = 1$, is presented in Sec. V, where the instability of algebraic solitons is proven. In particular, it is shown analytically that an algebraic soliton, which corresponds to a threshold between the exponential (i.e., sech-type) solitons and linear dispersive waves (see, e.g., [16] and references therein), transforms into one of the above-mentioned types of waves under the action of small perturbations. Finally, Sec. VI presents the concluding discussions.

II. NONLINEAR MODEL FOR THE DYNAMICS OF UNSTABLE SOLITONS

The standard approach of the soliton perturbation theory is applied to analyze the problems related to the dynamics of solitons under the action of either external or internal perturbations [8]. The external perturbations are given by additional (usually small) corrections to the basic nonlinear equation, while the internal ones can be presented, for instance, by interactions with other solitons or smooth inhomogeneous or/and weakly nonstationary radiation fields. Here we deal with a *different physical situation* considering *unstable* solitons in the "unperturbed" model (1). The unstable solitons evolve under the action of *their "own" perturbations* exponentially growing due to the instability. However, it turns out that even in this case we are still able to use a standard asymptotic analysis and, similarly to the general scheme of the soliton perturbation theory [8], reduce the soliton dynamics described by the original NLS model to a governing nonlinear equation for the adiabatic evolution of the soliton parameter.

In this section we present the basic idea of our approach, describing the soliton dynamics near the instability threshold. We consider the model (1) with *a general* nonlinear function $f(|\Psi|)$ and suppose that it has a stationary soliton solution in the form

$$\Psi_s(x,t) = \Phi(x;\omega)e^{i\omega t}. \tag{4}$$

Here we call ω the soliton propagation constant. The real function $\Phi(x;\omega)$ satisfies the following ordinary differential equation:

$$\frac{d^2\Phi}{dx^2} - \omega\Phi + f(\Phi)\Phi = 0. \tag{5}$$

We assume that the function Φ is even, and it vanishes exponentially as $|x| \to \infty$.

Now we consider the evolution of the soliton (4) under the action of an (infinitesimally small) initial perturbation of its shape. It is clear that the soliton changes its parameters as a result of the instability development and, in particular, the soliton parameter ω varies in time, $\omega = \omega(t)$. Our analytical method is based on the fact that near the instability threshold the growth rate is small. Thus we can assume that the instability-induced evolution of the perturbed soliton is, first, slow in t and, second, almost adiabatic. The first assumption allows us to use the asymptotic theory, whereas the second assumption implies that the localized part of the solution is close to the profile given by the soliton solution of Eq. (5) but with parameters slowly varying in time similarly to Ref. [8].

We expand the solution to the original model (1) in an asymptotic series,

$$\Psi(x,t) = \phi(x,t)\exp\left(i\int_0^t \omega(t')dt'\right),$$
$$\phi(x,t) = \Phi(x;\omega(t)) + \phi_1(x,t) + \phi_2(x,t) + \cdots. \tag{6}$$

If the assumption of the slowly varying perturbations is satisfied, i.e., $|\partial\phi/\partial t| \ll |\omega\phi|$, the NLS equation (1) can be reduced, with the help of the familiar multiscale asymptotic analysis, to a system of coupled linear inhomogeneous equations for the functions ϕ_1, ϕ_2, and so on. For instance, in the first-order approximation the correction ϕ_1 defined as

$$\phi_1 \equiv i\left(\frac{d\omega}{dt}\right) w_1(x;\omega)$$

is given by a solution of the linear equation,

$$\mathcal{L}_0 w_1 \equiv \frac{d^2 w_1}{dx^2} - \omega w_1 + f(\Phi)w_1 = -\frac{\partial\Phi}{\partial\omega}. \tag{7}$$

Because the eigenfunction of the linear operator $\mathcal{L}_0$ is proportional to the soliton solution $\Phi(x;\omega)$ (see, e.g., [10]), Eq. (7) can be easily solved analytically,

$$w_1(x;\omega) = -\Phi(x;\omega)\int_0^x \frac{dx'}{\Phi^2(x';\omega)} \times \int_0^{x'} \Phi(x'';\omega)\frac{\partial\Phi(x'';\omega)}{\partial\omega}dx''. \tag{8}$$

However, the solution (8) diverges exponentially unless the following (solvability) condition is satisfied:

$$\frac{dN_s(\omega)}{d\omega} = 0, \quad N_s(\omega) = \frac{1}{2}\int_{-\infty}^{+\infty}\Phi^2(x;\omega)dx. \tag{9}$$

This condition is nothing but the well known criterion for the onset of the soliton instability [10–12] known as the Vakhitov-Kolokolov criterion. Therefore the assumption of slow adiabatic variation of the soliton parameters can be justified only in the vicinity of the instability threshold, where the condition (9) is satisfied, at least approximately. Then the perturbation (8) is spatially (exponentially) localized and we can proceed with the asymptotic series to analyze the next, second-order approximation which leads to the following linear equation for ϕ_2:

$$\mathcal{L}_1\phi_2 \equiv \frac{d^2\phi_2}{dx^2} - \omega\phi_2 + [f(\Phi) + f'(\Phi)\Phi]\phi_2 = \frac{d^2\omega}{dt^2}w_1 + \left(\frac{d\omega}{dt}\right)^2\frac{\partial w_1}{\partial\omega} - \frac{1}{2}\left(\frac{d\omega}{dt}\right)^2 f'(\Phi)w_1^2, \tag{10}$$

where $f'(\Phi) \equiv df(\Phi)/d\Phi$.

Solutions of the linear equation (10) are spatially localized for any value of the soliton propagation constant $\omega(t)$ because the eigenfunction of the operator $\mathcal{L}_1$ is odd with respect to x (it is proportional to $\partial\Phi/\partial x$), while the right-hand side of Eq. (10) is even with respect to x. Therefore we can proceed further and consider the third-order approximation where the solvability condition produces a third-order differential equation for $\omega(t)$. However, here we prefer to avoid this cumbersome technique which involves higher-order approximations and, instead, we use a different but equivalent method to derive a nonlinear equation for the soliton propagation constant ω which is based on the conserved quantities (see, e.g., [8] and discussions therein).

Let us consider the conserved quantities for spatially localized solutions of Eq. (1),

$$N_0 = \frac{1}{2}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2dx = \frac{1}{2}\int_{-\infty}^{+\infty}|\Psi(x,0)|^2dx, \tag{11}$$

$$H_0 = \frac{1}{2}\int_{-\infty}^{+\infty}\left\{\left|\frac{\partial\Psi(x,t)}{\partial x}\right|^2 - F(|\Psi(x,t)|^2)\right\}dx = \frac{1}{2}\int_{-\infty}^{+\infty}\left\{\left|\frac{\partial\Psi(x,0)}{\partial x}\right|^2 - F(|\Psi(x,0)|^2)\right\}dx, \tag{12}$$

where $F(u) = \int_0^u f(\sqrt{u})du$. Now, we substitute the expansion (6) into the first conserved quantity (11) and neglect the terms of the fourth- (and higher-) order approximations. As a result, we obtain the following expression which involves the soliton propagation constant $\omega(t)$ and its derivatives:

$$M_s(\omega)\frac{d^2\omega}{dt^2} + \frac{1}{2}\frac{dM_s(\omega)}{d\omega}\left(\frac{d\omega}{dt}\right)^2 + N_s(\omega) = N_0, \tag{13}$$

where $N_s(\omega)$ is defined in Eq. (9) and $M_s(\omega)$ can be calculated through the stationary soliton solutions according to the formula

$$M_s(\omega) = \int_{-\infty}^{+\infty}\left[\frac{1}{\Phi(x;\omega)}\int_0^x \Phi(x';\omega)\frac{\partial\Phi(x';\omega)}{\partial\omega}dx'\right]^2 dx > 0. \tag{14}$$

The details of the calculation of the coefficients in Eq. (13) are given in the Appendix.

Because N_0 is conserved, the expansion (13) can be regarded as a differential equation for $\omega(t)$. We can readily find the first integral of Eq. (13) (see comments in the Appendix) and present it in the form

$$\frac{1}{2}M_s(\omega)\left(\frac{d\omega}{dt}\right)^2 + U_s(\omega) = E, \tag{15}$$

$$U_s(\omega) \equiv \int_{\omega_0}^{\omega} [N_s(\omega') - N_0]\, d\omega', \tag{16}$$

where E is the integration constant, and ω_0 is the value of the soliton parameter ω corresponding to the initial energy, i.e., it is defined through the equation $N_0 = N_s(\omega_0)$. We note that the first integral (15) can also be found from the second conserved quantity (12) and, therefore, the integration constant E can be expressed through the initial value of the system Hamiltonian, $E = H_0 - H_s(\omega_0)$, where $H_s(\omega)$ is the soliton energy satisfying the differential equation $dH_s(\omega) = -\omega dN_s(\omega)$.

Thus we obtain the remarkable result that in the generalized NLS-type model (1) the dynamics of solitons near the instability threshold can be described by the generic model (15) which is equivalent to the energy conservation law for an effective particle of the mass $M_s(\omega)$ with the coordinate ω which moves under the action of the "external" potential $U_s(\omega)$. The difference between the soliton energy $N_s(\omega)$ and the initial value N_0 of the input pulse (i.e., slightly perturbed soliton) creates an effective potential force, whereas the difference between the soliton Hamiltonian $H_s(\omega_0)$ and the initial value H_0 defines the energy of the particle. They determine completely the subsequent evolution of the unstable soliton.

Our asymptotic model is valid provided the soliton parameter ω is selected near the instability threshold (9) defined through the derivative of the energy $N_s(\omega)$. Therefore, in a general case, we can evaluate the function $N_s(\omega)$ near its extremum point by the standard approximation through a Taylor series up to the quadratic nonlinear term. This allows us to reduce Eq. (13) to the following quadratic equation of the second order:

$$M_s(\omega_0)\frac{d^2\Omega}{dt^2} + N_s'(\omega_0)\Omega + \frac{1}{2}N_s''(\omega_0)\Omega^2 = 0, \tag{17}$$

where $\Omega \equiv (\omega - \omega_0)$ is a deviation of the soliton propagation constant ω from the unperturbed value ω_0, and this deviation is assumed to be small. In Eq. (17) we neglect all the terms of higher orders, for example, the terms proportional to Ω^3 and $\Omega\ddot{\Omega}$, where $\dot{\Omega} \equiv d\Omega/dt$. Note that a quadratic equation similar to Eq. (17) has been recently derived, using the direct approach described just before Eqs. (11) and (12), in the problem of the soliton instability in a diffractive optical $\chi^{(2)}$ medium governed by phase-matched resonant quadratic interactions [17].

The first two terms in Eq. (17) give immediately the result of the linear stability analysis discussed, for example, in the review paper [10]. Indeed, because the soliton mass $M_s(\omega_0)$ is always positive, Eq. (17) immediately implies that the soliton is exponentially unstable for $N_s'(\omega_0) < 0$, and the growth rate of this instability λ can be found as $\lambda = \sqrt{-N_s'(\omega_0)/M_s(\omega_0)}$. For $N_s'(\omega_0) = 0$, we have a critical case when the growth rate of the linear (exponential) instability λ exactly vanishes. However, even in this degenerate case the soliton is still unstable with respect to small perturbations which grow on the soliton profile according to a *power-law* dependence [18]. On the other hand, the nonlinear (quadratic) term in Eq. (17) allows us to consider not only linear but also long-term (nonlinear) dynamics of unstable solitons and, moreover, to identify all possible scenarios of the soliton instability which, according to Eq. (17), are determined by the type of the extremum point of the function $N_s(\omega)$ (i.e., by the sign of the derivative N_s''). In the next sections we apply our general analytical model (17) to investigating the soliton instability in the generalized NLS equation (1) with two power-law nonlinearities (3). In some degenerate cases, when the coefficients in Eq. (17) *vanish* or *diverge*, we return back to the analysis of the general asymptotic model (13).

III. DIFFERENT SCENARIOS OF SOLITON DYNAMICS

A. Model and soliton solutions

Let us analyze the general characteristic features of the soliton instability dynamics for the model (1) with the power-law nonlinear function (3). As a matter of fact, the parameters a and b in Eq. (3) can be easily normalized by a trivial scaling transformation. Therefore we take the basic model in a form more convenient for our subsequent analysis,

$$i\frac{\partial\Psi}{\partial t} + \frac{\partial^2\Psi}{\partial x^2} - \sigma(p+2)|\Psi|^p\Psi + (p+1)|\Psi|^{2p}\Psi = 0, \tag{18}$$

where $\sigma = \pm 1$. As is well known (see, e.g., [15,16]), the NLS equation (18) possesses an explicit analytical solution in the form of the sech-type soliton (4) with the envelope $\Phi(x;\omega)$ defined as

$$\Phi(x;\omega) = \left(\frac{\omega}{\sqrt{1+\omega}\cosh[p\sqrt{\omega}x] - \sigma}\right)^{1/p}. \tag{19}$$

The soliton solution (19) corresponds to different types of nonlinear waves in the special limits of the parameter ω. For $\omega \gg 1$, the solution (19) asymptotically approaches the sech-type soliton supported by a higher-order nonlinear term in the model (18),

$$\Phi(x;\omega \gg 1) = \left(\frac{\sqrt{\omega}}{\cosh[p\sqrt{\omega}x]}\right)^{1/p}. \tag{20}$$

On the other hand, for $\omega \to 0$ and $\sigma = +1$ the solution (19) transforms into *an algebraic soliton* with the power-law asymptotics [16]

$$\Phi(x;\omega \to 0) = \left(\frac{2}{1+p^2x^2}\right)^{1/p}. \quad (21)$$

We note that both nonlinear terms in Eq. (18) are equally important for the existence of the algebraic solitons. The expression (21) for two particular cases, $p = 1$ and $p = 2$, has been recently found in Ref. [14] and the authors of Ref. [14] claimed the stability of these stationary solutions in both cases. However, the analytical model discussed below (see Sec. V) reveals *a weak instability* of algebraic solitons (21) for $p \leq 1$, and their *absolute instability* for $p > 1$ (see also the discussions in [16]).

For the case $\sigma = -1$, the limit $\omega \to 0$ does not change the exponential character of the soliton asymptotics for $|x| \to \infty$ but it reduces the soliton solution (19) to the form (20), i.e., to the soliton supported by the lower-order nonlinear term in Eq. (18).

B. Soliton stability

Now we analyze the stability of the soliton solutions (19) in the framework of the nonlinear equation (17). First, we define the soliton energy $N_s(\omega)$,

$$N_s(\omega;p,\sigma) = \frac{1}{p}\omega^{\frac{4-p}{2p}} I(\omega;p,\sigma),$$
$$I(\omega;p,\sigma) = \int_0^{+\infty} \frac{dx}{[\sqrt{1+\omega}\cosh(x) - \sigma]^{2/p}}. \quad (22)$$

The numerical investigation of the integral $I(\omega;p,\sigma)$ for various p shows that the extremum points (9) exist for $1 \leq p \leq 2$, at $\sigma = +1$, and for $2 \leq p \leq 4$, at $\sigma = -1$. On the plane (ω,p), the critical curve $\omega = \omega_c(p)$ defined by the condition $N_s'(\omega) = 0$ separates the instability and stability domains, as is shown in Figs. 1(a) and 1(b) for both signs of σ. According to the linear stability analysis, all solitons are unstable for $\omega < \omega_c(p)$ at $\sigma = +1$. For $\sigma = -1$, the situation is reversed, and the solitons are unstable for $\omega > \omega_c(p)$ [see Fig. 1(b)].

The type of the extremum point is determined by the sign of the second-order derivative $N_s''(\omega)$ along the curve $\omega = \omega_c(p)$. Using simple algebra, we can express this quantity through the soliton energy $N_s(\omega)$,

$$\left.\frac{d^2N(\omega)}{d\omega^2}\right|_{\omega=\omega_c(p)} = \frac{(p-1)(2-p)}{2p^2\omega(1+\omega)} N_s(\omega). \quad (23)$$

As follows from Eq. (23), for $\sigma = +1$ the extremum point is always a minimum because the value $N_s''(\omega_c)$ is positive for $1 < p < 2$ [the soliton energy $N_s(\omega)$ is positive for any ω, see the definition, Eq. (9)]. On the other hand, for $\sigma = -1$ the extremum point is always a maximum since the value N_s'' is negative for $2 < p < 4$. The typical profiles of the function $N_s(\omega)$ in the intermediate regions of the parameter p are presented in Fig. 2(a), for $\sigma = +1$ and in Fig. 3(a), for $\sigma = -1$.

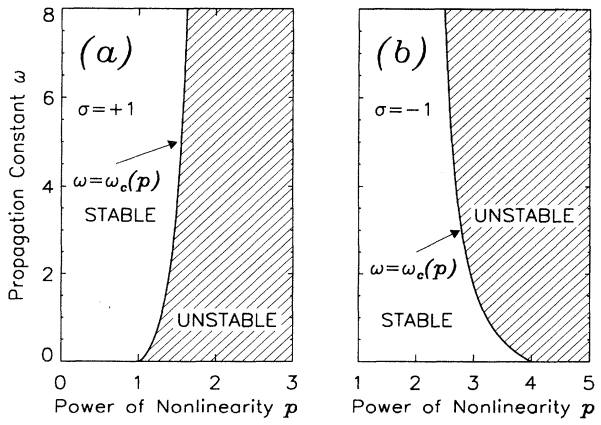

FIG. 1. Region of linear stability of the sech-type solitons (19) on the parameter plane (ω,p) for (a) $\sigma = +1$ and (b) $\sigma = -1$. Stability and instability domains are separated by the critical curve $\omega = \omega_c(p)$.

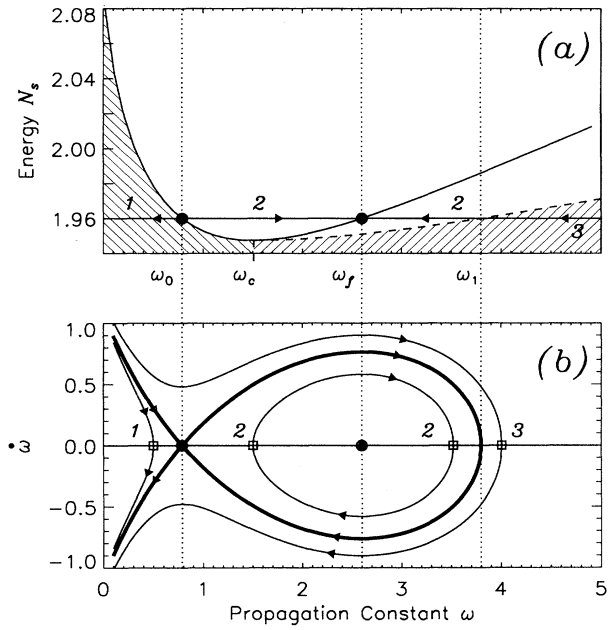

FIG. 2. (a) Characteristic dependence of the soliton energy $N_s(\omega)$ for $\sigma = +1$ and $1 < p < 2$ (here we select $p = 1.35$). Minimum point ω_c corresponds to the instability threshold and ω_0 and ω_f to the unstable and stable solitons at $N_s(\omega) = N_0 = 1.96$, respectively. Shaded regions 1 and 3 correspond to the soliton decay, and the region 2 to the periodic amplitude oscillations around the stable soliton. (b) Phase plane $(\omega,\dot{\omega})$ of the analytical model (13) describing different regimes of the soliton dynamics: curves 1 and 3—soliton decay; curve 2—amplitude oscillations around the stable soliton with $\omega = \omega_f$. Separatrix curve is shown by a bold curve.

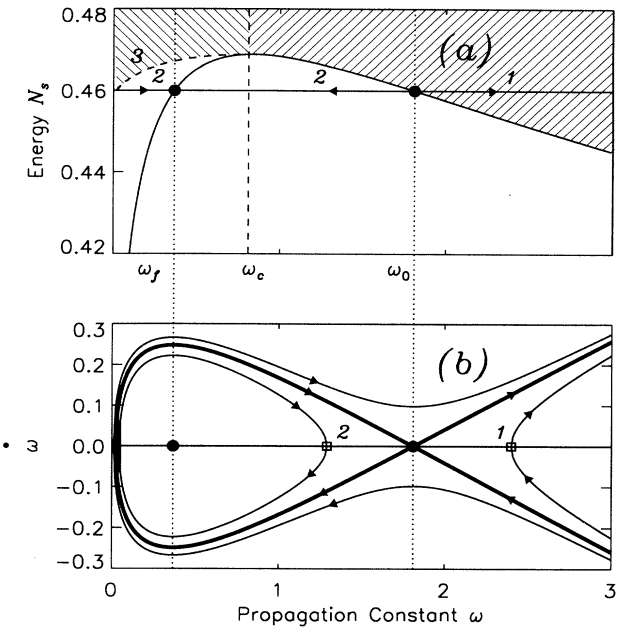

FIG. 3. (a) Characteristic dependence of the soliton energy $N_s(\omega)$ for $\sigma = -1$ and $2 < p < 4$ (here we select $p = 3.3$). The points ω_f, ω_c, and ω_0 have the same meaning as in Fig. 2(a). Shaded region determines the soliton collapse, while the region 2 corresponds to the periodic amplitude oscillations. (b) Phase plane $(\omega, \dot{\omega})$ of the analytical model (13) describing different regimes of the soliton dynamics: curve 1—soliton collapse; curve 2—amplitude oscillations around the stable solution with $\omega = \omega_f$.

In the special cases $p = 1$ and $p = 2$, we can calculate the integral $I(\omega; p, \sigma)$ defined by Eq. (22) in elementary functions and find N_s in an explicit form. For the case $\sigma = +1$ it is given by

$$N_s(\omega; 1, 1) = \pi + \sqrt{\omega} - \tan^{-1}(\sqrt{\omega}), \tag{24}$$

and

$$N_s(\omega; 2, 1) = \frac{1}{2}\left[\pi - \tan^{-1}(\sqrt{\omega})\right]. \tag{25}$$

It is obvious from Eq. (24) that the function $N_s(\omega)$ is monotonically increasing for $p = 1$ from a finite value $N_s(\omega \to 0) = \pi$. For $p < 1$, the curve $N_s(\omega)$ remains qualitatively similar to that described by Eq. (24) but with a slope which is always *positive*, including the critical point $\omega = 0$. This indicates the absolute stability of the whole branch of the (bright) sech-type solitons (19) for $p < 1$. On the other hand, the function $N_s(\omega)$ is monotonically decreasing for $p = 2$ from the value $N_s(\omega \to 0) = \pi/2$ to the value $N_s(\omega \to \infty) = \pi/4$. The slope of the function $N_s(\omega)$ is always *negative* for $p > 2$, and this indicates the absolute instability of the sech-type solitons for this case. A similar analysis shows that the solitons (19) for $\sigma = -1$ are stable for $p < 2$ and absolutely unstable for $p > 4$.

Next, we calculate numerically the coefficients $M_s(\omega_c)$ and $N_s''(\omega_c)$ of the quadratic model (17) at the stability threshold curve $\omega = \omega_c(p)$, using the formulas (14) and (23), and present them in Figs. 4(a) and 4(b), respectively. It turns out that the reduction to the quadratic model (17) fails for special critical cases, i.e., when (i) $p \to 2$, $\omega_c \to \infty$ at any σ, (ii) $p \to 1, \omega_c \to 0$ at $\sigma = +1$, and (iii) $p \to 4$, $\omega_c \to 0$ at $\sigma = -1$. In the first case, the coefficients M_s and N_s'' in Eq. (17) vanish, while in the other two cases they are diverging.

The first failure of our asymptotic reduction is related to a scaling invariance of the function $N_s(\omega)$ in the limit $p \to 2, \omega \to \infty$ [see Eq. (25)]. It is well known that such an invariance is a characteristic feature of models displaying the critical collapse phenomenon [18–20]. On the other hand, the divergence of all coefficients in Eq. (17) in the case (ii) is explained by a critical phenomenon of another type, when the sech-type solitons (19) with exponentially decaying asymptotics transform into algebraic solitons (21) with power-law asymptotics. In this case, the soliton energy becomes also independent of the parameter ω, and it tends to a nonzero critical value [see Eq. (24)]. Finally, we note that the case (iii) can also be related to the scaling invariance of the dependence $N_s(\omega)$, similarly to the case (i) but existing in the limit $p \to 4, \omega \to 0$. In all the cases, instead of the quadratic equation (17), we should consider the more general analytical model defined by Eqs. (13) or (15) in order to obtain a correct description of the long-term soliton dynamics. The special critical cases are analyzed in Secs. IV and V. However, in the intermediate region of the

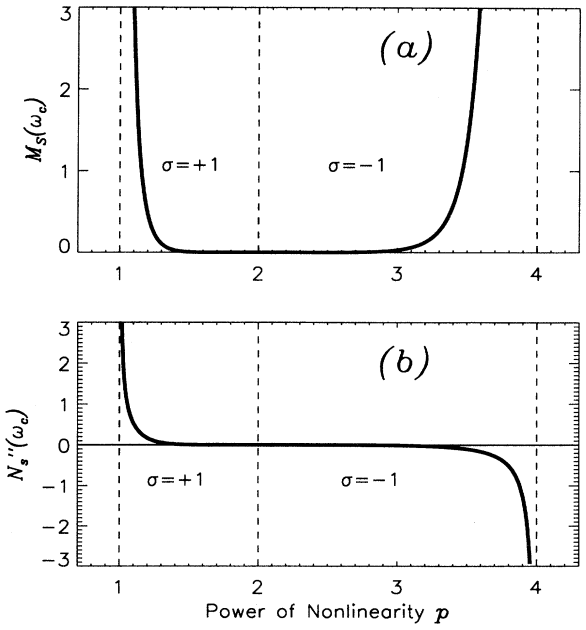

FIG. 4. (a) Effective mass $M_s(\omega_c)$, and (b) second-order derivative of the soliton energy, $N_s''(\omega_c)$, calculated for the soliton solutions (19) of the model (18) for both $\sigma = +1$ and $\sigma = -1$ (shown on the same plot), vs the power of nonlinearity p.

parameter p the nonlinear model (17) is a sufficient and very useful tool for investigating the instability dynamics and long-term evolution of bright solitons.

C. Soliton decay and oscillation for $1 < p < 2$

1. Analytical results

In this subsection, we consider the model (18) for $1 < p < 2$ and $\sigma = +1$, for which the derivative $N_s''(\omega_c)$ is positive [see Fig. 2(a)]. It is convenient to analyze the dynamical system (17) on the phase plane $(\Omega, \dot{\Omega})$ or, equivalently, the system (13) on the plane $(\omega, \dot{\omega})$, where $\dot{\omega} \equiv d\omega/dt$. When the initial value of the energy invariant N_0 exceeds the extremum value of $N_s(\omega_c)$, the corresponding phase plane is presented in Fig. 2(b).

The standard initial-value problem to the original NLS model (18) is imposed on a quasistationary solitonlike pulse when the first-order correction ϕ_1 to the soliton shape Φ is removed from the series (6) in an initial time instant. In the analytical model (13) this corresponds to the initial condition lying on the axis $\dot{\omega}(0) = 0$. For such initial conditions the integral curves given by Eq. (15) reveal three different types of soliton dynamics. They are depicted by the curves 1, 2, and 3 in Figs. 2(a) and 2(b).

If the amplitude of the initially perturbed soliton is taken to be smaller than the amplitude of the (unstable) stationary solution [i.e., $\omega < \omega_0$, curve 1 in Figs. 2(a) and 2(b)], the instability leads to a decrease of the soliton parameter ω (which is proportional to the soliton amplitude) and this process results in spreading of the soliton. As there are no new stable equilibrium states in the dependence $N_s(\omega)$ for $\omega < \omega_0$ which can be realized at the same value of N_0 [see Fig. 2(a)], the soliton spreading cannot be stabilized by the nonlinearity-induced self-focusing, and the soliton transforms into a small-amplitude wave packet which finally decays due to dispersive (or diffractive) properties of the wave medium.

On the other hand, if the initially perturbed soliton has an amplitude slightly bigger than that of the stationary soliton solution [i.e., $\omega_0 < \omega < \omega_1$, where $\omega_1 \approx \omega_0 + 3|N_s'(\omega_0)|/N_s''(\omega_0)$; curve 2 in Figs. 2(a) and 2(b)], the exponentially growing instability "pushes" the soliton into the region of stability where there exists *a stable stationary* soliton solution with the other (final) value of the parameter ω, $\omega_f \approx \omega_0 + 2|N_s'(\omega_0)|/N_s''(\omega_0)$, which corresponds to the same value of the energy invariant N_0 [see Fig. 2(a)]. However, due to the inertial nature of the soliton evolution, which is characterized by a nonzero value of the soliton mass M_s, and the absence of essential radiation (i.e., effective radiation friction), the direct transition from the unstable state to the stable one is impossible and, instead, the long lived periodic oscillations are observed around the stable equilibrium state with $\omega = \omega_f$. Thus near the stable state we predict the existence of periodic amplitude oscillations which are long lived due to a nontrivial oscillating *internal mode* of a bright soliton in the generalized NLS model (1). The frequency Ω_s of this internal mode can be estimated from Eq. (17) in a linear approximation as $\Omega_s = \sqrt{N_s'(\omega_f)/M_s(\omega_f)}$.

It is important to emphasize that, according to our analytical model (17), the periodic oscillations of the soliton amplitude must disappear for *larger deviation* from the stable equilibrium state [$\omega > \omega_1$, curve 3 in Figs. 2(a) and 2(b)]. In spite of the fact that larger deviation of the soliton amplitude does not remove the soliton from the region of stability, the evolution of perturbations is predicted to lead again to decreasing of the soliton amplitude. As soon as the parameter ω enters the unstable region, the soliton finally spreads out due to dispersion (or diffraction). *Thus soliton perturbations with larger amplitudes lead also to a decay of solitons near the instability threshold.* Moreover, when the initial energy N_0 decreases, the parameter domain corresponding to periodic oscillations of the soliton amplitude becomes smaller and finally disappears at $N_0 = N_s(\omega_c)$. Therefore all perturbed solitons spread out provided $N_0 \leq N_s(\omega_c)$.

2. Numerical results

To confirm the existence of the three dynamical regimes for the evolution of unstable solitons discussed in the previous subsection, we carry out a direct numerical integration of the generalized NLS model with two powerlike nonlinearities (18) selecting, for definiteness, $p = 1.35$. The initial pulse at $t = 0$ is taken in the form of a weakly perturbed soliton,

$$\Psi(x,0) = \Phi(x;\omega) + (a + bx^2)\frac{d^2\Phi(x;\omega)}{dx^2}, \tag{26}$$

where $\Phi(x;\omega)$ is the soliton solution (19) which has a perturbation multiplied by the factor $(a + bx^2)$. The parameter a is the amplitude of the perturbation, and the parameter b is introduced to keep the initial value of the energy, N_0, unchanged. The latter condition is necessary *to compare* different regimes of the soliton dynamics in the framework of the generalized NLS equation (18) with those described by the analytical model (13) and its reduced version (17).

In numerical simulations, we have found all predicted regimes of the soliton instability dynamics. Figures 5(a) and 5(b) show several characteristic examples of the soliton evolution for the effective propagation constant $\omega(t)$ which is determined numerically as the first-order derivative of the soliton phase calculated at the pulse maximum. Each dependence corresponds to different values of the perturbation parameter a, and the results are in good agreement with the predictions of the analytical model (13) [see the trajectories on the phase plane shown in Fig. 2(b)].

If the amplitude of the perturbed soliton is taken *smaller* than the amplitude of the unstable soliton [point 1 in Fig. 2(b)], the soliton propagation constant decreases monotonically, and it vanishes in a finite time [curve 1 in Fig. 5(a)]. For slightly *larger* amplitudes, the devel-

opment of the soliton instability leads to the excitation of periodic oscillations of the soliton amplitude around the stable soliton corresponding to the same value of the energy N_0 [curve 2 in Fig. 5(a)]. The long lived oscillations of the soliton intensity $|\Psi|^2$ are shown in Fig. 5(c). We note that these oscillations are mostly radiationless and, therefore, they are observed for many periods without visible changes. Such a behavior of the soliton is explained by the existence of periodic orbits within the separatrix loop [see curve 2 in Fig. 2(b)] in our analytical model.

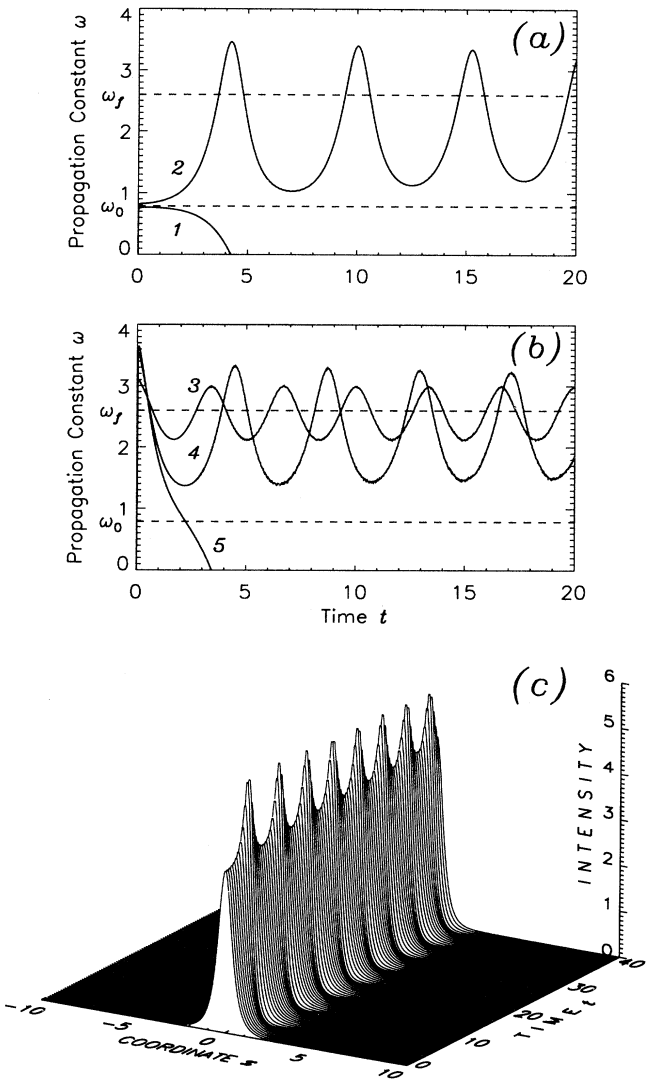

FIG. 5. Results of numerical simulations of the generalized NLS equation (18) at $\sigma = +1$ and $p = 1.35$ (see text). (a) Initial conditions are selected near the unstable soliton corresponding to the propagation constant ω_0 [see also Figs. 2(a) and 2(b)]. (b) Initial conditions are selected near the stable soliton corresponding to the propagation constant ω_f. (c) Long-lived periodic oscillations of the soliton intensity corresponding to the curve 2 in Fig. 5(a).

If we take a small initial perturbation near the stable soliton, we observe quasiharmonic small-amplitude oscillations of the effective propagation constant near the value ω_f corresponding to the stable soliton [see curve 3 in Fig. 5(b)]. As a matter of fact, this long lived oscillating dynamics is generated by a nontrivial internal mode existing in the vicinity of the stable soliton. When the amplitude of the initial perturbation increases, the period of the oscillations grows, and such oscillations become strongly nonlinear [curve 4 in Fig. 5(b)].

Finally, if the initial value of the soliton propagation constant exceeds the value ω_1 corresponding to the separatrix trajectory shown in Fig. 2(b) by a bold curve, the soliton amplitude permanently decreases, and finally the soliton decays [curve 5 in Fig. 5(b)]. We would like to emphasize again that the latter type of soliton dynamics is rather unexpected because the initial perturbed soliton pulse has been taken *inside* the stability domain.

Thus, the direct numerical simulations of the dynamics of the unstable and stable solitons in the generalized NLS equation (18) completely confirm the predictions given by our analytical model (13).

D. Soliton collapse for $2 < p < 4$

In this subsection, we analyze the case $\sigma = -1$ and $2 < p < 4$. Because the model (18) with the higher-order power-law nonlinear term ($f(|\Psi|) \sim |\Psi|^{2p}$) displays a collapse singularity for $p \geq 2$ [4], it is natural to expect that the soliton perturbations with $\omega > \omega_c(p)$ will lead to collapse as well. Indeed, this is confirmed by the results following from our quadratic model (17), where $N_s''(\omega_c) < 0$ [see Fig. 3(a)]. The corresponding types of soliton dynamics for the case $N_0 < N_s(\omega_c)$ are shown on the phase plane $(\omega, \dot{\omega})$, Fig. 3(b). We can see that those perturbations which are bounded by the separatrix generate periodic oscillations of the soliton amplitude near a stable soliton state [curve 2 in Figs. 3(a) and 3(b)]. Note that now the stable stationary solitons have *smaller* amplitudes than the unstable ones, i.e., $\omega_f < \omega_0$. On the other hand, perturbations lying outside this interval in ω have to grow infinitely. It can be easily shown that the model (13) displays the formation of singularities in a finite time (blowup) from such initial perturbations. Moreover, all perturbations with $N_0 \geq N_s(\omega_c)$ lead to collapse. However, when the deviation of the soliton propagation constant ω from the threshold value ω_c becomes larger, correct quantitative analysis based on the model (13) becomes impossible.

Numerical simulations of the generalized NLS equation (18) carried out at $p = 3.4$, similarly to those described in Sec. III C 2, display the same regimes of soliton dynamics as predicted by our nonlinear model (17). These are shown in Figs. 6(a) and 6(b). Evolution of the unstable solitons leads to collapsing solutions if the amplitude of the perturbed soliton is larger than that of the unstable one [curves 1 and 2 in Fig. 6(a)]. On the other hand, perturbed solitons with smaller amplitudes spread out [see curves 3 and 4 in Fig. 6(a)] in spite of the fact that the theory predicts oscillations around a stable state with

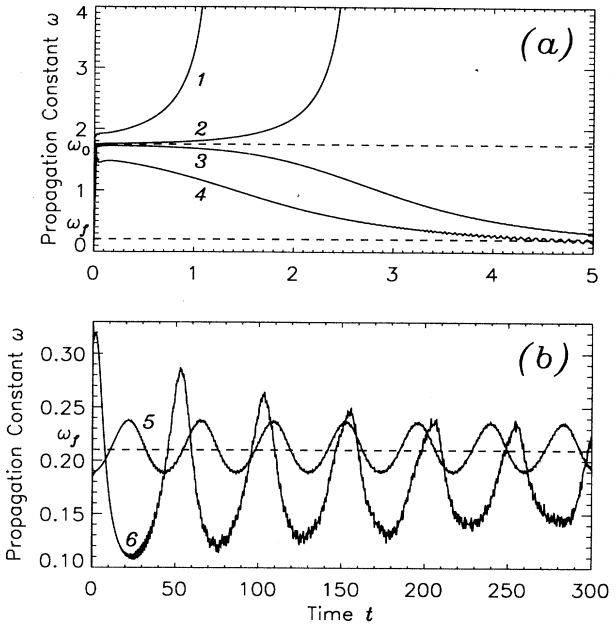

FIG. 6. The same as in Fig. 5 but for $\sigma = -1$ and $p = 3.4$ [see also Figs. 3(a) and 3(b)].

$\omega = \omega_f$. This disagreement is observed due to the fact that the separatrix, shown by a bold curve in Fig. 3(b), is very close to the critical value $\omega = 0$ where the soliton width grows infinitely and its amplitude becomes very small. As a result, the soliton transforms into dispersive linear waves, and finally disappears.

If we select the amplitude of the perturbed soliton near the stable state corresponding to ω_f, small-amplitude periodic oscillations are clearly observed [curve 5 in Fig. 6(b)]. For larger amplitudes, the oscillations become nonlinear, their period increases, and the minimum amplitude approaches the line $\omega = 0$ where strong radiation is generated. This effect results in fluctuations of the soliton propagation constant measured in numerical simulations near the minimum amplitude [see curve 6 in Fig. 6(b)]. We note also that radiation-induced losses observed in this case create a rapid decay of the oscillation amplitude.

IV. CRITICAL SOLITON COLLAPSE

As has been mentioned in Sec. III B, the case $p \to 2$, $\omega_c \to \infty$ is special, because in this case the coefficients $M_s(\omega_c)$ and $N_s''(\omega_c)$ vanish. Moreover, as follows from Eq. (25), for large ω the energy invariant $N_s(\omega)$ becomes independent of ω, i.e., $N_s(\omega) \to N_{cr} = \pi/4$ for $\omega \gg 1$. The latter feature implies the existence of scaling invariance of the soliton energy N_s, and this property corresponds to the phenomenon known as *critical collapse* (see, e.g., [18–20]). For the critical collapse the formation of singularities is possible only for initial values of N_0 larger than a certain value N_{cr}. Otherwise, i.e., for $N_0 < N_{cr}$, the solitons spread out due to radiation of the linear dispersive waves. Moreover, for the critical value $N_0 = N_{cr}$ the NLS equation (1) with a solely fifth-order nonlinearity [$f(|\Psi|) \sim |\Psi|^4$] possesses a self-similar localized solution describing an exact transformation of the stationary solitons into collapsing or decaying solutions. In this section we show that all the phenomena known for the critical collapse can be directly extracted from our analytical model (13).

Expanding Eq. (22) into a Taylor series in $\omega^{-1/2}$ for large ω, we can obtain the asymptotic representation of the soliton energy $N_s(\omega)$,

$$N_s(\omega; p, \sigma) = \frac{\sqrt{\pi}\Gamma(1/p)\omega^{\frac{2-p}{2p}}}{2p\Gamma(1/p + 1/2)} \times \left[1 + \frac{2\sigma\Gamma^2(1/p + 1/2)}{\sqrt{\omega}\Gamma^2(1/p)} + O\left(\frac{1}{\omega}\right)\right], \quad (27)$$

where $\Gamma(z)$ is the gamma function. Substitution of Eq. (27) into Eq. (9) allows us to find the asymptotic behavior of the threshold stability curve $\omega = \omega_c(p)$ for $p \to 2$,

$$\sqrt{\omega_c(p)} = \frac{4\sigma}{\pi(2-p)}. \quad (28)$$

Henceforth we consider only the case $\sigma = +1$. The other case, $\sigma = -1$, can be analyzed in a similar manner but it does not display qualitatively new features in the soliton dynamics. From Eq. (27) it is clear that for any $p < 2$ the function $N_s(\omega)$ is similar to that shown in Fig. 2(a) but with the slope approaching zero for larger ω and $p \to 2$. As a result, the soliton dynamics described by the analytical model (13) in this limiting case is essentially the same as that discussed above in Sec. III C and presented in Fig. 2(b). Nevertheless, the amplitude of the internal oscillations near the stable equilibrium state tends to infinity as $p \to 2$. Therefore, although the formation of singularities is formally impossible for $p < 2$, small perturbations of the unstable solitons have to lead to very large amplitudes of the soliton oscillations which look, at least initially, like those generated due to the development of the collapse instability (see also the discussions in [4]).

Let us consider the critical case $p = 2$, $\omega_c \to \infty$ in detail. It implies that we investigate now the soliton dynamics in the critical power-law NLS model

$$i\frac{\partial\Psi}{\partial t} + \frac{\partial^2\Psi}{\partial x^2} + 3|\Psi|^4\Psi = 0. \quad (29)$$

In this case, the condition (9) is identically satisfied for the soliton solutions (20) at $p = 2$ and the analytical model (13) becomes nonlinear only due to the dependence of the soliton mass M_s on ω, which is given by

$$M_s(\omega) = \frac{m}{\omega^3}, \quad (30)$$

where $m = \pi^3/512$. Substitution of Eq. (30) into the first integral (15) leads to the equation

$$\frac{m}{2}\left(\frac{d\omega}{dt}\right)^2 = E\omega^3 + \Delta\omega^4, \tag{31}$$

where $\Delta = N_0 - N_{cr}$ and $E = H_0$. A general solution of Eq. (31) can be found in an analytic form,

$$\omega = -\frac{2mE}{2m\Delta - E^2(t-t_0)^2}. \tag{32}$$

First, we select the integration constant t_0 to be zero in order to satisfy the initial condition $\dot{\omega}(0) = 0$. It follows from (32) that the two constants are related as $E = -\omega(0)\Delta$. If now the initial value of the pulse energy is less than the critical value (i.e., $\Delta < 0$ and $E > 0$), the solution (32) describes a monotonic, power-law decay of the solitons with different initial amplitudes (curve 1 in Fig. 7). In the opposite case, i.e., when $\Delta > 0$ and $E < 0$, the soliton evolution results in the formation of singularities in a finite time, $t_c = \sqrt{2m\Delta}/|E|$ (curve 2 in Fig. 7).

In the special cases $\Delta = 0$ or $E = 0$, the evolution of the soliton depends on the type of small perturbations at $t = 0$ because the initial condition $\dot{\omega}(0) = 0$ (on $t_0 = 0$) cannot be satisfied any more. In these cases, the solutions of Eq. (31) can be rewritten in the form

$$\omega = \omega(0)\left(\frac{t_0}{t_0 - t}\right)^{\gamma}, \tag{33}$$

where $\gamma = 2$ for $\Delta = 0$ ($E > 0$) and $\gamma = 1$ for $E = 0$ ($\Delta > 0$). These solutions describe either soliton collapse, for $t_0 > 0$ and $\dot{\omega}(0) > 0$, or soliton decay, for $t_0 < 0$ and $\dot{\omega}(0) < 0$. In the case $N_0 = N_{cr}$ ($\gamma = 2$), these solutions are shown in Fig. 7 by the corresponding dashed lines. Thus, depending on a weak initial perturbation, the solitons with slightly *increased* amplitudes collapse whereas the solitons with slightly *decreased* amplitudes spread out. These phenomena indicate a weak, power-law instability of the bright solitons in the critical power-law NLS equation (29) (see [18,19]).

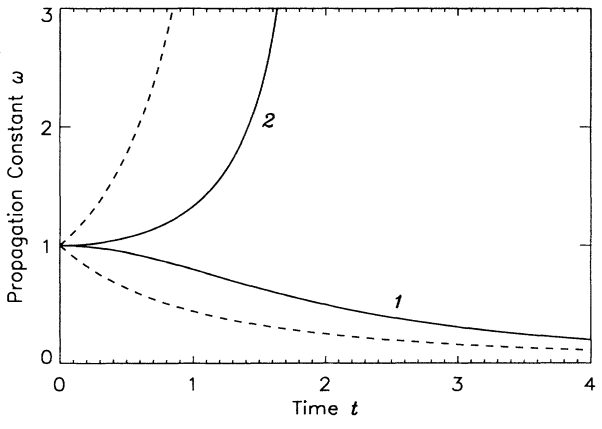

FIG. 7. Evolution of the soliton propagation constant ω for the critical collapse described by Eq. (32). Solid curve 1 shows the case $N_0 < N_{cr}$, and curve 2, for $N_0 > N_{cr}$. Dashed curves present the corresponding solutions given by Eq. (33) for the case $N_0 = N_{cr}$ ($\gamma = 2$).

It is important to note that, in the case of the critical collapse, our asymptotic results completely coincide with the *exact self-similar solutions* of the power-law NLS equation (29) found in Refs. [19,20]. In the primary variables, the exact solution has the form

$$\Psi(x,t) = [\omega(t)]^{1/4}\psi(X)\exp[i\theta(X,t)], \tag{34}$$

$$\theta(X,t) = \int_0^t \omega(t')dt' - \frac{X^2}{8\omega^2}\left(\frac{d\omega}{dt}\right),$$

where $X = \sqrt{\omega(t)}x$, the dependence $\omega(t)$ is given by Eq. (32), and the function $\psi(X)$ satisfies the differential equation

$$\frac{d^2\psi}{dX^2} + 3\psi^5 - \psi + \frac{\Delta}{8m}X^2\psi = 0. \tag{35}$$

For $\Delta = 0$, this equation has a localized solution corresponding to the stationary soliton $\psi(X) = \Phi(X;1)$. For nonzero but small Δ, we can expand Eq. (35) into the asymptotic series valid for $|X| \ll 2\sqrt{2m}/\sqrt{|\Delta|}$. In this case, our expansions (6) include the terms generated by this asymptotic series, and also a formal expansion of the exponential factor in Eq. (34) into a Taylor series.

It is obvious that solutions to Eq. (35) are not exponentially localized as $|X| \to \infty$ for $\Delta > 0$ due to the last, oscillatory-type term. Such improper self-similar solutions have been investigated for the two-dimensional NLS equation in Ref. [20], where it has been shown that they still describe the initial regime of the wave collapse. For larger time intervals the collapse is accompanied by generation of a radiation field, and its approximation by the exact self-similar solutions becomes worse. These difficulties are beyond the lowest order of our asymptotic expansion which allows us, anyway, to reconstruct the scaling laws of the initial dynamics for collapsing solitons.

Thus the analytical model (15) gives an adequate description of the evolution of initially solitonlike perturbations in the dependence of the initial values of the integral quantities N_0 and H_0. We have found that the solitonlike perturbations are collapsing when their energy N_0 exceeds the critical value N_{cr} realized on the unstable soliton solution. In the opposite case, the solitonlike solutions are decaying. A similar conclusion follows from the model (15) with (27) for $p > 2$ but in this latter case there is no universal critical value for the energy. As a matter of fact, for any N_0 at $p > 2$ there is a value of the amplitude of the input solitonlike pulse which separates two very different scenarios of soliton evolution, either collapse or decay. This critical amplitude coincides with the amplitude of the unstable stationary soliton.

V. INSTABILITY OF ALGEBRAIC SOLITONS

In this section we consider the other critical case, $p \to 1$ and $\omega_c \to 0$, when the quadratic model (17) becomes invalid because of the divergence of the coefficients $M_s(\omega_c)$ and $N_s''(\omega_c)$ [see Figs. 4(a) and 4(b)]. In this case, the sech-type soliton (19) transforms into the algebraic soliton (21) with slowly decaying, power-law asymptotics. Algebraic soliton solutions to the NLS equation (18) have been recently demonstrated numerically in Ref. [16] to be *inherently unstable* due to a resonant interaction with infinitely long linear waves. The numerical simulations carried out in Ref. [16] for the model (18) with $p \leq 1$ reveal a slow transformation of an algebraic soliton (for which $\omega = 0$) either to a sech-type soliton (with positive ω), if an initial perturbation increases the soliton amplitude, or to linear dispersive waves (corresponding to negative ω), if the perturbation decreases the soliton amplitude. In that sense, the algebraic soliton (21) at $p \leq 1$ is a degenerate solution which separates the sech-type solitons and linear dispersive waves.

Now we will show that the critical phenomenon observed numerically in Ref. [16] can be described analytically in the framework of our model (13) for the case $p \to 1$. Moreover, for the case $p < 1$ we find an explicit solution of the linear equation (describing small deviations of the algebraic soliton profile) which is responsible for slow, powerlike instability and nonlinear transformation of the algebraic soliton into either a sech-type soliton or dispersive waves.

First, we obtain the asymptotic expansion of the soliton energy (24) in the limit $\omega \to 0$,

$$N_s = \pi + \frac{1}{3}\omega^{3/2} + O(\omega^{5/2}). \tag{36}$$

Matching the value N_s'' found from Eqs. (23) and (36) we obtain the asymptotic result for the critical curve $\omega = \omega_c(p)$ for $p \to 1$,

$$\sqrt{\omega_c(p)} = 2\pi(p-1). \tag{37}$$

Next, expanding the function $M_s(\omega)$ into a Loran series as $\omega \to 0$ along the curve (37), we find the leading order of this series in the form

$$M_s = \frac{m}{\omega^{3/2}}. \tag{38}$$

Here m is a constant which can be found numerically, $m \approx 0.15$. Now we substitute Eqs. (36) and (38) into the first integral (15) and obtain the equation

$$\frac{m}{2}\left(\frac{d\omega}{dt}\right)^2 = E\omega^{3/2} + \Delta\omega^{5/2} - \frac{2}{15}\omega^4, \tag{39}$$

where $\Delta = N_0 - N_{cr}$ and the critical value N_{cr} of the soliton energy corresponds to the algebraic soliton, $N_{cr} \equiv N_s(\omega \to 0) = \pi$.

The model (39) describes the evolution of the sech-type soliton which is very close to the algebraic soliton, i.e., in the limit of small ω. Because solutions to this equation cannot be found in an explicit form, we analyze them on the phase plane $(\omega, \dot{\omega})$. For the subcritical case $\Delta > 0$, the phase plane is presented in Fig. 8(a). All trajectories are located in two regions. The first region is limited by a separatrix loop shown in Fig. 8(a) by a bold line, and it describes periodic oscillations of the perturbed soliton near the stable equilibrium state $\omega = \omega_f = (3\Delta)^{2/3}$ [curve 2 in Fig. 8(a)]. This equilibrium state corresponds to a stable sech-type soliton with the propagation constant ω_f, so that $N_s(\omega_f) = N_0$. The region outside of the separatrix loop describes a permanent decreasing of the soliton propagation constant ω from its initial value $\omega(0) \geq \omega_1$ to the value $\omega = 0$ corresponding to the algebraic soliton [curve 3 in Fig. 8(a)]. This transformation occurs for a finite time but the further evolution of such a decaying soliton remains unclear in the framework of our analytical model. It is important to mention that the soliton dynamics in the critical case $p = 1$ is qualitatively similar to that in a general case, $1 < p < 2$ [cf. Figs. 2(b) and 8(a)] but the region 1 in Fig. 2(b) disappears in the limit $\omega_0 \to 0$. This similarity of the soliton dynamics can be explained by the similarity of the dependence of the soliton energy N_s on the propagation constant ω.

For the case $\Delta < 0$ the integral curves are shown in Fig. 8(b). In this case, there are no stable sech-type solitons, and all initial perturbations decay into the algebraic soliton with $\omega = 0$ for a finite time.

Finally, let us consider the special case $\Delta = 0$. For small ω we can neglect the last term in Eq. (39) and obtain the simple analytical solution,

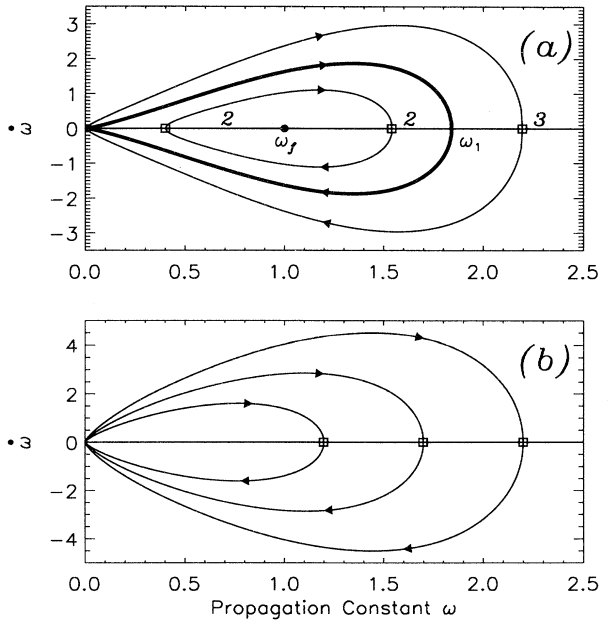

FIG. 8. Phase trajectories of the dynamical system (39) describing transformations of the algebraic soliton for (a) $N_0 > N_{cr}$ and (b) $N_0 < N_{cr}$.

$$\omega = \omega_0 \left(1 - \frac{t}{t_0}\right)^4. \tag{40}$$

For $t_0 < 0$ Eq. (40) describes slow evolution of the initial soliton with an infinitely small ω_0 to a stable sech-type soliton. For the other case, $t_0 > 0$, the soliton propagation constant ω vanishes for a finite time $t = t_0$. In both cases, the solution (40) indicates a weak, powerlike instability of the soliton with an infinitely small ω_0.

When the soliton propagation constant ω approaches zero, our asymptotic theory described in Sec. II becomes invalid because the assumption of the fast, exponential decay of the soliton asymptotics at $|x| \to \infty$ is violated. Therefore we are not able to consider the further evolution of the algebraic soliton in the framework of our approach. The reason for that is the generation of a smooth radiation field which finally decays into linear dispersive waves. Indeed, this process is clearly seen from the numerical results for the model (18) at $p = 1$ reported in Ref. [16]. On the other hand, a slow evolution of the soliton of slightly larger amplitude must be stabilized at larger values of ω where the corrections to the function $N_s(\omega)$ become important, and this leads to the amplitude oscillations around a stable state corresponding to a sech-type soliton, as we have discussed above (see also Ref. [16]).

Thus the algebraic soliton (21) at $p = 1$ displays basically the same critical properties as the sech-type solitons in the critical collapse models (see Sec. IV). Note that the similar critical properties remain valid also for the algebraic solitons for $p < 1$ because the energy $N_s(\omega)$ does not vanish at $\omega \to 0$ when the sech-type soliton (19) transforms into the algebraic solitary wave (21) (for the case $\sigma = -1$, $N_s \to 0$ for $\omega \to 0$ at $p < 4$). Instead, the soliton energy realizes a minimum value exactly at the algebraic soliton, which is equal to

$$N_{cr} \equiv N_s(\omega \to 0; p, 1) = \frac{\sqrt{\pi} 2^{2/p-1} \Gamma(2/p - 1/2)}{p\Gamma(2/p)}, \qquad p < 4. \tag{41}$$

It is clear that any soliton perturbation with $N_0 < N_{cr}$ will lead to dispersion, and any with $N_0 > N_{cr}$ to oscillations near the stable sech-type soliton. For the critical case, $N_0 = N_{cr}$, both scenarios can occur depending on the type of perturbation. Moreover, we can find an explicit solution to the NLS equation linearized around the algebraic soliton background which indicates both types of the soliton dynamics as a weak instability of the algebraic solitons. This solution is just an expansion of the soliton solutions in the form (4) and (19) in a Taylor series for small ω,

$$\Psi = \Phi + \omega\,(it\Phi + \Phi_1) + O(\omega^2), \tag{42}$$

where Φ is given by Eq. (21) and Φ_1 is

$$\Phi_1 \equiv \left.\frac{\partial \Phi}{\partial \omega}\right|_{\omega=0} = \frac{(3 - 6p^2x^2 - p^4x^4)}{12p(1 + p^2x^2)}\Phi. \tag{43}$$

For $p < 1$ the first correction Φ_1 is spatially localized and the expansion (42) gives a solution of the linearized problem which grows in time. Of course, such solutions exist also for sech-type solitons but they do not usually indicate a real instability because it is always possible *to renormalize* the soliton propagation constant, obtaining another sech-type soliton which belongs to the same family of localized solutions. However, the algebraic soliton has no such (free) parameter. Therefore, the development of a small perturbation (42) changes drastically the form of the algebraic soliton, transforming it into a sech-type soliton, if the initial perturbation gives a correction with $\omega > 0$, or into linear dispersive waves, if the initial perturbation leads to $\omega < 0$. Thus our analytical results prove the hypothesis of Ref. [16] that algebraic solitons, which are the degenerate case of sech-type solitons when $\omega \to 0$, are inherently but weakly unstable for $p < 1$.

VI. CONCLUSIONS

We have shown that the long-term dynamics of (bright) solitons in the generalized nonlinear Schrödinger equation depends crucially on the type of balance between nonlinearity and dispersion (or diffraction). When this balance is destroyed and bright solitons may become unstable, they transform into other (stable) solitons corresponding to the same value of the integrals of motion $N(\omega)$. If dispersion dominates, the instability of the solitons leads to their slow decay into small-amplitude dispersive wave packets. Otherwise, the soliton amplitude grows and, for some types of nonlinearity, it results in a pulse collapse. In addition, near the instability threshold we have found that the stable solitons have *an internal oscillating mode.* The existence of this mode is responsible for long lived oscillating behavior of the perturbed stable solitons. However, unlike the integrable NLS equation possessing exact breatherlike oscillating solutions, the amplitude oscillations in the nonintegrable models (1) are *quasistationary* and they decay slowly due to emission of a very small amount of radiation.

The analytical theory developed in this paper seems to be a rather powerful and universal tool for studying the soliton instabilities in the problems of one- (and many-) dimensional wave propagation. This approach can give a full analytical description of the long-term dynamics of slightly perturbed unstable solitons for the NLS-type model (1) with a rather general nonlinearity. For example, our analytical theory immediately explains the interesting phenomenon which is known as *bistability* of optical solitons (see Refs. [5,6]). For this phenomenon, the soliton energy is presented by *a three-valued function* $N_s(\omega)$. It is clear that near the instability threshold the quadratic approximation of this function is not valid and, instead, we should use the approximation in the form of a cubic polynomial. This leads to a *cubic*, rather than quadratic, nonlinear model similar to Eq. (17) which describes the transformation of the unstable solitons at the intermediate branch of the function $N_s(\omega)$ to one of stable solitons of larger or smaller amplitudes and periodic long lived oscillations near a stable soliton state.

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APPENDIX

Here we demonstrate explicitly how to calculate the coefficients in Eq. (13). To do this, we substitute the asymptotic expansion (6) into Eq. (11) and obtain the following equation:

$$N_0 = N_s(\omega) + \int_{-\infty}^{+\infty} \left\{ \Phi\phi_2 + \frac{1}{2}\left(\frac{d\omega}{dt}\right)^2 w_1^2 \right\} dx, \quad \text{(A1)}$$

where N_0 is the conserved energy of the initially perturbed soliton, $N_s(\omega)$ is defined in Eq. (9), and the functions w_1 and ϕ_2 can be found from the linear inhomogeneous equations (7) and (10). Now we use the fact that these functions are exponentially localized. Under this condition, we can simplify the integral terms of Eq. (A1) expressing Φ from Eq. (5), integrating by parts, and using Eq. (10),

$$\begin{aligned}\int_{-\infty}^{+\infty} \Phi\phi_2 dx &= \int_{-\infty}^{+\infty} \phi_2 \left\{ \frac{d^2}{dx^2}\left(\frac{\partial\Phi}{\partial\omega}\right) - \omega\frac{\partial\Phi}{\partial\omega} + [f(\Phi) + f'(\Phi)\Phi]\frac{\partial\Phi}{\partial\omega} \right\} dx \\ &= \int_{-\infty}^{+\infty} \left\{ \frac{d^2\omega}{dt^2} w_1 \frac{\partial\Phi}{\partial\omega} + \left(\frac{d\omega}{dt}\right)^2 \left[\frac{\partial w_1}{\partial\omega}\frac{\partial\Phi}{\partial\omega} - \frac{1}{2} f'(\Phi) w_1^2 \frac{\partial\Phi}{\partial\omega} \right] \right\} dx. \end{aligned} \quad \text{(A2)}$$

As a result of this straightforward procedure, we obtain a second-order differential equation for the function $\omega(t)$,

$$N_0 = N_s(\omega) + M_s(\omega)\frac{d^2\omega}{dt^2} + K_s(\omega)\left(\frac{d\omega}{dt}\right)^2, \quad \text{(A3)}$$

where the coefficient $M_s(\omega)$ can be transformed to the form given by Eq. (14) with the help of Eq. (8) and integration by parts. As for the other coefficient $K_s(\omega)$, it allows the further transformation,

$$\begin{aligned} K_s(\omega) = \frac{1}{2}\frac{dM_s(\omega)}{d\omega} + \frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{\partial w_1}{\partial\omega}\frac{\partial\Phi}{\partial\omega} - w_1\frac{\partial^2\Phi}{\partial\omega^2} \right. \\ \left. + w_1^2 - f'(\Phi) w_1^2 \frac{\partial\Phi}{\partial\omega} \right) dx \, . \end{aligned} \quad \text{(A4)}$$

Next, we express the function $\partial^2\Phi/\partial\omega^2$ from Eq. (7) and find that the second integral is *identically equal to zero.* Finally, Eq. (A3) with $K_s(\omega) = (1/2)(dM_s/d\omega)$ coincides with Eq. (13), and this simplified form allows us to find the first integral to this equation as Eq. (15). It is important to emphasize that for an *arbitrary* coefficient $K_s(\omega)$ Eq. (A3) cannot be further integrated.

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