Large time nonequilibrium dynamics of a particle in a random potential

Leticia F. Cugliandolo*
Service de Physique de L'Etat Condensé, Commissariat à l'Energie Atomique, Orme des Merisiers, Centre d'Etudes Nucléaires de Saclay, 91191, Gif-sur-Yvette Cedex, France

Pierre Le Doussal†
CNRS, Laboratoire de Physique Theorique de l'Ecole Normale Superieure, 24 rue Lhomond, 75231 Cedex 05, Paris, France

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We study the nonequilibrium dynamics of a particle in a general N-dimensional random potential when $N \to \infty$. We demonstrate the existence of two asymptotic time regimes: (i) stationary dynamics and (ii) slow aging dynamics with violation of equilibrium theorems. We derive the equations obeyed by the slowly varying part of the two-time correlation and response functions and obtain an analytical solution of these equations. For short-range correlated potentials we find that (i) the scaling function is nonanalytic at similar times and this behavior crosses over to ultrametricity when the correlations become long range and (ii) aging dynamics persists in the limit of zero confining mass with universal features for widely separated times. We compare the numerical solution to the dynamical equations and generalize the dynamical equations to finite N by extending the variational method to the dynamics.

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I. INTRODUCTION

A major theoretical challenge, as well as an important issue for numerous experimental systems, is to understand the nonequilibrium dynamics of elastic manifolds, with or without internal periodic structure, in quenched random media [1]. The corresponding models, such as the model of an interface subjected to quenched impurities or the sine-Gordon model with phase randomness, have been studied for some time. Some progress has been made in the description of the statics [2–6], but little is still known about the dynamics, especially about the nonequilibrium features of the relaxations. It is by now well established that these systems exhibit glassy behavior that presents similarities to the glassiness of spin glasses, such as slow relaxations, history dependence, and strong sensitivity to changes in the external parameters [7,8].

Recently, progress has been made in solving the nonequilibrium dynamics of mean-field models of spin-glasses [9–13]. A method to study analytically the large time nonequilibrium dynamics has been introduced [10,11]. This purely dynamical method has, on the one hand, some formal connections with the replica symmetry breaking schemes introduced to study the static properties, i.e., the Gibbs measure, of these models. On the other hand, the method yields interesting additional information specific to the dynamics. It allows us to find a solution that exhibits two asymptotic time regimes: a stationary dynamics regime for large but similar times and a slow aging regime for widely separated times. Most important is that it allows one to establish contact with the essentially nonequilibrium experimental observations, namely, the slow relaxations and the aging effects [7]. It is then natural to extend this method to study the dynamics of a broader class of systems with slow relaxations, including the random manifold problem.

For problems such as the manifold in a random medium, the mean-field limit corresponds to the dimension of the embedding space N going to infinity, i.e., it is represented by a field theory with a large number of components. In this case one can derive a closed set of dynamical equations and the above dynamical method can be applied, yielding exact results. Realistic systems, however, are embedded in a finite-dimensional space, i.e., N is finite, and one must make some approximation to obtain closed dynamical equations. A way to obtain these equations is to extend the Gaussian variational approximation (GVA) to the dynamics.

The model of a manifold of internal dimension D embedded in a random medium of dimension N is described, in terms of an N component displacement field $\boldsymbol{\phi}$, $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)$, by the Hamiltonian [4,5]

$$H = \int d^D x \, \left(\frac{c}{2} \, [\nabla \boldsymbol{\phi}(x)]^2 + V(\boldsymbol{\phi}(x), x) + \frac{\mu}{2} \, \boldsymbol{\phi}^2 \right) \,, \tag{1.1}$$

where μ is a mass that effectively constraints the manifold to fluctuate in a restricted volume of the embedding space and V is a Gaussian random potential with correlations

$$\overline{V(\boldsymbol{\phi}, x) V(\boldsymbol{\phi}', x')} = -N \delta^D(x - x') \, \mathcal{V}\left(\frac{(\boldsymbol{\phi} - \boldsymbol{\phi}')^2}{N} \right) \,. \tag{1.2}$$

*Electronic address: leticia@amoco.saclay.cea.fr
†Electronic address: ledou@physique.ens.fr

Our aim is to study the nonequilibrium dynamics of this model for a general random potential. The dynamics that we consider is the Langevin dynamics

$$\frac{\partial \boldsymbol{\phi}(x,t)}{\partial t} = -\frac{\delta H}{\delta \boldsymbol{\phi}(x,t)} + \boldsymbol{\eta}(x,t), \tag{1.3}$$

with $\langle \eta_\alpha(x,t)\eta_\beta(x',t')\rangle = 2T\, \delta_{\alpha\beta}\, \delta^D(x-x')\delta(t-t')$.

To summarize, in this paper we concentrate on the problem of a particle moving in an N-dimensional random potential. It corresponds to the limit of a manifold with zero-internal dimension [14,15] $D = 0$. The zero-dimensional case is a necessary step before studying the model in finite D and a good trial ground for the method. Schematically we do the following. First, we study analytically the exact dynamical equations in the mean-field limit $N \to \infty$. Second, we derive the dynamical equations for finite N within the extended GVA and apply the same dynamical method to the approximated set of equations. We shall present the analysis of the finite-dimensional models ($D > 0$) with the necessary extensions of the method in a separate work [16].

The quantities of interest in the long-time nonequilibrium dynamics are the two-time correlation and response functions. In the $D = 0$ model we define

$$C(t,t') = \frac{1}{N}\, \overline{\langle \boldsymbol{\phi}(t)\boldsymbol{\phi}(t')\rangle}, \quad R(t,t') = \frac{1}{N}\, \left.\overline{\frac{\delta\langle \boldsymbol{\phi}(t)\rangle}{\delta \boldsymbol{f}(t')}}\right|_{\boldsymbol{f}=0}, \tag{1.4}$$

where $f(t')$ is a small perturbation applied at time t'. For the models we consider here it is also convenient to use the mean-squared displacement correlation function

$$\begin{aligned} B(t,t') &= C(t,t) + C(t',t') - 2C(t,t') \\ &= \overline{\langle [\phi(t) - \phi(t')]^2 \rangle}\ . \end{aligned} \tag{1.5}$$

A. Infinite-dimensional models $N \to \infty$

It is well known that large-N limits lead to simplifications in field theories and this property has been used to study problems in very different areas of theoretical physics [17,18]. Thus it is natural to study this limit for the present model.

There are several models that can be studied for $N \to \infty$ and $D = 0$ corresponding to different choices of the random potential correlation $\mathcal{V}(z)$. A common choice is

$$\mathcal{V}(z) = \frac{(\theta + z)^{1-\gamma}}{2(1-\gamma)}\ . \tag{1.6}$$

In what follows we shall refer to this model as the power-law model. There are two physically distinct cases. If $\gamma < 1$ the correlations grow with the distance z and the potential is called "long range." If $\gamma > 1$ the correlations decay with the distance and the potential is called "short range." The statics of these models has been studied in detail by Engel [19] with the replica trick. In this work the author has applied a general procedure that uses the (static) Gaussian variational method complemented by the replica analysis proposed by Mézard and Parisi [5] to the zero-dimensional case. The short-range case is considered to be solved with a one-step replica symmetric ansatz while the long-range case needs a full replica symmetry breaking scheme.

One can also define spherically constrained models letting the mass μ become a time-dependent function $\mu(t)$ related to the Lagrange multiplier enforcing the N-spherical constraint. Model (1.1) then represents a manifold of internal dimension D embedded on the surface of an N-dimensional sphere that can be related to spherically constrained spin-glass models.

In particular, the p-spin spherical spin-glass model defined by the Hamiltonian [20]

$$V(\phi) = \sum_{i_1<\cdots<i_p}^{N} J_{i_1,\ldots,i_p}\phi_{i_1}\cdots\phi_{i_p}, \tag{1.7}$$

with the additional spherical constraint $\sum_{i=1}^{N}\phi_i^2 = N$ and $J_{i_1,\ldots,i_p}$ taken from a Gaussian distribution, is contained in this family of models. It is recovered for $D = 0$ and the correlation of the random potential $\mathcal{V}$ given by

$$\mathcal{V}(x) = -\frac{1}{2}\left(1 - \frac{x}{2}\right)^p\ . \tag{1.8}$$

The replica analysis yields an exact replica symmetric solution [21] when $p = 2$. However, this is a marginal case and an infinitesimal departure from $p = 2$ makes the exact solution be one-step replica symmetry breaking [20]. The nonequilibrium dynamics has been studied in Refs. [22,13] for $p = 2$ and in Ref. [10] when $p \geq 3$.

We study these models as follows [10–13]. The thermodynamic limit $N \to \infty$ is taken initially and then the limit of large times is eventually taken. The dynamics starts at a finite initial time $t_0 = 0$. The initial condition is chosen at random from a Gaussian distribution. The models we study have many ergodic components and the system *is not able* to jump from one component to another since, by definition, the barriers separating different ergodic components are divergent and hence unsurmountable. However, almost all the initial conditions do not lead the system to equilibrium at a large time. A clear realization of this is observed when studying the dynamics of the $p = 2$ spherical spin glass [13] starting from a generic initial condition for which the system evolves indefinitely without ever reaching any of the two equilibrium states.

The two assumptions that describe quantitatively this scenario are the *weak ergocity breaking* hypothesis [23,10] and the *weak long-term memory* hypothesis [10,11]. The former states that even in the limit of large time the particle is always able to escape from its previous positions and never gets stuck in a local equilibrium. The latter states that at a sufficiently large time t the particle forgets any small perturbation applied during any previous *finite* time interval. For instance the "remanent magnetization" associated with a field applied during a finite time interval $[0, t_w]$ decays to zero for a sufficiently large t. For mean-field spin-glass models the relaxations slow down as time passes, though the systems do not reach any kind of equilibrium. There is no equilibration time $t_{\rm eq}$ such that for all the subsequent times the "equilibrium-dynamics" theorems, i.e., time-translation invariance for all the correlation functions and the fluctuation-dissipation theorem (FDT), hold. The systems do not visit the equilibrium states at any time. Importantly enough, this scenario is what is observed during experimental times in real spin glasses. Despite being nonequilibrium, mean-field spin-glass models as well as real spin glasses reach an asymptotic (large times) regime for which some general features can be demonstrated. Generalizations of the equilibrium theorems to the large time nonequilibrium relaxations have been proposed in Refs. [10,11] and we shall apply them here to study model (1.1) for a general $\mathcal{V}$.

It is important to note that this scenario is completely different from the one proposed by Sompolinsky and Zippelius [24,25] to study *the equilibrium properties* of mean-field spin-glass models using a dynamical approach. In the so-called Sompolinsky dynamics one takes a large but *finite* system (finite N). One then chooses the initial time to be $-\infty$, in such a way that one assumes that at finite times the system has arrived at a certain equilibrium state. One takes the inverse order of limits as above, namely, $\lim_{N\to\infty}\lim_{t\to\infty}$, to allow for jumping from one ergodic component to another [25].

The dynamical analysis of Sompolinsky and Zippelius [24], which had been originally proposed to study the decay *inside an equilibrium state,* formally carries through to the nonequilibrium analysis. However, its interpretation must be changed. In a phase-space description, in the nonequilibrium approach, instead of representing the decay inside an equilibrium state it represents the rapid decay of correlation and response functions inside some trapping region of phase space.

Later, always referring to equilibrium dynamics, Sompolinsky [25] proposed a formalism to study also the interstate dynamics. In this approach time translational invariance is assumed to hold for all times and hence nonequilibrium and aging effects *are not* captured. Moreover, the method has some problems since when N is finite and times diverge with respect to N quantities such as the autocorrelation function become non-self-averging [26]. The power model (1.1)–(1.6) has been studied previously using the Sompolinsky dynamics by Kinzelbach and Horner [27,28].

An analysis of the out of equilibrium relaxation of the power model (1.1) at $D = 0$ with long-range correlations given by Eq. (1.6) has been carried out by Franz and Mézard [12]. The authors have solved numerically the mean-field dynamical equations and have proposed a scenario of nonoverlapping time domains to account for their numerical results.

In this paper we find that, in the low-temperature phase, for large times $t \geq t'$ such that $(t-t')/t' \ll 1$, the particle undergoes stationary dynamics $B(t,t') = B_F(t-t')$, where $B_F(t-t')$ grows from 0 to the limiting value q. For larger time separation there is an aging regime where $B(t,t')$ further grows slowly from q to b_0. The behavior in the aging regime depends on the range of the correlations of the potential.

(i) For long-range correlations we find that the one-time quantities converge to the values predicted by the statics. This confirms the numerical observations of Franz and Mézard [12] for model (1.1) with $\mathcal{V}$ given by (1.6) and $\gamma < 1$. We demonstrate that $B(t,t')$ satisfies ultrametricity in time and we explicitly compute the rate of violation of the FDT at large times.

(ii) For short-range correlated potentials we find that one-time quantities that involve the aging regime, such as the asymptotic energy density or b_0, differ from their equivalents in the statics. The dynamical phase diagram is different from the equilibrium one and, in particular, the transition in the dynamics survives in the limit of vanishing mass $\mu \to 0$. We find that in the aging regime $B(t,t') = \hat{B}[h(t')/h(t)]$, where generically the scaling function $\hat{B}[\lambda]$ is nonanalytic when $\lambda \to 1$ $(t'/t \to 1)$. This feature is one of the main results of this paper. We find that $\hat{B}[\lambda] - q \propto (1-\lambda)^{\alpha}$ with a nontrivial exponent α, which depends continuously on temperature. Furthermore, in the limit of vanishing mass $\mu \to 0$ we show that, for *all* short-range models, $\hat{B}[\lambda] \sim q + \ln(1/\lambda)$ for $\lambda \ll 1$. This remarkable form implies that the response function at large time separation becomes a function of t' only. This is consistent with some numerical results that we present.

The parameter γ in the definition of $\mathcal{V}$, Eq. (1.6), tunes from long-range correlated potentials ($\gamma \leq 1$) to short-range correlated potentials ($\gamma > 1$). Interestingly enough, this allows us to explicitly show how the ultrametric dynamical solution is approached when $\gamma \to 1$ from above.

The limit of zero mass raises some fundamental questions. For $\mu > 0$, $C(t,t)$ reaches a finite limit when $t \to \infty$. By contrast, at strictly $\mu = 0$, $C(t,t)$ grows unboundedly with time. Although the corresponding unbounded diffusion process deserves further study [16], we show that the method still applies and the limit $\mu \to 0$ can safely be taken for quantities such as $B(t,t')$ and $R(t,t')$.

B. Finite-dimensional models $N < \infty$

The model (1.1) with finite N (and $D = 1,2,3$) has several physically interesting realizations, such as the problem of a manifold pinned by impurities for which the correlator is of the form (1.6). It arises in the study of interfaces in a random field as well as glassy phases of vortices in high-T_c superconductors. Similarly, the sine-Gordon model with random phase disorder (RSGM) arises in the study of several problems including quenched disorder. For example, it describes the glass transition of a surface of a crystal deposited on a disordered two-dimensional substrate. It is also related to the vortex-free XY model in a random field. It is defined in terms of a $N = 1$ phase field ψ by

$$H = \int d^D x \left(\frac{c}{2}\,[\nabla\psi(x)]^2 - \zeta_1(x)\cos[\psi(x)] - \zeta_2(x)\sin[\psi(x)] + \frac{\mu}{2}\,\psi^2 \right), \tag{1.9}$$

where $\overline{\zeta_i(x)\zeta_j(x')} = \delta_{ij}\delta(x-x')$. This model corresponds to the choice of correlator

$$\overline{V(\psi,x)V(\psi',x')} = -\delta^D(x-x')\cos(\psi-\psi') \ . \quad (1.10)$$

The statics of the RSGM has been studied using renormalization group techniques [2], a Gaussian approximation (variational method) complemented by the replica trick [6], and with extensive numerical simulations [3].

The statics of model (1.1) for general D and N has been studied using a Hartree, i.e., a Gaussian variational, approximation by Mézard and Parisi [5]. The corresponding mean-field equations are identical to those for N infinite, apart from a replacement of the random potential correlator $\mathcal{V}(x)$ by $\hat{\mathcal{V}}(x)$ defined as $\hat{\mathcal{V}}(\langle\phi\rangle_0^2) \equiv \langle\mathcal{V}(\phi^2)\rangle_0$, where $\langle\ \rangle_0$ denotes a Gaussian average over ϕ. In this paper we extend the GVA to the dynamics by performing a Gaussian decoupling approximation in the exact dynamical equations of motion. As detailed in Appendix A, this shows explicitly that the same replacement of $\mathcal{V}(x)$ by $\hat{\mathcal{V}}(x)$ holds in the dynamics. For instance, the RSGM dynamics is described by Eq. (1.6) with $\mathcal{V}(x) \to \hat{\mathcal{V}}(x) = -\Delta\exp(-x/2)$.

The zero-dimensional version $D = 0$ has been studied as a toy model of the above problems [14,15]. Even if one does not expect, strictly speaking, a glass transition in $D = 0$, this model exhibits at low temperatures several features of a glass that are present in its higher-dimensional versions. In the statics perturbation theory breaks down because of the large number of metastable states and the importance of rare fluctuations. In the dynamics with $\mu > 0$, there is a finite ergodic time $t_{\rm eq}$ beyond which equilibrium dynamics is established for all subsequent times. This time, however, can be very large since barriers grow as powers of μ^{-1} and N. Therefore even in this simple case, there could be an aging regime at intermediate times. If this is the case it is likely that the methods of the present paper, using the GVA, will provide a basis for describing this nonequilibrium regime.

The present zero-dimensional model has also been applied to describe the motion of dislocations or kinks in presence of disorder [29] and, furthermore, when $N = 1$ it is similar to a model describing the dynamics of droplet size in random magnets [30]. In the limit of a zero mass, this model corresponds to the problem of diffusion in a random potential, which has been extensively studied. In the particular case $\gamma = 1/2$ $N = 1$ the model (1.1)–(1.6) is the celebrated Sinai model [31]. More generally, it is known [32] that $\phi^2(t) \sim \ln(t)^{2/(1-\gamma)}$ for any finite N. However, the two-time correlations and thus the aging properties have not been studied previously analytically (see also Ref. [34]), except in the case of an applied force by Feigel'man and Vinokur [33], who showed that broad distributions of trapping times is a possible mechanism for aging.

To summarize, in this paper we study analytically the long-time behavior of model (1.1) with general $\mathcal{V}$ in the simplest case of zero internal dimensions. We use the formalism of Refs. [10,11] and we concentrate on the low-temperature phases. We also reproduce and explain as explicitly as possible some of the necessary calculations. The paper is organized as follows. In Sec. II we present the general mean-field dynamical equations for the zero internal dimension case. In Sec. III we describe the separation of the asymptotic dynamics in a stationary regime for long but similar times and a nonstationary regime for long and very different times. We derive the dynamical equations for both regimes. In Sec. IV we review the extensions of the equilibrium dynamics theorems to the asymptotic nonequilibrium relaxation. In Sec. V we present the time-reparametrization-invariant equations for the aging regime. We also discuss the strategy we follow in Secs. VI and VII to find their solutions. In Sec. VI we use a "one-blob" ansatz to solve the short-range models and in Sec. VII we use an ultrametric ansatz to solve the long-range models. Finally, in Sec. VIII we present our conclusions.

II. MEAN-FIELD EQUATIONS IN THE LARGE TIME LIMIT

The general dynamical equations for the two-time response function $R(t,t')$ and correlation function $C(t,t')$ for the model defined by Eq. (1.1) in the limit $N \to \infty$ were derived by Franz and Mézard [12]. They were presented previously in Ref. [10] for the special case of the p-spin spherical model. The dynamical equations *assuming* time-translation invariance (TTI) at all times for all the two-time functions were presented in Refs. [27,28,35,36] for model (1.1) with short and long correlation, the p-spin spherical spin glass, and the RSGM, respectively. In these references the equilibrium dynamics has been studied in the manner of Sompolinsky.

In Appendix A we sketch a derivation of the general dynamical equations using the Martin-Siggia-Rose formalism complemented by a Gaussian approximation that becomes exact in the $N \to \infty$ limit. The mean-field dynamical equations in terms of C and R defined in Eq. (1.4) read

$$\begin{aligned}\frac{\partial R(t,t')}{\partial t} &= -\mu\, R(t,t') + 4\int_0^t ds\ \mathcal{V}''(B(t,s))\ R(t,s) \\ &\quad \times[R(t,t') - R(s,t')] \ , \quad (2.1)\\ \frac{\partial C(t,t')}{\partial t} &= -\mu\, C(t,t') + 2\int_0^{t'} ds\ \mathcal{V}'(B(t,s))\ R(t',s) \\ &\quad +4\int_0^t ds\ \mathcal{V}''(B(t,s))\ R(t,s) \\ &\quad \times[C(t,t') - C(s,t')] + 2T\ R(t',t) \ . \quad (2.2)\end{aligned}$$

We use the Ito convention $\lim_{\epsilon\to 0} R(t,t-\epsilon) = 1$, $R(t,t) = 0$. Using the fact that the last term in Eq.(2.2) vanishes for $t > t'$ one also has

$$\begin{aligned}\frac{1}{2}\frac{dC(t,t)}{dt} &= -\mu C(t,t) + T + 2\int_0^t ds\ \mathcal{V}'(B(t,s))\ R(t,s) \\ &\quad +2\int_0^t ds\ \mathcal{V}''(B(t,s))\ R(t,s) \\ &\quad \times[C(t,t) - C(s,s) + B(t,s)] \ . \quad (2.3)\end{aligned}$$

For these problems it is convenient to use the mean-squared displacement correlation function $B(t,t') = C(t,t) + C(t',t') - 2C(t,t') = \overline{\langle[\phi(t) - \phi(t')]^2\rangle}$ that satisfies, for $t > t'$,

$$\begin{aligned} \frac{1}{2}\frac{\partial B(t,t')}{\partial t} &= -\frac{\mu}{2}\,[C(t,t) - C(t',t') + B(t,t')] \\ &\quad +2\int_0^t ds\, \mathcal{V}'(B(t,s))\,[R(t,s) - R(t',s)] \\ &\quad +T + 2\int_0^t ds\, \mathcal{V}''(B(t,s))\,R(t,s) \\ &\quad \times[B(t,s) + B(t,t') - B(s,t')]\,. \end{aligned} \tag{2.4}$$

Note that one can use $B(t,t')$ $(t > t')$ rather than $C(t,t')$ since Eqs. (2.1), (2.3), and (2.4) are a complete description of the problem together with the identities $B(t,t) = 0\ \forall t$. In these equations there are two external parameters T and μ.

In deriving these equations a mean over the random initial condition $\phi(t=0)$ [with a Gaussian distribution of variance $C(0,0)$] is implicit. One could certainly obtain the dynamical equations for a particular initial condition $\phi(t=0) = \phi_0$ or a different distribution of initial conditions and attempt to study their effects in detail. We shall not do so here, but we shall restrict our analysis to the study of the above dynamical equations.

The time-dependent energy density [12] is

$$\mathcal{E}(t) = \frac{\mu}{2}\,C(t,t) - 2\int_0^t ds\, \mathcal{V}'(B(t,s))\,R(t,s)\,. \tag{2.5}$$

In order to obtain spherically constrained models one has to, on the one hand, let μ be a function of time $\mu(t)$ and, on the other hand, determine $\mu(t)$ by imposing explicitly the spherical constraint. If $C(t,t)$ is set to $\tilde{q}$, then Eq. (2.3) implies

$$\begin{aligned} \mu(t)\,\tilde{q} &= T + 2\int_0^t ds\,[\mathcal{V}'(B(t,s)) + \mathcal{V}''(B(t,s))\,B(t,s)] \\ &\quad \times R(t,s)\,. \end{aligned} \tag{2.6}$$

In particular, if $\tilde{q} = 1$ and $\mathcal{V}$ is given by Eq. (1.8) and we compare with the equation determining the Lagrange multiplier $z(t)$ that enforces the spherical constraint in the p-spin spherical spin glass [10], we obtain

$$T\,z(t) = \mu(t) + \frac{p-1}{2}\int_0^t ds\,[C(t,s)]^{p-2}\,R(t,s)\,. \tag{2.7}$$

The dynamics is described by the set (2.1)–(2.4) of non-linear coupled integro-differential equations that admit a unique solution for each "initial autocorrelation" $C(0,0)$. The dynamical set of equations is *causal.* One can then use a numerical algorithm to iterate the equations and construct the solution step by step in time. A numerical analysis of this type of the power model (1.1)–(1.6) with $\gamma < 1$ (long range) has been carried out in Ref. [12].

III. STATIONARY AND NONSTATIONARY DYNAMICS

In the high-temperature phase both TTI and the FDT hold. Equations (2.1)–(2.4) reduce to a single equation. In Refs. [27,28] the model (1.1)–(1.6) with short- and long-range correlations have been studied, respectively. The detailed analysis of the resulting equation shows that there is a critical curve $T_c(\mu)$, below which no solution with these characteristics exists. This marks the end of the high-temperature phase. Similar situations occur in the p-spin spherical model [35] and in the random sine-Gordon model [36].

The analytical study of the mean-field dynamical equations in the low-temperature phase requires the use of certain hypotheses. For a purely relaxational dynamics such as a Langevin process that approaches an equilibrium situation, one can show that the autocorrelation function is a monotonic function of the time difference τ and that the response function is a positive function of its argument τ. For a general nonequilibrium process the property of monotonicity is not obvious. However, for the kind of models we study it is a quite natural starting point to assume that the systems continuously drift away. As a consequence, though the displacement function *is not* an exclusive function of the time difference τ for all times, it is assumed to increase monotonically when the separation between the two times increases:

$$\frac{\partial B(t,t')}{\partial t} \geq 0, \quad \frac{\partial B(t,t')}{\partial t'} \leq 0\,. \tag{3.1}$$

The weak ergodicity breaking hypothesis [23,10] includes the assumption above and, if the particle moves in an infinite-dimensional space and there is a finite mass,

$$\begin{aligned} &\lim_{t\to\infty} B(t,t') = b_0 \quad \forall\ t' \text{ finite}\,, \\ &\lim_{\tau\to\infty}\lim_{t'\to\infty} B(t = \tau + t', t') = q\,, \end{aligned} \tag{3.2}$$

and

$$\lim_{t\to\infty} C(t,t) = \tilde{q}\,. \tag{3.3}$$

If the mass is zero from the start the particle diffuses and, in principle, both b_0 and $\tilde{q}$ are nontrivial time-dependent functions of t that tend to infinity when $t \to \infty$.

Hence, in the large-time limit, one can separate the relaxation that takes place for large and similar times such that $(t-t')/t' \ll 1$, during which the displacement grows from 0 to q, from the large and widely separated times relaxation during which the displacement further grows from q to b_0. The nature of the relaxations in these regimes is clearly different. The former is given by an "equilibrium" or stationary dynamics while the latter is given by a "nonequilibrium" or nonstationary dynamics with its possibly associated aging effects [10–12]. This hypothesis has been verified for different models with simulations, numerical analysis, and exact results. For instance, the Monte Carlo simulations of various finite-dimensional and mean-field spin-glass models [37], the numerical study of the model defined by (1.1) for long-range correlations [12], and the exact analytical results [13] for the model defined by Eqs. (1.7) and (1.8) when $p=2$ provide evidence for this scenario.

In the large-t' limit we thus separate the t dependence in two regimes: for small time differences $(t-t')/t' \ll 1$ there is equilibrium dynamics obeying the FDT and TTI

$$\begin{aligned} \lim_{t'\to\infty} R(\tau+t',t') &= r_F(\tau) \ , \\ \lim_{t'\to\infty} B(\tau+t',t') &= b_F(\tau) \ , \\ \lim_{t'\to\infty} C(\tau+t',t') &= c_F(\tau) \ . \end{aligned} \tag{3.4}$$

The response and displacement functions are related by the FDT

$$r_F(\tau) = \frac{1}{2T}\,\frac{\partial b_F(\tau)}{\partial\tau}\theta(\tau) = -\frac{1}{T}\,\frac{\partial c_F(\tau)}{\partial\tau}\,\theta(\tau) \ . \tag{3.5}$$

Note that, by definition, $\lim_{\tau\to\infty} b_F(\tau) = q$ and thus the displacement is bounded in the equilibrium dynamics regime.

The aging regime is defined by taking the limit of large times $t,t' \to \infty$ in such a way that the displacements $B(t,t')$ is fixed but larger than $b_F(\infty)$, i.e., $B(t,t') > q$. It corresponds to larger time differences and exhibits slow dynamics violating both the FDT and TTI

$$R(t,t') = r(t,t') \ , \quad B(t,t') = b(t,t') \ , \quad C(t,t') = c(t,t') \ . \tag{3.6}$$

The limiting values for b are then

$$\begin{aligned} &b_F(t,t) = 0, \qquad \lim_{\tau\to\infty} b_F(\tau) = q, \\ &b(t,t_-) = q, \qquad b(t,0_+) = b_0 \end{aligned} \tag{3.7}$$

[see Eqs. (3.2) and (3.3)]. We denote the initial time of the asymptotic regime $s=0_+$. It has to be understood as a very large time, as opposed to $s=0$, which is the actual initial time and hence finite. One thus has $b(t,0) \geq b(t,0_+) = b_0$. For fixed t the aging regime extends from $t'=0_+$ to $t'=t_-$ and the FDT regime from $t'=t_-$ to $t'=t$. We shall use $\lim_{t\to\infty} t_-/t = 1$. Calling q_0, q_1, and $\tilde{q}$ the large time limiting values of C

$$\begin{aligned} &C(t,t) = c_F(0) = \tilde{q}, \qquad \lim_{\tau\to\infty} c_F(\tau) = q_1 \ , \\ &c(t,t_-) = q_1 \ , \qquad c(t,0_+) = q_0 \ , \end{aligned} \tag{3.8}$$

it follows from these definitions that

$$q = 2\,(\tilde{q} - q_1), \qquad b_0 = 2\,(\tilde{q} - q_0) \ . \tag{3.9}$$

Note that the corresponding definitions in the statics are as follows [5,19]. One defines $q_{ab} = \frac{1}{N}\langle \phi_a \phi_b \rangle$, where a,b are replica indices. The equal time correlation $\tilde{q}^{\rm stat}$ corresponds to q_{aa}. Within a replica symmetry breaking the solution q_{ab} for $a \neq b$ is parametrized by $q(u)$, $0 < u < 1$. While the precise shape of $q(u)$ depends on the model, one has generally a constant $q(u) = q_1^{\rm stat}$ from $u_c < u < 1$ and $q(u) = q_0^{\rm stat}$.

The second and crucial assumption can be called weak long-term memory [10]. It states that the response to any small perturbation applied during a finite interval $[0,t_w]$ eventually decays to zero for a large enough subsequent time t. In other words, the integral of the response function $R(t,t')$ over t' in the interval $t' \in [t_1,t_2]$ with $t_2 - t_1$ finite vanishes for long enough t. In a spin system this means that the thermoremanent magnetization decays to zero for a long enough time after having switched-off the (small) magnetic field.

This separation in stationary and nonstationary dynamics allows us to write the dynamical equations (2.1)–(2.4) for the two regimes, separating cleanly each contribution. In what follows we shall refer to the former time regime $[(t-t')/t' \ll 1]$ as the FDT regime and to the latter time regime as the aging regime. In Sec. V we shall present some numerical evidence for the separation in these two distinct time regimes.

A. The FDT regime

The equations obeyed by the FDT part $[(t-t')/t' \ll 1]$ read

$$\begin{aligned} \frac{db_F(\tau)}{d\tau} = {} & 2T + (-\mu + M)\ b_F(\tau) \\ & -\frac{2}{T}\int_0^\tau d\tau'\ \mathcal{V}''(b_F(\tau-\tau')) \\ & \times \frac{\partial b_F(\tau-\tau')}{\partial(\tau-\tau')}\ b_F(\tau'), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \frac{dr_F(\tau)}{d\tau} = {} & (-\mu + M)\ r_F(\tau) \\ & -4\int_0^\tau d\tau'\ \mathcal{V}''(b_F(\tau-\tau'))\ r_F(\tau-\tau')\ r_F(\tau') \ , \end{aligned} \tag{3.11}$$

$$\begin{aligned} c_F(0) = \frac{1}{\mu}\Bigg(& T + \frac{q}{T}\ \mathcal{V}'(q) \\ & +2\lim_{t\to\infty}\int_0^t ds\ [\mathcal{V}''(b(t,s))\ r(t,s)\ b(t,s) \\ & +\mathcal{V}'(b(t,s))\ r(t,s)]\Bigg) \ , \end{aligned} \tag{3.12}$$

with

$$M \equiv 4\lim_{t\to\infty}\int_0^t ds\ \mathcal{V}''(B(t,s))\ R(t,s), \tag{3.13}$$

$$\begin{aligned} M_F &\equiv 4\int_0^\infty d\tau'\ \mathcal{V}''(b_F(\tau'))\ r_F(\tau') \\ &= \frac{2}{T}\,[\mathcal{V}'(q) - \mathcal{V}'(0)] \ , \end{aligned} \tag{3.14}$$

$$\overline{M} = M - M_F = 4\lim_{t\to\infty}\int_0^t ds\ \mathcal{V}''(b(t,s))\ r(t,s) \ . \tag{3.15}$$

It is easy to prove that because of the FDT relation

(3.5) the first two equations collapse and yield only one equation involving b_F. However, it also contains contributions from the aging regime through a single, *a priori* unknown quantity, the "anomaly" $\overline{M}$. We shall see that, conversely, the equations written in the aging regime do not include the details of the FDT solution. Thus, in order to find the full solution for the FDT regime the correct procedure is to first obtain the solution for the aging regime, compute the anomaly, insert it back into the FDT equation, and then solve for the FDT regime.

The equation for the FDT regime is formally identical to the equation one finds studying the equilibrium dynamics in the manner of Sompolinsky, though the meaning of the anomaly is different in the two contexts. In the nonequilibrium situation the anomaly represents the memory of the system and, as can be clearly seen from its definition (3.15), it is associated with an integration from the initial time of the asymptotic time regime up to the final time $t \to \infty$. (An alternative interpretation of the anomaly follows from the study of the nonequilibrium dynamics of finite-dimensional models, such as the evolution of a manifold in a random potential, and the formal comparison with the results from the replica analysis. See the discussion presented in Ref. [16]). In the equilibrium dynamics instead, the initial time is chosen to be $-\infty$ in such a way as to propose that the system is at some equilibrium state at a finite time and that it then is able to visit different ergodic components at diverging (with N) time scales. The anomalous term is then the contribution from all times starting at $-\infty$ and it is related to contributions from the barrier crossing.

Despite the different interpretation for the anomaly one can still borrow or at least formally compare some of the results concerning this equation from the previous works in which the equilibrium dynamics has been studied. The equilibrium analysis has been done in great detail by Kinzelbach and Horner [27,28] for the short- and long-range model (1.1) and by Cule and Shapir [36] for the RSGM. In the high-temperature phase one finds $\overline{M} = 0$. However, for $T < T_c(\mu)$ the FDT equation has *no solution* for arbitrary large time differences if one keeps $\overline{M} = 0$. There is a critical time-difference scale beyond which the FDT solution becomes unstable. It is then assumed that the equilibrium dynamics is "marginal," meaning that the anomaly is chosen to have the minimal value required to remove the instability of the FDT solution [24,27,28].

Here instead we shall first solve for the well-defined aging solution, compute the anomaly, and then return to the study of the asymptote (large time difference τ, i.e., $\lim_{\tau\to\infty}\lim_{\tau/t'\to 0}$) of the FDT regime. It turns out that, for the models studied here, the value of $\overline{M}$ obtained by this method coincides with the one obtained from the postulate of "marginal stability." Thus the detailed behavior in the FDT regime, such as the power law decay of the displacement to the asymptotic value q, also coincides.

B. The aging regime

In Appendix B we derive the set of equations for large and widely separate times $t > t' \gg 1$ and $q < B = b < b_0$. The self-consistent procedure is as follows. In this slowly varying time region, the time derivatives on the left-hand side of Eqs. (2.1)–(2.4) are assumed to be small compared to the right-hand side and so are then neglected. This is related to the weak ergodicity breaking hypothesis that states that, as time elapses, the relaxation of the systems slows down [23,10]. Typically, the integrals on the right-hand side go from $s = 0$ to $s = t$ or t'. In order to approximate them we start by separating the contribution of finite times, namely, integrals going from the actual initial time $s = 0$ to the starting time of the large time regime $s = 0_+$. These are assumed to be small and hence neglected. This assumption is related to the hypothesis of weak long-term memory [10] and it allows us to write the long-time equations exclusively in terms of the long-time functions b and r. Finally, we separate the large time-integration intervals to distinguish the FDT and aging contributions. We are typically left with integrals of slowly varying functions over the whole aging time regime or with integrals of products of slowly and rapidly varying functions over the short time intervals. We then keep the first kind of integral and we approximate the second kind of integral by replacing the slowly varying functions over the whole short integration interval by just the value of the slowly varying function at the integration border.

The large times dynamical equation for the response function r reads

$$
\begin{aligned}
0 = r(t,t') &\left(-\mu + 4\int_0^t ds\; \mathcal{V}''(b(t,s))\, r(t,s) - \frac{2q}{T}\, \mathcal{V}''(b(t,t'))\right) \\
&-4\int_{t'}^t ds\; \mathcal{V}''(b(t,s))\, r(t,s)\, r(s,t') \qquad (3.16)
\end{aligned}
$$

and that for the displacement correlation function b reads

$$
\begin{aligned}
0 = b(t,t') &\left(-\frac{\mu}{2} + 2\int_0^t ds\; \mathcal{V}''(b(t,s))\, r(t,s)\right) + T + \frac{q}{T}[\mathcal{V}'(q) - \mathcal{V}'(b(t,t'))] \\
&+2\int_0^t ds\; \mathcal{V}'(b(t,s))\, r(t,s) - 2\int_0^{t'} ds\; \mathcal{V}'(b(t,s))\, r(t',s) \\
&+2\int_0^t ds\; \mathcal{V}''(b(t,s))\, r(t,s)\, b(t,s) - 2\int_0^{t'} ds\; \mathcal{V}''(b(t,s))\, r(t,s)\, b(t',s) - 2\int_{t'}^t ds\; \mathcal{V}''(b(t,s))\, r(t,s)\, b(s,t')\,. \qquad (3.17)
\end{aligned}
$$

[N.B. We have dropped all the subindices $-,+$ in the limits of the integrals. In particular, we have dropped the $+$ subindex in the zero limits. The zeros and all the (time) integration variables should be interpreted as being in the asymptotic time regimes.] We shall call these two equations the r equation and the b equation, respectively.

The equation for $C(t,t)$ gives, at large t, another equation that contains contributions from the slowly varying parts as well as the FDT parts. In this large time limit, $C(t,t)$ is assumed to have reached its asymptotic value $\tilde{q}$. Thus

$$0 = -\mu\tilde{q} + T + \frac{q}{T}\mathcal{V}'(q) + 2\int_0^t ds\ \mathcal{V}'(b(t,s))\ r(t,s) + 2\int_0^t ds\ \mathcal{V}''(b(t,s))\ r(t,s)\ b(t,s)\ . \tag{3.18}$$

For completeness we give the equation for $c(t,t')$. Using similar methods one has

$$\begin{aligned} 0 = c(t,t')&\left(-\mu + 4\int_0^t ds\ \mathcal{V}''(b(t,s))\ r(t,s)\right) + \frac{q}{T}\mathcal{V}'(b(t,t')) \\ &+2\int_0^{t'} ds\ \mathcal{V}'(b(t,s))\ r(t',s) - 4\int_0^{t'} ds\ \mathcal{V}''(b(t,s))\ r(t,s)c(t',s) - 4\int_{t'}^t ds\ \mathcal{V}''(b(t,s))\ r(t,s)c(s,t')\ , \end{aligned} \tag{3.19}$$

which we shall call the c equation.

We can obtain some relations between the asymptotic values $q, \tilde{q}, b_0$ by considering particular values of the times t, t'. Letting $t' \to t_-$ in the above equations and using the limiting values (3.7)–(3.9) we find two conditions. Indeed, assuming that the integral in the r equation vanishes in this limit, the condition for $r(t,t_-)$ to be nonzero is that the term in large parentheses in the following equation vanishes:

$$0 = r(t,t_-)\left(-\mu + 4\int_0^t ds\ \mathcal{V}''(b(t,s))\ r(t,s) - \frac{2q}{T}\mathcal{V}''(q)\right)\ . \tag{3.20}$$

The solution $r(t,t_-) = 0$ corresponds to the high-temperature phase for which there is no nonequilibrium dynamics. The equation arising from the requirement of the vanishing term involves the anomaly: $\overline{M}$ is equal to the integral above when $t \to \infty$ [see Eq. (3.15)]. Hence, in order to have a nontrivial low-temperature solution, the anomaly must satisfy

$$\overline{M} = \mu + \frac{2q}{T}\mathcal{V}''(q)\ . \tag{3.21}$$

This equation is also the marginality condition that determines the anomaly in the equilibrium dynamics treatment. We see here how it arises naturally in the nonequilibrium approach. In addition, it is related to the condition of vanishing replicon eigenvalue in the static-replica approach (see Sec. VI).

Similarly, the b equation, when $t' \to t_-$, yields

$$0 = -\mu + \frac{2T}{q} + 4\int_0^t ds\ \mathcal{V}''(b(t,s))\ r(t,s). \tag{3.22}$$

Subtracting Eq. (3.22) from Eq. (3.20) yields

$$q^2\,\mathcal{V}''(q) = -T^2\ . \tag{3.23}$$

This equation determines q as a function of T and the potential correlation $\mathcal{V}$. Interestingly enough the equation is valid for all the potentials for $T < T_c$. It is also independent of the mass and of the dynamical solution of the model and hence q is a purely *geometrical* quantity related only to the potential correlation. Besides, it imposes a condition on $\mathcal{V}''$: it must be negative at q in order to let (3.23) have a sensible solution for q. The remaining equation, Eq. (3.20), is important to select the behavior of the models in the aging regime. Combining both equations, one obtains

$$q = \frac{2T}{\mu - \overline{M}}\ . \tag{3.24}$$

One does not obtain any new equation by letting $t' \to t_-$ in the c equation. One can check that when combined with Eq. (3.18) it gives back Eq. (3.22), as expected.

Letting $t' \to 0$ in the b equation, one finds

$$\begin{aligned} 0 = &\left(-\frac{\mu}{2} + 2\int_0^t ds\ \mathcal{V}''(b(t,s))\ r(t,s)\right) b_0 + T \\ &+\frac{q}{T}[\mathcal{V}'(q) - \mathcal{V}'(b_0)] \\ &+2\int_0^t ds\ [\mathcal{V}'(b(t,s)) + \mathcal{V}''(b(t,s))\ b(t,s)]\ r(t,s) \\ &-2\int_0^t ds\ \mathcal{V}''(b(t,s))\ r(t,s)\ b(s,0)\ , \end{aligned} \tag{3.25}$$

where $b_0 \equiv b(t,0)$. This is the equation that fixes b_0. It is important to note that this equation exists only for a strictly nonzero mass. Finally, the energy density (2.5) can be expressed using the separation of the FDT and aging regimes as

$$\mathcal{E}(t) = \frac{\mu}{2}\tilde{q} + \frac{1}{T}[\mathcal{V}(0) - \mathcal{V}(q)] - 2\int_0^t ds\ \mathcal{V}'(b(t,s))\ r(t,s)\ . \tag{3.26}$$

IV. EXTENSIONS OF THE DYNAMICAL THEOREMS TO THE NONEQUILIBRIUM REGIME

In this section we review the extensions of the equilibrium theorems for the nonequilibrium dynamics of systems with slow relaxations proposed in Refs. [10,11]. Since we shall mainly use in our calculations the displacement instead of the correlation function, we here reexpress these extensions in terms of the displacement function. As opposed to the case of mean-field spin-glass models, in this paper we deal with problems that do not have a normalized correlation function. However, if the mass is nonzero, the quadratic potential associated with it ensures that the equal time correlation function reaches a finite limit at large times. This indeed can be easily checked by solving numerically the dynamical equations. Hence, in these cases there is basically no difference between working with the displacement or with the correlation. We shall not discuss here in detail other further extensions associated with the massless cases in which there is unbounded diffusion and the equal time correlation functions do not reach asymptotically a finite limit [16]. We finally discuss the description of possible singularities at the extremities of the aging regime.

A. Definitions

One defines the function $\tilde{X}(t,t')$ as

$$R(t,t') = \tilde{X}(t,t')\,\frac{\partial B(t,t')}{\partial t'}, \tag{4.1}$$

with $t \geq t'$. In the FDT regime $\tilde{X}(t,t') = X_F = -1/(2T)$, while in the aging regime it measures the deviation from the FDT theorem.

The mean-squared separation $B(t,t')$ monotonically increases when the two times t and t' become further separated. It increases from $B(t,t) = 0$ to its maximum value $B(t,0)$, which explicitly depends on the initial value for the correlation $C(0,0)$: $B(t,0) \equiv C(t,t) + C(0,0) - 2C(t,0)$. Since we shall only consider large times, the maximum value that $B(t,t')$ can take is then $B(t,0_+) = b(t,0_+) \equiv b_0$. Note, however, that, in general, $B(t,0)$ may be different from $b(t,0_+)$, more precisely, $B(t,0) \geq b(t,0_+)$. If $B(t,0) > b(t,0_+)$ the weak long-term memory hypothesis requires that $\tilde{X}[t,0_+] = 0$. The monotonicity of $B(t,t')$ allows us to invert the function $B(t,t')$ and to write the function $\tilde{X}$, for instance, in terms of the smaller time and the displacement function $\tilde{X}(t,t') = X(B(t,t'),t')$. One then assumes that in the large-t' limit and for a given $B(t,t')$, $X(B(t,t'),t')$ approaches a function that depends only on $B(t,t')$, i.e.,

$$X(t,t') \sim X(B(t,t')) \qquad \text{for } t' \gg 1\ . \tag{4.2}$$

We shall also assume that $|X(B)|$ is a monotonically decreasing function of B. In Sec. V we shall present results from the numerical solution of the dynamical equations that support the assumption made in Eq. (4.2) and the further assumption of monotonicity. The property of monotonicity permits one to invert $X(B)$ to have a function $B(X)$. This function is useful to compare the asymptotic dynamical values with the equilibrium ones and it is somehow related [10] to the replica Parisi function $b(u) = 2[\tilde{q}^{\rm stat} - q(u)]$.

Similarly, for three given times, because of the monotonicity of the displacement B, one can always write the relation

$$B(t,t') = \tilde{f}(B(t,s),B(s,t'),t')\ . \tag{4.3}$$

One then assumes that when all times are large $t > s > t'$ and for a given $B(t,t')$, the explicit time dependence disappears and

$$B(t,t') = f(B(t,s),B(s,t')), \qquad t > s > t' \to \infty\ . \tag{4.4}$$

The function f connects any three correlation functions; it has been called a triangle relation. Formally, one can also define the reciprocal function $\overline{f}(B',B)$ such that

$$B(s,t') = \overline{f}(B(t,s),B(t,t'))\ . \tag{4.5}$$

Similar, and in the end equivalent, definitions of X and f could be given in terms of $C(t,t')$.

B. Properties of the triangle relations

Let us recall the general properties of the function $f(x,y)$. By definition $f(x,y)$ is associative $f(f(x,y),z) = f(x,f(y,z))$. Since $b(t,t')$ is an increasing function of the separation between t and t' one must have

$$f(x,y) \geq \max(x,y)\ . \tag{4.6}$$

One defines "fixed points" b_i^* such that $f(b_i^*,b_i^*) = b_i^*$. Clearly $f(x,x) > x$ for x between two fixed points. One can show that

$$f(b^*,x) = f(x,b^*) = x\ ,\quad b^* \leq x, \tag{4.7}$$

$$f(x,b^*) = f(b^*,x) = b^*\ ,\quad x \leq b^*. \tag{4.8}$$

Hence the relation between a fixed point b^* and any other point x is in this case "maximum." Between two fixed points $b_1^* < x < b_2^*$, one can show, under the assumption that $f(x,y)$ is *smooth enough* [38], i.e., that it has a formal series expansion, and using the fact that there exists an identity (b_1^*) and a zero (b_2^*),

$$f(x,y) = \begin{cases} f(y,x) & \text{for } f \text{ commutative} \qquad (4.9a)\\ \jmath^{-1}(\jmath(x)\jmath(y)) & \text{for } f \text{ isomorphic to the product} \qquad (4.9b)\end{cases}$$

with $\jmath(b_1^*) = 1$ and $\jmath(b_2^*) = 0$. This implies $y = \overline{f}(x,z) = \jmath^{-1}(\jmath(z)/\jmath(x))$, $z \geq x$.

It is useful to explicitly compute some derivatives of $f(x,y)$ and $\overline{f}(x,z)$ with respect to x, y, and z when they are isomorphic to the product and division. For $x, y \in (b_1^*, b_2^*)$ one finds

$$-\left.\frac{\partial \overline{f}(x,y)}{\partial x}\right|_{x=y\to b_1^*} = \left.\frac{\partial \overline{f}(x,y)}{\partial y}\right|_{x=y\to b_1^*} = 1 \,. \qquad (4.10)$$

Moreover, since $f(b^*,x) = x$ and $\overline{f}(b^*,x) = x$ for all $x > b^*$, then $\partial f(b^*,x)/\partial x = 1$ and $\partial \overline{f}(b^*,x)/\partial x = 1$ for all $x > b^*$. Instead, since $f(b^*,x) = b^*$ and $\overline{f}(b^*,x) = b^*$ for all $x < b^*$, then $\partial f(b^*,x)/\partial x = 0$ and $\partial \overline{f}(b^*,x)/\partial x = 0$ for all $x < b^*$. Thus $\partial f(b^*,x)/\partial x = \theta(x - b^*)$.

It is useful to note that the form (4.9b) can be expressed as

$$B(t',s) = \jmath^{-1}\left(\frac{\jmath(B(t,t'))}{\jmath(B(t,s))}\right) = \jmath^{-1}\left(\frac{h(t')}{h(s)}\right), \qquad (4.11)$$

where s is an arbitrary fixed time such that $t' < s < t$. For a given interval $t_1 < s < t_2$ with t_1,t_2 fixed the function $h(s)$ can always be defined as

$$h(s) = \frac{\jmath(B(t_2,s))}{\jmath(B(t_2,t_1))} h(t_1) \,. \qquad (4.12)$$

However, the requirement that f has a formal series expansion [38] turns out to be too restrictive and for certain dynamical problems the large time dynamical equation may not admit a solution with an analytic $\jmath$ or $\jmath^{-1}$ in the whole interval (b_1^*, b_2^*).

Indeed, studying the $p = 2$ spherical spin-glass model [13], one finds that the *exact* solution to the dynamical equations can be written, in the regime of large and widely separated times, as a function f isomorphic to the product as in Eq. (4.9), but with a badly behaved $\jmath(\lambda)$ when $\lambda \to 1$. This is indeed related to the fact that the time derivatives $\partial c(t,t')/\partial t = \partial c(t,t')/\partial t' = 0$ when $t' \to t_-$ implies that the inverse is not defined in this limit.

More precisely, the zero-temperature solution corresponds to [13]

$$\jmath^{-1}(\lambda) = 2\sqrt{2}\, \frac{\lambda^{3/4}}{(1+\lambda)^{3/2}} \,, \qquad (4.13)$$

with $\lambda = h(t')/h(t) = t'/t$. $\jmath^{-1}$ has a vanishing derivative when $\lambda \to 1$ and hence $\jmath'(\lambda)|_{\lambda\to 1} \to \infty$. $\lambda \to 1$ for $\jmath^{-1}$ corresponds to $b' \to b$ for f and $\overline{f}$ or $t' \to t_-$ for c and b; i.e., it is the beginning of the aging regime. One of the main results of the present paper is that many of the models studied here do not admit a smooth solution when $b' \to b$ but instead are solved in the aging regime by an ansatz still of the form (4.9) with a nonanalytic $\jmath^{-1}$ at $\lambda \to 1$.

C. General organization of fixed points

We concentrate on the analysis of the long-time dynamics for which the functions defined above have been proposed. Having neglected the time dependence in the functions X and f suggests that the evolution of the system should be measured in terms of the displacement value (or the correlation value) instead of in terms of the times. This implies that time scales are replaced by correlation scales. A correlation scale is the range of correlations between two fixed points of the function f. We call a "blob" a discrete correlation scale between two separate fixed points, for which the function f is proposed to satisfy Eq. (4.9).

It is clear that $B(t,t) = 0$ is a fixed point of f for all the models we consider. The separation of large times into close and separated amounts to assuming the existence of a fixed point at $B(t,t_-) = q$ that marks the end of the FDT correlation scale and the begining of the nonequilibrium correlation scales, the aging regime.

When the times are close to each other, the displacements are homogeneous functions of time $B(t,t') = B(t-t')$. Setting $\tau_1 = t - t'$, $\tau_2 = t - t''$, $\tau_3 = t'' - t'$, and $B_1 = B(\tau_1)$, $B_2 = B(\tau_2)$, and $B_3 = B(\tau_3)$, one can immediately show that a function f as defined above exists in the FDT regime. Indeed, the displacement functions can be inverted to give $\tau_2 = \tau_2(B_2)$, and $\tau_3 = \tau_3(B_3)$, and then using $\tau_1 = \tau_2 + \tau_3$ we have $B_1 = B(\tau_1) = B(\tau_2 + \tau_3) = B(\tau_2(B_2) + \tau_3(B_3))$. Thus one finds that in the FDT regime $B(\tau) = \jmath^{-1}[\exp(\tau)]$. The function X is just the constant associated with the FDT in this correlation scale $X = -1/(2T)$.

Another particular value of the displacement function is b_0. Since it is defined as the maximum value the displacement function can take in the large time dynamics (for $\mu > 0$), it must necessarily be a fixed point of f. The displacement cannot go beyond this value. Moreover, $b(t,0_+) = b(t',0_+) = b_0$.

The strategy is to solve the large time dynamical equations viz., the r, b, and c equations [Eqs. (3.16) and (3.17)], using the general properties of the functions X and f. The dynamical equations for each particular model, i.e., for each particular $\mathcal{V}(x)$, will determine the form of X and f.

V. REPARAMETRIZATION INVARIANT EQUATIONS FOR THE AGING REGIME

In this section we write the dynamical equations for the aging regime Eqs. (3.16)–(3.19) in terms of the functions X and f defined and decribed in the preceding section. We discuss the implications of those assumptions and we also present numerical evidence for them. We finally describe the strategy we shall follow in the two following sections to find an analytic solution to the nonstationary dynamics.

Neglecting the time derivatives in the dynamical equations and the explicit dependence on time of the functions X and f amounts to introducing an artificial time-reparametrization invariance into the large time dynamical equations. The solutions we shall find are, as a consequence, invariant under time reparametrizations. This problem has already been encountered when solving the asymptotics of mean-field spin-glass models. The question on how to select the unique actual solution from the time-reparametrization invariant family of solutions has not been answered yet. Indeed, this is already a well-known problem in the theory of nonlinear one-time differential equations. It is sometimes called the *matching problem* and it has been solved only for some particular cases. How to match the solutions we find for short time differences with one representative of the family of large time-differences solutions remains a difficult, open problem.

We can thus completely eliminate the times in the r, b, and c equations and rewrite them just in terms of the displacement b. With this aim we define

$$F[b] \equiv -\int_b^q db'\ X(b'),\quad \overline{M}[b] \equiv 4\int_b^q db'\ \mathcal{V}''(b')X(b')\ . \tag{5.1}$$

The function $\overline{M}[b]$ is related to the anomaly $\overline{M} = \overline{M}[b_0]$; see Eq. (3.15). These functions could be also defined in such a way as to contain the FDT regime. The upper limit in the integrals should then be set to $b' = 0$. In this case the second integral would be related to M, Eq. (3.13).

We can then rewrite the r equation (3.16) as

$$\begin{aligned}0 = \frac{\partial}{\partial t'}\left((-\mu+\overline{M})\ F[b(t,t')] + \frac{q}{2T}\ \overline{M}[b(t,t')]\right)\\ -4\mathcal{V}''(b(t,t'))\ r(t,t')\ F[q]\\ -4\frac{\partial}{\partial t'}\int_{b(t,t')}^q db'\ \mathcal{V}''(b')\ X[b']\ F[\overline{f}(b',b(t,t'))]\ .\end{aligned} \tag{5.2}$$

Using $F[q] = 0$ and integrating over t' one finds

$$\begin{aligned}0 = F[b]\ \left(-\mu+\overline{M}\right) + \frac{q}{2T}\ \overline{M}[b]\\ +4\int_q^b db'\ \mathcal{V}''(b')\ X[b']\ F[\overline{f}(b',b)],\end{aligned} \tag{5.3}$$

the integration constant being zero, as can be seen by evaluating the resulting equation at $b = q$. This equation is in fact equivalent to a first integral of the r equation.

The b equation (3.17) can be rewritten as

$$\begin{aligned}0 = \left(-\mu+\overline{M}\right)\ b + 2T + \frac{2q}{T}\ [\mathcal{V}'(q) - \mathcal{V}'(b)] + 4\mathcal{V}'(b_0)\ F[b_0'] + 4\int_{b_0}^q db'\ \mathcal{V}'(b')\ X[b'] + 4\int_{b_0}^b db'\ \mathcal{V}''(b')\ F[\overline{f}(b,b')]\\ +4\int_{b_0}^q db'\ \mathcal{V}''(b')\ b'\ X[b'] - 4\int_{b_0}^b db'\ \mathcal{V}''(b')\ \overline{f}(b,b')\ X[b'] + 4\int_q^b db'\ \mathcal{V}''(b')\ \overline{f}(b',b)\ X[b']\ ,\end{aligned} \tag{5.4}$$

where we denote $b_0' = b(t',0)$. For $t > t' > 0$, $b(t,0) \geq b(t',0)$. Since in the large time t' limit $b(t',0) = b_0'$ must be a fixed point, for all $t' > 0$, $b_0 = b(t,0) = f(b(t,t'), b(t',0)) = b(t',0) = b_0'$.

Both equations (5.3) and (5.4) are evidently time-reparametrization invariant. In the same way we can rewrite the c equation (3.19) in terms of b.

The b equation (5.4) evaluated in $b \to q$ and $b \to b_0$ gives back Eqs. (3.22) and (3.25), respectively, which in terms of the functionals F and $\overline{M}$ read

$$0 = (-\mu+\overline{M})\, q + 2T\ , \tag{5.5}$$

$$\begin{aligned}0 = -\mu b_0 + 2T + \frac{2q}{T}\ [\mathcal{V}'(q) - \mathcal{V}'(b_0)] + 4\mathcal{V}'(b_0)F[b_0]\\ +4\int_{b_0}^q db'\ \frac{\partial}{\partial b'}[\mathcal{V}'(b')\ b']\ X[b'].\end{aligned} \tag{5.6}$$

Again, the latter exists only if $\mu \neq 0$ from the start.

The solution to the aging regime amounts, in this language, to solving Eqs. (5.3) and (5.4). Our strategy is as follows.

(a) We propose the existence of only one discrete correlation scale (one blob) with a constant $X[b] = x$. We propose a completely regular $\overline{f}$; by this we mean that the integration of a finite function of b' times $\overline{f}(b',b)$ between b and q tends to zero when $b \to q$. We find that only the p-spin spherical spin glass and the limit of vanishing mass of the RSGM are solved by an ansatz of this type.

(b) We propose the existence of only one discrete scale, but we allow for nonanalytic solutions around $b \sim q$. This implies, in particular, that the integral above contributes in the limit $b \to q$. We find the conditions that the potential correlation must satisfy in order to allow for such a solution. Model (1.1) with short-range correlations and the massive RSGM are solved in this way.

(c) We propose an ultrametric ansatz for all the correlations in the aging regime, namely $\overline{f}(b',b) = \max(b',b)$, $b',b > q$. We show the condition that the potential correlation must satisfy to admit this ansatz as a solution. Model (1.1) with long-range correlations belongs to the family of models with an ultrametric dynamical solution. We present this analysis in Sec. VII.

In principle, it is possible to show that the dynamical equations do not admit another type of solution with, for example, two discrete scales (two blobs) and different constant x inside each of the blobs or many other possibilities. This is a rather extensive task that we do not do in this paper. We study just the three cases we described above.

Numerical evidence for the two regimes

The numerical solution of the full set of dynamical equations demonstrates the existence of the two separate time regimes and also the existence of a function $X(B)$

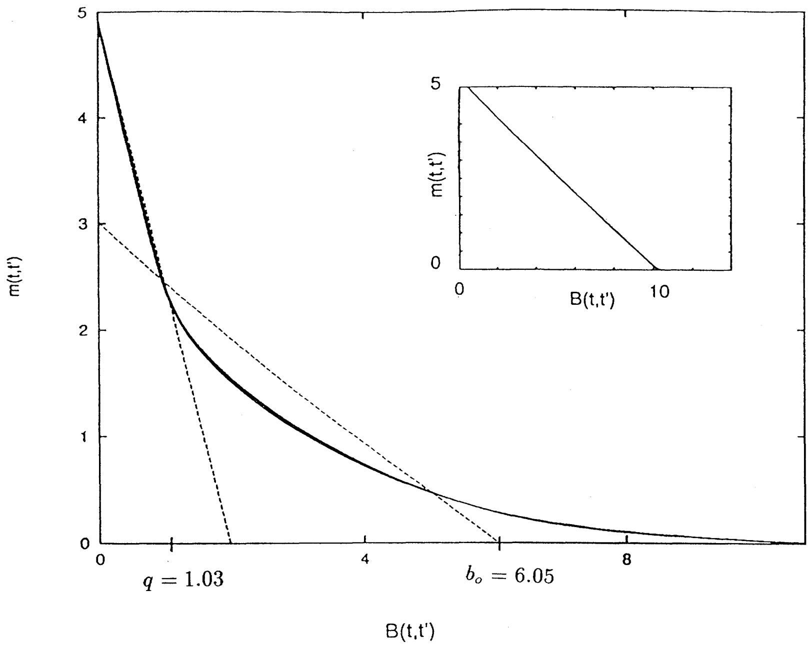

FIG. 1. $m(t,t')$ vs $B(t,t')$ for the long-range power law model. $\gamma = 0.5$, $\mu = 0.2$, and $\theta = 2.5$. For the full lines $t = 200, 300, 400$. In the main plot $T = 0.2$ while in the inset $T = 1.0 > T_c$. Note that with the choice $C(0,0) = 0$ one has automatically $C(t,0) = 0$ and thus $B(t,0) = C(t,t)$.

for large times. One convenient way to show this is by ting the equivalent of the thermoremanent magnetization of a magnetic system

$$m(t,t') \equiv \int_0^{t'} dt'' \, R(t,t'') \tag{5.7}$$

vs the displacement function $B(t,t')$. If $R(t,t') = X(B(t,t'))\, \partial B(t,t')/\partial t'$ for large t', then $X(B)$ is the local slope of these curves.

In Figs. 1 and 2 we plot $m(t,t')$ vs $B(t,t')$ for the power law model with long $\gamma = 0.5$ and short $\gamma = 1.5$ correlations, respectively. The external parameters are $\mu = 0.2$, $\theta = 2.5$, and $T = 0.2$ and we choose $C(0,0) = 0$. Three curves for three times t are included in both figures: $t = 200, 300, 400$ (1000, 1500, and 2000 iterations of the algorithm with a step $h = 0.2$; all our numerical results have been performed with $h = 0.2$). They show clearly two B regimes: for $B \leq q$ the slope is constant and equal to $-1/(2T) = -0.25$ (the FDT holds). For $B > q$ and if the potential correlation is long range the slope is not constant, while if it is short range the slope is still compatible with a constant $|x| < |-1/(2T)|$. The breaking point q and end point b_0 of the displacement can be read from the figures and, as we shall see in Secs. VI and VII, they coincide nicely with the analytical predictions. Note that it is important to keep a finite mass in doing these plots since for μ finite one can ensure that b_0 stays finite.

The curves for different total times t do not exactly lie upon each other, showing that the numerical $X(B)$ still depends on t. We expect that for sufficiently large times they will converge to the analytical prediction that we describe in Secs. VI and VII and include in these figures as broken lines. We can see that the numerical convergence is indeed rather slow. In the insets we plot the same curves for the same choice of parameters, but for the high-temperature phases $T = 1$ and $t = 400$. In both figures the slope is -0.5 for all B.

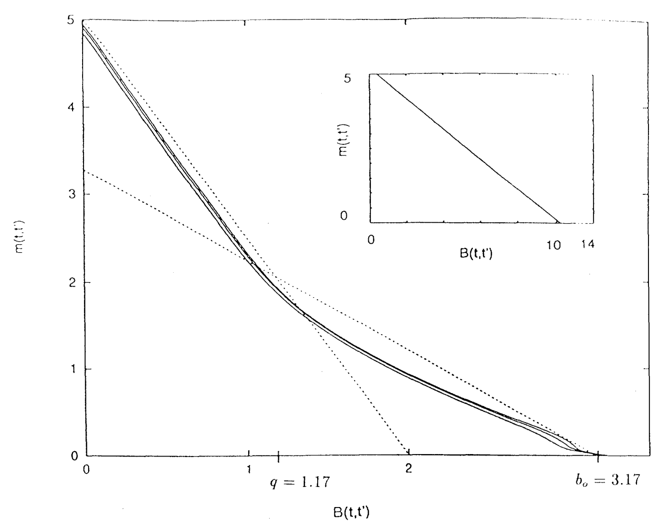

FIG. 2. $m(t,t')$ vs $B(t,t')$ for the short-range power law model. $\gamma = 1.5$, $\mu = 0.2$, and $\theta = 2.5$. For the full lines $t = 200, 300, 400$. $T = 0.2$, while in the inset $T = 1.0 > T_c$. $q \sim 1.17$ and $b_0 \sim 3.17$.

VI. SOLUTIONS TO THE DYNAMICAL EQUATIONS: SHORT-RANGE POTENTIALS

In this section we apply a one-blob ansatz to the aging regime equations and we show that those associated with the short-range potentials are solved in this way.

A. Asymptotic values: Comparison with the statics

We start by studying the asymptotic values q, b_0, and x. Though a deep understanding of the relation between the dynamical $X[b]$ and the statical replica function $x[q]$ is still missing, it is by now clear that, at least formally, they play similar roles. We compare here the aymptotic values q and b_0 and the parameter $x = X[b]$ inside the blob to the static [19] $q^{\rm stat}$, $b_0^{\rm stat}$, and $x^{\rm stat}$.

Equations (3.22) and (3.25) with $X[b] = x$ are identical to the ones obtained within the replica analysis [19] as the extremum conditions for the static q and b_0. This is a consequence of an algebraic relation between dynamical and static (replica) approaches [39]. Instead, Eq. (3.21), dynamical in origin, is formally equal to the condition to have a vanishing replicon eigenvalue of the replica treatment. It does not coincide with the third equation arising from the replica recipe, i.e., the one obtained maximizing the free energy with respect to x. As a consequence, the dynamical value for x does not coincide with the static one. This implies that also the values for q and b_0 differ since x enters in their equations. The dynamical phase diagram turns out to be different from the static; the dynamical transition temperature is higher than the static critical temperature for a range of masses. Moreover, the dynamical asymptotic energy is higher than the equilibrium energy. This discrepancy has already been observed for the special case of the p-spin spherical spin glass and it seems to be related to a peculiar organization of the metastable states of the Thouless-Anderson-Palmer [41] free-energy landscape, with the existence of a threshold (higher than the equilibrium free energy, below which infinite barriers appear [10,42]). A more detailed analysis along these lines is in order to determine if this is at the origin of the difference between static and dynamical asymptotic values in these models too.

If the model is massive, the particle is constrained to move in a restricted region of the space and hence b_0 and $\tilde{q}$ are finite. If we take the limit $\mu \to 0$ afterward we shall see that $b_0 \to \infty$ in such a way that μb_0 remains finite. Instead the case of a strictly massless model is more delicate. Let us now obtain the equations that fix q, b_0, and x for a massive model. Equation (3.23) determines q. When x is constant the equations that fix b_0 and x follow from Eqs. (5.5) and (5.6) and read

$$\frac{\mathcal{V}'(q) - \mathcal{V}'(b_0)}{b_0 - q} = \frac{\mu T}{2q} \,, \tag{6.1}$$

$$b_0 - q = -\frac{1}{2Tx}\left(\frac{2T}{\mu} - q\right) \,, \tag{6.2}$$

while Eq. (6.3) gives $\tilde{q}$,

$$\tilde{q} = \frac{1}{\mu}\left(T + \frac{q}{T}\,\mathcal{V}'(q) + 2x[q\mathcal{V}'(q) - b_0\mathcal{V}'(b_0)]\right) \,. \tag{6.3}$$

The anomaly $\overline{M}$ can be immediately computed,

$$\overline{M} = 4x[\mathcal{V}'(q) - \mathcal{V}'(b_0)], \tag{6.4}$$

and can be used to solve the FDT equations (see Sec. III).

The asymptotic energy density reads

$$\mathcal{E}(t) = \frac{\mu}{2}\,\tilde{q} + \frac{1}{T}\,[\mathcal{V}(0) - \mathcal{V}(q)] - 2x[\mathcal{V}(q) - \mathcal{V}(b_0)] \tag{6.5}$$

and thus it is determined by the asymptotic values q, b_0, and x. Since x does not coincide with the static replica, the asymptotic dynamical energy differs from the static one. We shall show below some numerical results on this point (Fig. 7).

B. The μ-T phase diagram

In Fig. 3 we present the phase diagram for the power law model with $\gamma = 1.5$ and $\theta = 2.5$. The equation for q, Eq. (3.23), determines a first critical temperature (independent of the mass) given by

$$T_c^{(1)} = \sqrt{-(q^*)^2\mathcal{V}''(q^*)} \,, \tag{6.6}$$

with q^* the value of q at which $q^2\mathcal{V}''(q)$ attains its maximum, i.e.,

$$\frac{q^*\mathcal{V}'''(q^*)}{2\mathcal{V}''(q^*)} = -1 \,. \tag{6.7}$$

The glass phase cannot extend beyond $T = T_c^{(1)}$ since there is then no solution to Eq. (3.23).

Just from their definitions $b_0 \geq q$. This condition determines a critical line $T_c^{(2)}(\mu)$. Equation (6.1), when $b_0 \to q$, yields

$$T_c^{(2)}(\mu) = \frac{q(T)\mu}{2} \,, \tag{6.8}$$

where $q(T)$ is the solution to Eq. (3.23). The general equation for this line is

$$\mathcal{V}''\left(\frac{2T_c^{(2)}}{\mu}\right) = -\frac{\mu^2}{4} \,. \tag{6.9}$$

This same equation is found in the replica treatment as the condition to have a vanishing replicon eigenvalue.

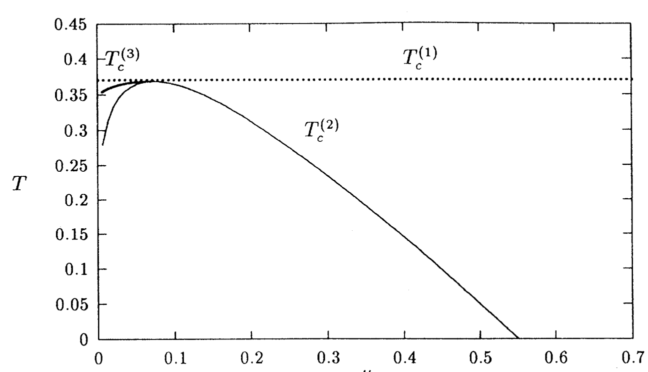

FIG. 3. Phase diagram for the power law model with $\gamma = 1.5$ and $\theta = 2.5$. The glassy phase, with aging dynamics, is located below the curve $\max(T_c^{(2)}, T_c^{(3)})$.

Statically this determines the line below which the replica symmetric solution becomes unstable and hence the transition line from the replica symmetric solution to the one-step replica symmetry-breaking solution [19]. Equation (6.9) delimits a domain in the μ-T plane where the aging solution exists. When $T \to 0$ we find a finite critical $\mu = \mu_c = 2\sqrt{|\mathcal{V}''(0)|}$ beyond which the above equation has no solution. When $\mu \to 0$ and the potential is short-range [i.e., $\mathcal{V}''(x)$ decreases faster than $1/x^2$ at infinity] $T_c^{(2)}(\mu) \to 0$.

For the power law model the explicit formula for $T_c^{(2)}(\mu)$ is

$$T_c^{(2)}(\mu) = \frac{\mu}{2}\left[\left(\frac{2\gamma}{\mu^2}\right)^{1/(\gamma+1)} - \theta\right] . \tag{6.10}$$

If $\gamma > 1$, $T_c^{(2)}(\mu \to 0) \to 0$ and for $\gamma = 1$, $T_c^{(2)}(\mu \to 0) = 1/\sqrt{2}$.

Returning to the general potential correlation and evaluating x from Eq. (6.2) at the critical line $T_c^{(2)}$, $b_0 = q+\epsilon$, $\epsilon \to 0$, and one obtains

$$x = x_c = \frac{q}{4T}\,\frac{\mathcal{V}'''(q)}{\mathcal{V}''(q)} , \tag{6.11}$$

with $q = 2T_c^{(2)}/\mu$ on the critical line. One sees that as long as the function $q^2\,\mathcal{V}''(q)$ is increasing (small values of q, $q < q^*$), one automatically has $|x_c| < 1/(2T)$. Thus, even if the transition is continuous for q and b_0 there is a jump in x from the value x_c below the critical line $T_c^{(2)}$ to the high-T phase value $x = x_F = -1/(2T)$.

The value $x_c = -1/(2T)$ is reached exactly at the temperature $T_c^{(2)} = T_c^{(1)}$ where q attains its maximal value. Equations (6.6) and (6.9) determine the tricritical point (μ_c^*, T_c^*) at which $b_0 = q$ and $x = -1/(2T)$ simultaneously. For $\mu < \mu_c^*$ the nature of the transition changes. The phase boundary is now determined by the condition $x_c = -1/(2T)$, while b_0 at the transition is no longer equal to q. Setting $x = -1/(2T)$ in the equations (6.1) and (6.2), one obtains $b_0 = 2T_c^{(3)}/\mu$ at the transition and the following equations for the transition temperature:

$$\mathcal{V}'(q) - \mathcal{V}'\left(\frac{2T_c^{(3)}}{\mu}\right) = \frac{(T_c^{(3)})^2}{q} - \frac{\mu T_c^{(3)}}{2} , \tag{6.12}$$

with q given, as always, by Eq. (3.23). The transition is now continuous in x but discontinuous in q. $T_c^{(3)}(\mu)$ is well above the transition line for the statics and it reaches a finite limit at $\mu \to 0$, as opposed to $T_c \to 0$ when $\mu \to 0$ in the statics. For the potential correlation (1.6) one finds that when $\mu \to 0$, $[T_c^{(3)}(\mu)]^2 \to \frac{\theta^{1-\gamma}}{2(\gamma-1)}(\frac{\gamma-1}{\gamma})^\gamma$ and $q = \theta/(\gamma-1)$.

For model (1.1) with $\mathcal{V}$ given by Eq. (1.6) and $\gamma > 1$, these conditions imply a phase diagram identical to the one described by Kinzelbach and Horner in Ref. [27] for the equilibrium dynamics. It is not clear why it should be a difference between static and equilibrium dynamic transitions (and furthermore between static and asymptotic equilibrium dynamic energies) especially if one remembers that the initial assumption of the latter approach is that the evolution starts at an equilibrium state. In the nonequilibrium evolution there is *a priori* no reason for the two transition temperatures to coincide and indeed one knows this happens in real systems as glasses. We shall return to this discussion later when we shall present the numerical results for the asymptotic energy-density vs the static one.

C. Equations within a blob with $X[b] = x$

Using $X[b] = x$,

$$F[b] = -x(q-b), \quad \overline{M}[b] = 4x\,[\mathcal{V}'(q) - \mathcal{V}'(b)] , \tag{6.13}$$

and the first integral of the r equation (5.3) and the equation (5.4) read

$$\begin{aligned} 0 = (b-q)\,&\{-\mu + 4x\,[\mathcal{V}'(q) - \mathcal{V}'(b_0)]\} \\ &+4q\left(\frac{1}{2T} + x\right)[\mathcal{V}'(q) - \mathcal{V}'(b)] \\ &+ 4x\int_q^b db'\,\mathcal{V}''(b')\,\overline{f}(b',b) , \end{aligned} \tag{6.14}$$

$$\begin{aligned} 0 = b\,&\{-\mu + 4x\,[\mathcal{V}'(q) - \mathcal{V}'(b_0)]\} \\ &+2T + 4q\left(\frac{1}{2T} + x\right)[\mathcal{V}'(q) - \mathcal{V}'(b)] \\ &+ 4x\int_q^b db'\,\mathcal{V}''(b')\,\overline{f}(b',b) . \end{aligned} \tag{6.15}$$

We see that these equations coincide if we use $\mu - \overline{M} = 2T/q$. We shall concentrate then on the r equation (6.14).

D. Analytic ansatz at $t' \sim t$

We can now show that a regular solution at $t' \to t_-$ cannot exist for all the models we consider. In fact, if $\overline{f}$ is regular at $b \to q$ one can evaluate Eq. (6.14) and its variations with respect to b in this limit. If $\overline{f}$ and all its variations with respect to b are smooth when $b \to q$, one can set the integral in each equation to zero (the limits collapse and the integrands are regular by hypothesis) when evaluating at $b \to q$. However, if this is so, one gets new conditions on x, q, and b_0 that are not always compatible.

More precisely, the first new equation arises from the second variation of the r equation with respect to b. When $b \to q$ it implies

$$x = \frac{q}{2T}\,\frac{\mathcal{V}'''(q)}{\mathcal{V}''(q)} , \tag{6.16}$$

where we have used $\partial\overline{f}(b',b)/\partial b|_{(b'\to b)\to q} = 1$ and we have set the remaining integral to zero. For general models, i.e., general $\mathcal{V}$'s, this condition is not compatible with the x arising from Eq. (6.2) above.

In an alternative way, one can investigate the equation that the potential correlation $\mathcal{V}$ has to satisfy in order to allow for a regular solution to the dynamical equation. Combining Eqs. (3.23), (6.1), and (6.2) with Eq. (6.16) yields

$$\frac{\mathcal{V}'(q)-\mathcal{V}'(b_0)}{b_0-q}=-\frac{\mathcal{V}''(q)}{1-(b_0-q)\,\frac{\mathcal{V}'''(q)}{\mathcal{V}''(q)}}\,. \qquad (6.17)$$

We can now consider the temperature T as a varying parameter and then q as the independent variable in a differential equation for $\mathcal{V}(q)$. Setting $y(q)\equiv\mathcal{V}'(q)$ and $y_0\equiv\mathcal{V}'(b_0)$ we have

$$-(y-y_0)\,[y'-(b_0-q)y'']=(b_0-q)\,y'^2\,. \qquad (6.18)$$

The potential correlation has to satisfy this equation in order to allow for a regular solution.

We have checked that from the potential correlations described in the Introduction only the one associated with the p-spin spherical model with $p\geq 2$ admits a solution with a well-behaved $\jmath^{-1}$ when $\lambda\to 1$ (the case $p=2$ is particular because the first derivative of $\jmath^{-1}$ is zero). The x arising from the second variation of Eq. (6.14) given by Eq. (6.16) is compatible with the x arising from Eq. (6.2). All the higher variations are trivial and do not give further constraints. Conversely, if one perturbs the original p-spin Hamiltonian with an arbitrary small term associated with a different p in the manner of Ref. [40], one immediately sees that a regular solution for the dynamics is no longer allowed. One can say that the p-spin spherical spin glass is somehow a marginal model.

The problem for a general $\mathcal{V}$ is in fact so constrained that it is not hard to believe that the potential correlation associated with the p-spin spherical spin glass is the unique function having a regular solution.

E. Nonanalytic solution at $t'\sim t$

Using $\mu-\overline{M}=2T/q$, the remaining equation (6.14) reads

$$0=-\frac{T}{q}\,(b-q)-2q\left(x+\frac{1}{2T}\right)\,[\mathcal{V}'(b)-\mathcal{V}'(q)]$$
$$+2x\int_q^b db'\,\mathcal{V}''(b')\,\jmath^{-1}\left(\frac{\jmath(b)}{\jmath(b')}\right)\,. \qquad (6.19)$$

It is useful now to remember the original time dependence inside the integral using

$$b(t,t')=\jmath^{-1}\left(\frac{h(t')}{h(t)}\right)\,. \qquad (6.20)$$

Thus, defining a new variable u and a new function $B(u)$ as [10]

$$\exp(u)\equiv h(t),$$
$$b(t,t')=q+B(u-u')$$
$$\equiv\jmath^{-1}\left(\frac{h(t')}{h(t)}\right)=\jmath^{-1}\{\exp[-(u-u')]\}\,. \qquad (6.21)$$

The variable $u-u'=\ln[h(t)/h(t')]$ is the natural variable for the problem. When t increases it increases from $u-u'=0$ at $t=t'$ to $u-u'=\infty$ and the function $B(u-u')$ then grows from 0 to b_0-q. Eq. (6.19) becomes

$$0=B(u)\mathcal{V}''(q)-\mathcal{V}'(q+B(u))+\mathcal{V}'(q)-\frac{2xq}{T}\,\mathcal{V}''(q)$$
$$\times\int_0^{B(u)}dB(u')\,\mathcal{V}''(q+B(u'))\,B(u-u') \qquad (6.22)$$

or, integrating by parts the last term,

$$0=-B(u)\mathcal{V}''(q)+\mathcal{V}'(q+B(u))-\mathcal{V}'(q)$$
$$+\frac{2xq}{T}\,\mathcal{V}''(q)\left[-\mathcal{V}'(q)B(u)\right.$$
$$\left.+\frac{\partial}{\partial u}\int_0^u du'\,\mathcal{V}'(q+B(u'))\,B(u-u')\right]\,. \qquad (6.23)$$

This equation represents the homogenized version of Eq. (6.19). It is important to note that it *is not homogeneous in time* and thus its solutions are not either. The homogeneity holds at the level of nonlinear functions of the correlation functions and not at the level of times. It has no free parameters. q and x are fixed by Eqs. (3.23), (6.1), and (6.2). The invariance of the aging regime equations under time reparametrizations $t\to h(t)$ implies the invariance of Eqs. (6.22) and (6.23) under dilatations of u: $u\to\eta u$.

Equation (6.23) can be solved exactly for some special choices of $\mathcal{V}$ such as the p-spin spherical model and the RSGM in the limit of vanishing mass. The solutions are $B(u)=2q_{PS}\,[1-\exp(-\eta u)]$ and $B(u)=\eta u$, respectively. We shall describe these cases in detail at the end of this section. In what follows we shall instead analyze the small- and large-u behavior of $B(u)$ for the general potential correlation. A powerful method to study both limits, small and large u, can be developed by using Laplace transform techniques. We do not describe it in the present paper, but we shall do so, applying it to the case of finite dimensions $D>0$, in Ref. [16].

Small u

We solve Eq. (6.22) around $u\sim 0$, i.e., $b\sim q$, which corresponds to $B(u)\sim 0$, by first expanding in powers of $B(u)$ and then proposing a series expansion of $B(u)$ in terms of u^α. α is a positive exponent that should be determined by the equation. An exponent α smaller than one related to an irregularity at $u\sim 0$.

The expansion around $B(u)\sim 0$ yields

$$0=\sum_{n=2}^{\infty}\left[\frac{\mathcal{V}^{(n+1)}(q)}{\mathcal{V}''(q)\,n!}\,[B(u)]^n\right.$$
$$\left.+\frac{2xq}{T}\,\frac{\mathcal{V}^{(n)}(q)}{(n-1)!}\,\frac{\partial}{\partial u}\int_0^u du'\,[B(u')]^{n-1}\,B(u-u')\right]\,.$$
$$(6.24)$$

It is easy to see that the zeroth- and first-order equations in $B(u)$ are immediately satisfied. The second-order equation is, however, nontrivial:

$$0 = \frac{\mathcal{V}'''(q)}{\mathcal{V}''(q)} [B(u)]^2 + \frac{4xq}{T} \mathcal{V}''(q) \frac{\partial}{\partial u} \int_0^u du' \, B(u') B(u - u') \,. \tag{6.25}$$

We now proceed by proposing a formal series expansion for the function $B(u)$ around $u \sim 0$ in terms of u^α

$$B(u) = \mathcal{B}(u^\alpha) = \sum_{m=0}^{\infty} b_m u^{m\alpha} \tag{6.26}$$

and then solving Eq. (6.24) term by term in the double series expansion. More precisely, Eq. (6.25) at first order in u^α implies

$$0 = \left(\frac{\mathcal{V}'''(q)}{\mathcal{V}''(q)} + \frac{4xq}{T} \mathcal{V}''(q) \frac{[\Gamma(1+\alpha)]^2}{\Gamma(1+2\alpha)} \right) (b_1 u^\alpha)^2 \,. \tag{6.27}$$

This equation fixes the function

$$\rho(\alpha) \equiv \frac{[\Gamma(1+\alpha)]^2}{\Gamma(1+2\alpha)} \tag{6.28}$$

and hence the exponent α. From its definition, the limiting values of ρ are

$$\begin{aligned} \lim_{\alpha\to\infty} \rho(\alpha) &= 0 \,, \\ \rho(1) &= 1/2 \,, \\ \lim_{\alpha\to 0} \rho(\alpha) &= 1 \,. \end{aligned}$$

For a general $\mathcal{V}$

$$\rho(\alpha) = -\frac{T}{4xq} \frac{\mathcal{V}'''(q)}{[\mathcal{V}''(q)]^2} \,. \tag{6.29}$$

This last equation fixes $\rho(\alpha)$ and thus α. We shall see below, in some concrete examples, how the limiting values in (6.29) select the kind of potential that allows for a solution of the one-blob type.

The first coefficient b_1 is not fixed as a consequence of the reparametrizations invariance. In Appendix D we describe how we can obtain the coefficients $b_m, m \geq 2$, in terms of b_1 and then reconstruct the series for $B(u)$. In Figs. 4 and 5 we show $B(u)$ for model (1.1) with $\gamma = 1.5$, $\theta = 2.5$, and $T = 0.2$. Figure 4 corresponds to the massless case and Fig. 5 to a massive case with $\mu = 0.2$. The different curves in both figures correspond to having approximated the series for $B(u)$ with a sum with different numbers of terms. We see that both in the massless and massive cases $B(u)$ can be obtained in this simple way up to reasonable values of u^α. We have checked, however, that in the massive case a Padé approximation gives $B(u \to \infty) \sim b_0 - q$, as expected. In the massless case we obtain the analytic asymptotic behavior of $B(u)$ in Sec. VI F and find that it is in good accord with the tendency of the curves in Fig. 4.

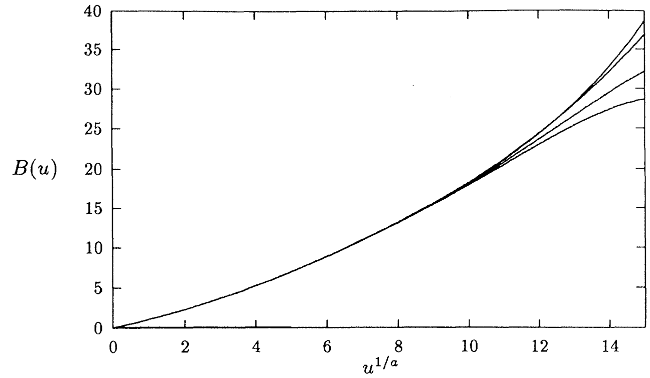

FIG. 4. $B(u)$ vs $u^{1/a}$ with $1/a = \alpha$, $\gamma = 1.5$, $\theta = 2.5$, $T = 0.2$, and $\mu = 0$. From top to bottom the number of coefficients are 7, 9, 10, 8.

Assuming that the potential correlation $\mathcal{V}$ allows for a solution of this type, we can study the behavior of α when we approach the transition lines $T_c^{(2)}$ and $T_c^{(3)}$. Approaching the transition line $T_c^{(2)}$ from below, $b_0 \to q$ and $x \to q/(4T)\, \mathcal{V}'''(q)/\mathcal{V}''(q)$. Thus $\mathcal{V}''(q) \to -\mu^2/4$, $q \to 2T/\mu$, and

$$\rho(\alpha) \to 1 \qquad \Rightarrow \qquad \alpha \to 0 \,, \tag{6.30}$$

using the limiting values (6.29). The aging part of the dynamical solution becomes very steep near equal times, while the total variation $b_0 - q$ vanishes when we approach the transition line $T_c^{(2)}$.

Approaching now the transition line $T^{(3)}$ from below, $x \to -1/(2T)$ and then

$$\rho(\alpha) \to \frac{T^2}{2q} \frac{\mathcal{V}'''(q)}{[\mathcal{V}''(q)]^2} \leq 1 \tag{6.31}$$

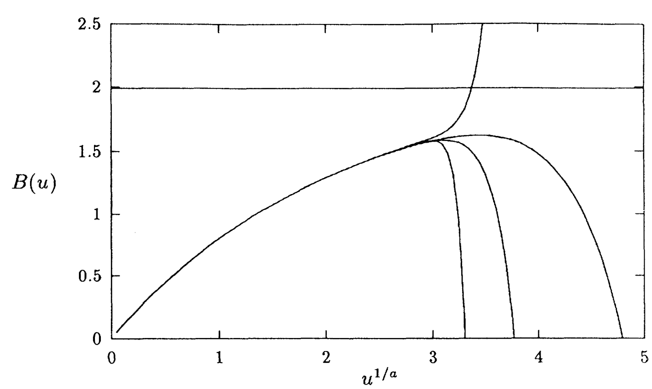

FIG. 5. $B(u)$ vs $u^{1/a}$ where $1/a = \alpha$, $\gamma = 1.5$, $\theta = 2.5$, $T = 0.2$, and $\mu = 0.2$. The curve pointing upward has 30 coefficients while the curves pointing downward have 50,20,10 from left to right. The straight line is $B = b_0 - q$.

and the exponent α as well as the aging function $B(u)$ reaches a nontrivial limit when approaching the transition line at a fixed μ. At $T_c^{(2)} = T_c^{(3)}$ one has $\rho(\alpha) \to 1$. Summarizing, at early times in the aging regime—early epochs—the displacement is

$$b(t,t') \sim q + b_1 \ln^\alpha \left(\frac{h(t)}{h(t')}\right) \sim q + b_1 \left(\frac{h(t)}{h(t')} - 1\right)^\alpha . \quad (6.32)$$

F. Solution for $b \sim b_0$, large-u behavior for the massless models

The analysis of Eq. (6.22) or (6.23) for large u is delicate and we shall not pursue it in all generality. We shall describe instead what happens in the limit of a vanishing mass for a general potential correlation.

In the limit $\mu \to 0$ Eqs. (6.1), (6.3), and (6.2) imply

$$b_0 \mu \sim -\frac{1}{x},$$

$$\mathcal{V}'(q) - \mathcal{V}'(b_0) \sim -\frac{T}{2qx}, \quad (6.33)$$

$$\mu \tilde{q} \sim \frac{q\mathcal{V}'(q)}{T} - \frac{T\mathcal{V}(q)}{q\mathcal{V}'(q)} .$$

Thus $b_0 \to \infty$, $\mathcal{V}'(b_0) \to 0$, and $\mathcal{V}'(q) \sim -T/(2qx)$. Using this result Eq. (6.22) simplifies:

$$\mathcal{V}'(q) - \mathcal{V}'(q + B(u)) = -\frac{\mathcal{V}''(q)}{\mathcal{V}'(q)} \frac{\partial}{\partial u} \int_0^u du'\, \mathcal{V}'(q + B(u'))\, B(u - u') . \quad (6.34)$$

Let us now study the asymptotic behavior of $B(u)$ at large u. Since $B(u) \to \infty$ one must have

$$\int_0^u du'\, \mathcal{V}'(q + B(u'))\, B(u - u') \sim \frac{\mathcal{V}'(q)^2}{\mathcal{V}''(q)}\, u . \quad (6.35)$$

Defining $I \equiv \int_0^\infty du'\, \mathcal{V}'(q + B(u'))$ and assuming I to be a convergent integral immediately implies $B(u) \sim \eta u$ with $\eta I = \frac{\mathcal{V}'(q)^2}{\mathcal{V}''(q)}$. One can check that this is the only possibility for short-range models. It implies

$$b(t,t') \sim q + \ln \left(\frac{h(t)}{h(t')}\right) \quad (6.36)$$

in the limit $h(t')/h(t) \ll 1$ for *all* short-range models. This form that turns out to be exact for the RSGM in the massless limit (see Sec. VI G) for all u implies that $r(t,t')$ is only a function of t' for widely separated times. We have checked this result numerically. In Fig. 6 we plot the response function $R(t,t')$ vs the smaller time t' for four times t in the massless case. The parameters are $T = 0.2$ and $\theta = 2.5$.

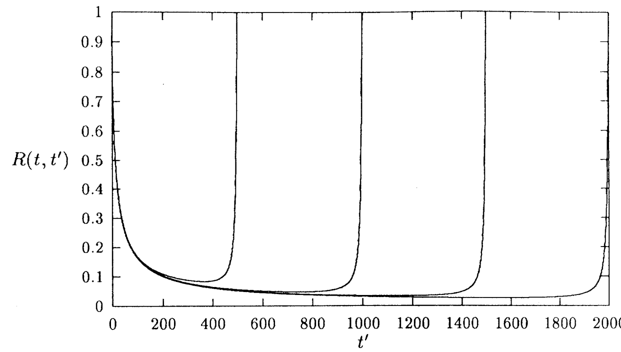

FIG. 6. Response function $R(t,t')$ vs t' for $t = ih$, $h = 0.2$, and $i = 100, 200, 300, 400$. $\mu = 0$, $T = 0.2$, and $\gamma = 1.5$.

In the present section we are letting $\mu \to 0$ and using all the equations including (6.1). Note that an identical equation (and solution) for $B(u)$ would be obtained by setting $\mu = 0$ first and then setting $b_0 = \infty$, abandoning Eq. (6.1).

A very interesting quantity to calculate for the short-range models in particular in the massless limit is the asymptotic energy density

$$\mathcal{E}_\infty(\mu \to 0) = \mu \tilde{q} + \frac{1}{T} [\mathcal{V}(0) - \mathcal{V}(q)] - 2x\mathcal{V}(q) . \quad (6.37)$$

As we shall see in the following subsection, the asymptotic energy density of the massive power law model with short-range correlations (1.6) is higher than the equilibrium energy density obtained with the replica analysis [19]. Moreover, within the dynamic approach we find a nontrivial solution when $\mu \to 0$ with a finite asymptotic energy density while statically the solution does not exist in the massless limit since $T_c(\mu \to 0) \to 0$ in the statics. This difference between asymptotic and equilibrium values seems to be characteristic of models solved by 1RSB ansatz statically and it is related to the fact that the dynamic $B(X)$ is not equal to static Parisi function $b(u)$.

G. Some particular models

The power law model

For model (1.1) $\rho(\alpha)$ becomes

$$\rho(\alpha) = -\frac{T}{xq} \frac{1+\gamma}{2\gamma} (\theta + q)^\gamma . \quad (6.38)$$

In Fig. 2 we show the analytical prediction (broken straight lines) for this model with $\gamma = 1.5$, $\mu = 0.2$, $T = 0.2$, and $\theta = 2.5$. The full lines are the numerical solution to the dynamical equations for $t = 200, 300, 400$ from Eqs. (3.23)–(6.2) $q = 1.17$ and $b_0 = 3.17$. The FDT decay and q are quite rapidly reached by the numerical solution. Instead, the convergence towards the aging theoretical curve and b_0 is slower. Note that the end of the full lines is at $b(t,0)$, which is slightly larger than $b_0 = 3.17$, as expected.

It is interesting to see what happens to α when $\gamma \to 1+$, i.e., when γ approaches the critical value $\gamma_{\rm cr} = 1$ at which the potential correlation becomes long range. The limit $\gamma \to 1+$ of Eq. (6.38) above yields

$$\lim_{\gamma\to 1+} \rho(\alpha) = -\frac{1}{\sqrt{2}x} = 1 \,. \tag{6.39}$$

The last equality holds since $x \to -1/\sqrt{2}$ when $\gamma \to 1+$. Thus

$$\gamma \to 1+ \qquad \Rightarrow \qquad \alpha \to 0 \tag{6.40}$$

everywhere in the low-temperature phase. Since inside the low-temperature phase $b_0 - q$ reaches a (T- and μ-dependent) nonzero limit and the aging function reaches a steplike function. This is an illustration of the way in which the ultrametric solution is reached when $\gamma \to 1+$, as discussed in Sec. VI H. It can be shown that for $\gamma < 1$, $\rho(\alpha)$ becomes larger than 1 and thus no solution exists.

When the mass μ tends to zero the asymptotic values q and b_0, as well as x, can be easily obtained. They are given by the equations

$$q^2\,(\theta+q)^{-(1+\gamma)} = \frac{2}{\gamma}\,T^2, \quad b_0\mu \sim -\frac{1}{x} \sim \frac{q}{T}\,(\theta+q)^{-\gamma}, \tag{6.41}$$

$$\mu\,\tilde{q} \sim \frac{T(\theta+q)}{q}\,\left(\frac{1-2\gamma}{\gamma(1-\gamma)}\right) . \tag{6.42}$$

When $\mu \to 0$, $b_0 \to \infty$ and $\tilde{q} \to \infty$ in such a way that $b_0\mu$ and $\tilde{q}\mu$ remain finite. Thus, in the zero mass limit

$$\rho(\alpha) \sim \frac{1+\gamma}{2\gamma} \,. \tag{6.43}$$

Reconsidering the limiting behavior of ρ [cf. Eq. (6.29)], we see that

$$\begin{aligned} \gamma \to \infty \quad &\Rightarrow \quad \rho \to 1/2 \quad \Rightarrow \quad \alpha \to 1, \\ \gamma \to 1 \quad &\Rightarrow \quad \rho \to 1 \quad \Rightarrow \quad \alpha \to 0, \\ \gamma < 1 \quad &\Rightarrow \quad \rho > 1 \quad \Rightarrow \quad \alpha \not\exists. \end{aligned} \tag{6.44}$$

Thus α decreases from $\alpha = 1$ to $\alpha = 0$ for short-range models. For $\gamma < 1$ (long-range potential) $\rho > 1$ but, from its definition (6.28), $\rho(\alpha)$ cannot be greater than 1. Hence we have demonstrated that the long-range potentials of type (1.6) do not have a one-blob solution, at least in the massless limit, for any temperature. This demonstration can be extended to show that indeed the long-range model does not admit a one-blob solution even when the mass is finite.

At the zero-temperature limit we can obtain the explicit dependence of q, b_0, and x on the parameters θ and γ:

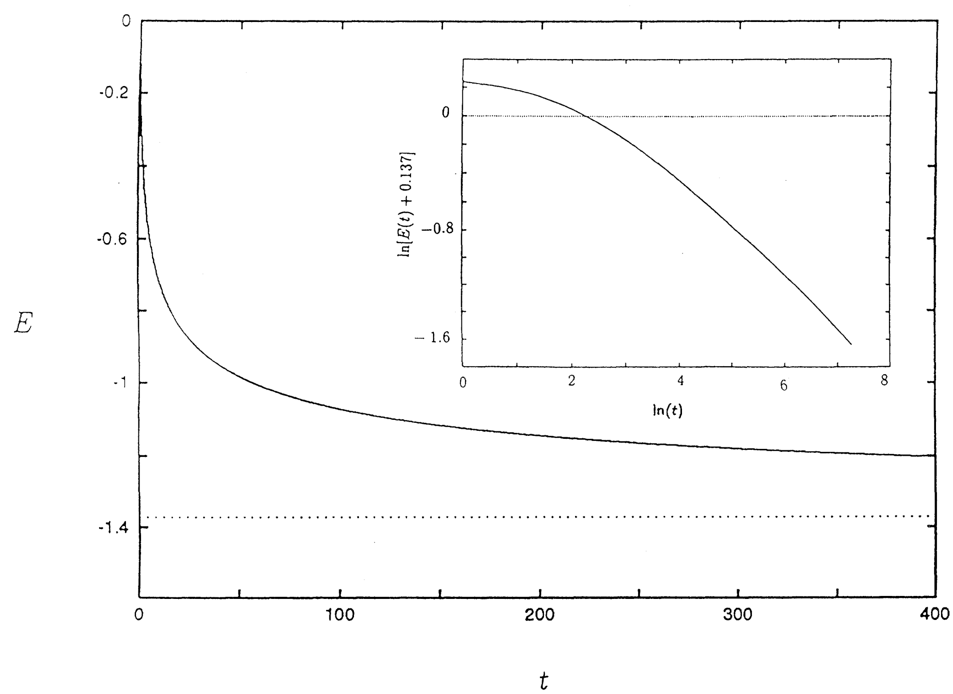

FIG. 7. Energy-density decay: $\mathcal{E}(t)$ vs t for the short-range model with $\gamma = 1.5$, $\theta = 2.5$, and $T = 0.2$. Inset: $\ln[\mathcal{E}(t) + 0.137]$ vs $\ln t$.

$$
\begin{aligned}
q &\sim T\,\sqrt{\frac{2}{\gamma}}\,\theta^{\frac{1+\gamma}{2}} \ll 1\ , \\
x &\sim -\sqrt{\frac{\gamma}{2}}\,\theta^{\frac{\gamma-1}{2}} \sim 1, \\
b_0\mu &\sim \sqrt{\frac{2}{\gamma}}\,\theta^{\frac{1-\gamma}{2}} \sim 1\ , \\
\tilde{q}\,\mu &\sim \frac{1}{\sqrt{2\gamma}}\,\theta^{(1-\gamma)/2}\,\left(\frac{1-2\gamma}{1-\gamma}\right) \sim 1\ .
\end{aligned} \tag{6.45}
$$

Note that x remains finite at zero temperature.

The asymptotic energy density for the power law model is

$$
\begin{aligned}
\mathcal{E}_\infty = \mu\tilde{q} &+ \frac{1}{T}\,\frac{1}{2(1-\gamma)}\,[\theta^{1-\gamma} - (\theta+q)^{1-\gamma}] \\
&- x\,\frac{1}{(1-\gamma)}\,[(\theta+q)^{1-\gamma} - (\theta+b_0)^{1-\gamma}]\ ;
\end{aligned} \tag{6.46}
$$

in the massless case it reads

$$
\begin{aligned}
\mathcal{E}_\infty(\mu\to 0) = &\frac{T}{\gamma q}(\theta+q) \\
&+ \frac{1}{2T(1-\gamma)}\,[\theta^{1-\gamma} - (\theta+q)^{1-\gamma}]\ .
\end{aligned} \tag{6.47}
$$

In Fig. 7 we plot the energy-density decay for the massless power law model with parameters $\gamma = 1.5$, $\theta = 2.5$, and $T = 0.2$. Equation (6.47) implies $\mathcal{E}_\infty \sim -0.137$. In the inset we plot $\ln[\mathcal{E}(t) + 0.137]$ vs $\ln t$. One can see that the energy density approaches its asymptotic value with a power law. Finally, at zero temperature we have

$$
\mathcal{E}_\infty(\mu\to 0, T\to 0) = \sqrt{\frac{2}{\gamma}}\,\theta^{(1-\gamma)/2}\ . \tag{6.48}
$$

H. The zero mass random sine-Gordon model

In the limit of zero mass, Eqs. (6.1) and (6.2) can be easily solved. The maximum displacement b_0 tends to infinity in this limit but the product μb_0 stays finite; we obtain

$$
q^2\,\exp(-q/2) = \frac{4T^2}{\Delta},\quad \mu b_0 \sim -\frac{1}{x} \sim \frac{4T}{q},\quad \tilde{q}\mu \sim \frac{4T}{q}\ . \tag{6.49}
$$

Therefore, $\rho(\alpha)$ satisfies

$$
\rho(\alpha) = \frac{1}{2} \qquad \Rightarrow \qquad \alpha = 1\ . \tag{6.50}
$$

In addition, this is an especially simple problem that can be solved completely from the original equations. Indeed, the solution is

$$
b(t,t') = q + \ln\left(\frac{h(t)}{h(t')}\right),\quad r(t,t') = \frac{q}{4T}\,\frac{h'(t')}{h(t')}. \tag{6.51}
$$

The exponent $\alpha = 1$ is consistent with this solution since it translates into

$$
B(u) = \eta u\ , \tag{6.52}
$$

which is a solution of Eq. (6.34). The triangular relation function is thus $f(x,y) = x + y - q$. Note that a particular solution corresponds to $h(t) = t^\delta$, which we speculate is a good candidate for being the solution selected by matching at small times.

The p-spin spherical model

It is interesting to recall the known results for this model and compare them with those obtained with the general treatment described above. For spherically constrained models $b_0 = 2$. In the case of the p-spin spherical model, with $p \geq 2$, Eq. (3.23) implies $p(p-1)/2\ (1-q_{PS})^2\ q_{PS}^{p-2} = T^2$. The mass μ in Eqs. (6.1) and (6.2) has to be interpreted as the constant long-time limit of the time-dependent $\mu(t)$ related to the Lagrange multiplier enforcing the spherical constraint; cf. Eq. (2.7). Solving for x one gets $x = -1/(2T)\ (p-2)(1-q_{PS})/q_{PS}$. These equations coincide with those found in Ref. [10]. Equation (6.22) is solved, for all u, by $B(u) = 2q_{PS}[1 - \exp(-\eta u)]$ with η undetermined. Hence the function $\jmath$ defined in Eq. (6.9) that relates any three displacements in the aging regime is $\jmath(b) = q_{PS}(1 - b/2)$ and the triangle relation is just $f(x,y) = xy/q_{PS}$.

If we study Eq. (6.23) for $u \sim 0$ as described in Sec. VI G 3 from Eq. (6.29), $\rho(\alpha)$ is

$$
\rho(\alpha) = \frac{1}{2} \qquad \Rightarrow \qquad \alpha = 1 \qquad \text{for} \quad p > 2\ . \tag{6.53}
$$

The exact solution in the whole interval $u \in [0,\infty]$ is approximated for small u as $B(u) = 2q_{PS}[1 - \exp(-\eta u)] \sim 2q_{PS}\eta u$ and we recover the exponent $\alpha = 1$ in this limit. The case $p = 2$ is particular. Both the third derivative in the numerator of Eq. (6.29) and x are zero. One cannot naively use Eq. (6.29) to determine α. Equation (6.23) can be used also to check that the large-u analysis of it is delicate.

I. From short-range models to long-range models: The appearance of the ultrametric solution

Let us study in detail the limit $\gamma \to 1$ from above for the power law model. In general, we have solved the short-range models with an ansatz consisting of a constant x and a triangular relation $f(x,y) = \jmath^{-1}[\jmath(x)\,\jmath(y)]$, with $\jmath^{-1}(x)$ a nonanalytic function at $x = 1$:

$$
\jmath^{-1}(1+\epsilon) = q + c\epsilon^\alpha + O(\epsilon^{2\alpha})\ , \tag{6.54}
$$

which is different for different models, i.e., different $\mathcal{V}$. This implies, for f,

$$f(q+z,q+z') = q + (z^{1/\alpha} + z'^{1/\alpha})^\alpha + O(z^2, z'^2, zz') \,. \tag{6.55}$$

In Sec. VI G we have also shown when γ tends to one the exponent α for the power law model tends to 0. Hence, when $\gamma \to 1$,

$$f(q+z,q+z') \sim q + \max(z,z') + \text{(higher-order terms)} \tag{6.56}$$

and we see explicitly how the ultrametric solution is approached when the potential correlation range is increased. The FDT violation x stays constant when $\gamma \to 1$ from above and this is also valid when the limit is approached from below, as can be seen from Eq. (7.11).

VII. SOLUTIONS TO THE DYNAMICAL EQUATIONS: LONG-RANGE POTENTIALS

We shall apply the ultrametric ansatz to Eqs. (5.3) and (5.4). The ultrametric ansatz means

$$\begin{aligned} b &= f(b',b'') = \max(b',b'') \,, \\ b'' &= \overline{f}(b',b) = \max(b',b) \end{aligned}$$

for all b, b', b'' inside the interval $[q, b_0]$. Note that $\partial\overline{f}(b',b)/\partial b = 1$ and $\partial\overline{f}(b',b)/\partial b' = 0$ since $b > b'$. Furthermore, $\partial^{(k)}\overline{f}(b',b)/\partial b^{(k)} = \partial^{(k)}\overline{f}(b',b)/\partial b'^{(k)} = 0$ $\forall k \geq 1$. In the dynamic equations (5.3) and (5.4) only $\overline{f}$ enters. We can then take variations of these equations with respect to b safely.

Inserting the ultrametric form of $\overline{f}$ in Eq. (5.3) and in its variation with respect to b and using Eq. (5.5) yields

$$\begin{aligned} 0 &= \frac{2q}{T}\left(\mathcal{V}''(q)F[b] + \frac{1}{4}\overline{M}[b]\right) - F[b]\overline{M}[b] \,, \\ 0 &= X[b]\left[\frac{2q}{T}\left[\mathcal{V}''(q) - \mathcal{V}''(b)\right] - 4\overline{M}[b] + 4\mathcal{V}''(b)F[b]\right] . \end{aligned} \tag{7.1}$$

Solving these equations for $F[b]$ and $\overline{M}[b]$, we obtain

$$\begin{aligned} F[b] &= \mp\frac{q}{2T}\left[\frac{\mathcal{V}''(b) - \sqrt{\mathcal{V}''(b)\mathcal{V}''(q)}}{\mathcal{V}''(b)}\right] , \\ \overline{M}[b] &= \pm\frac{2q}{T}\left[\mathcal{V}''(q) - \sqrt{\mathcal{V}''(b)\mathcal{V}''(q)}\right] . \end{aligned} \tag{7.2}$$

The second variation of Eq. (5.3) combined with the above expressions for $F[b]$ and $\overline{M}[b]$ gives

$$X[b] = \mp\frac{q}{4T}\,\frac{\mathcal{V}'''(b)}{\mathcal{V}''(b)}\sqrt{\frac{\mathcal{V}''(q)}{\mathcal{V}''(b)}} \,. \tag{7.3}$$

The sign must be chosen in such a way that $X[b]$ is negative; it depends on the relative signs of $\mathcal{V}'''$ and $\mathcal{V}''$. In the following we shall consider $\mathcal{V}'' < 0$ and $\mathcal{V}''' > 0$. This is the case for the power law model with long-range correlations. We shall then choose the lower signs in the expressions above. If $\overline{f}$ is a maximum, all the integrals appearing in the higher variations of the equation with respect to b in the limit $b \to q$ vanish.

We require that $|X|$ be a decreasing or constant function of its argument b. This indeed selects the models that can be solved by an ultrametric ansatz. Imposing $|X[b]| < |X[q]|$ for $b > q$ yields

$$1 > \frac{\eta(b)}{\eta(q)} \quad \text{with} \quad \eta \equiv \frac{\mathcal{V}'''}{(-\mathcal{V}'')^{3/2}} \,. \tag{7.4}$$

For instance, if the function $\mathcal{V}$ is of the form (1.6), we have

$$\left(\frac{\theta+q}{\theta+b}\right)^{(1-\gamma)/2)} \leq 1 \,. \tag{7.5}$$

If $\gamma > 1$ this inequality cannot be satisfied. This means that model (1.1) with short-range correlations cannot be solved by an ultrametric ansatz. Instead, if $\gamma \leq 1$, the condition is satisfied for all b and model (1.1) with long-range correlations can be solved by an ultrametric ansatz.

As a second example we can examine the RSGM. Equations (1.10) and (7.4) imply

$$\frac{\exp(b/4)}{\exp(q/4)} \leq 1 \,. \tag{7.6}$$

Since $b \geq q$, then this imposes $q = b$ and the RSGM does not admit an ultrametric solution.

Finally, we study the p-spin model. In this case, $1 - B/2 = C$ and the correlation function decays from q when $t' \to t$ to 0 when $t' \to 0$. Then

$$\left(\frac{q}{c}\right)^{p/2} \leq 1 \,. \tag{7.7}$$

Since $q \geq c$ the inequality is not satisfied.

In conclusion, from the particular cases we presented in the Introduction, only model (1.1) with long-range interactions, i.e., $\gamma \leq 1$, admits an ultrametric solution. More generally, Eq. (7.4) selects the potentials that admit an ultrametric solution.

Formula (7.3) for X, together with the ultrametric form of f, represents the full solution for the model. Due to the time-reparametrization invariance we have included in the equations we cannot go beyond these expressions and obtain the explicit time dependence of the displacement b. We can check that Eq. (3.17) is also solved by this ansatz.

We can now explicitly obtain q, b_0, and $\tilde{q}$ and compare them with the static results obtained with the replica approach [5,19]. Equation (3.23) determines q and with $X[b]$ given by Eq. (7.3), after a bit of algebra, we obtain b_0 from Eq. (5.5):

$$\mathcal{V}''(b_0) = -\frac{\mu^2}{4} \,. \tag{7.8}$$

For the power law model with $\gamma \leq 1$ q and b_0 are given by

$$\frac{q^2}{(\theta+q)^{1+\gamma}} = \frac{2T^2}{\gamma}, \quad (\theta+b_0)^{-(\gamma+1)/2} = \frac{\mu}{\sqrt{2\gamma}} \tag{7.9}$$

and $X[b]$ is

$$X[b] = -\frac{1}{2}\,\frac{(\gamma+1)}{\sqrt{2\gamma}}\,(\theta+b)^{(\gamma-1)/2}\,. \tag{7.10}$$

In the limit $\gamma \to 1^-$ we have

$$X[b] = x = -\frac{1}{\sqrt{2}}, \tag{7.11}$$

which coincides with the constant x obtained for the short-range models when $\gamma \to 1^+$. Note that x is then a constant in all the low-temperature phase (independent of T and μ).

One can check that these equations for q and b_0 coincide with the equilibrium equations obtained with a full replica symmetry breaking ansatz [19,12]. One can also check that the expression we obtained for $X(B)$ is formally identical to the Parisi function $x(q)$ of the replica treatment. Hence the asymptotic dynamical values, in particular the energy density, coincide with the static values, though the particle *is not* visiting any equilibrium state.

In Fig. 1 we confront the analytical predictions (broken curves) to the numerical solution to the dynamical equations plotting $m(t,t')$ vs $b(t,t')$ for $t = 200, 300, 400$. $\gamma = 0.5$, $T = 0.2$, $\mu = 0.2$, and $\theta = 2.5$ in the figure. q and the FDT relaxation converge rather quickly to the predicted value and straight line behavior. However, the aging regime is still very far from the asymptotic analytical curve (for this choice of parameters the curve for the aging regime is unfortunately very similar to a straight line). The convergence for the long-range model is slower than for the short-range model (see Fig. 2 and the discussion in Sec. VI G). We observe that the evolution of the numerical curves as t increases is, though exceedingly slow, towards the dotted line. This family of long-range models behaves in every respect as the Sherrington-Kirkpatrick model of a spin glass [11].

The discussion of the phase diagram for power law models with $\gamma < 1$ is very different from the short-range case. The function $q^2\mathcal{V}''(q)$ is increasing monotonical, thus there is no $T_c^{(1)}$. The critical temperature $T_c(\mu)$ is determined by the condition $b_0 = q$ that implies $\mathcal{V}''(2T_c/\mu) = -\mu^2/4$. This equation coincides with Eq. (6.9) and hence its solution is $T_c^{(2)}(\mu)$ given by Eq. (6.10). We can check that below this line $|X[b]| < 1/2T$ since

$$-2TX(q) = -\frac{q}{2}\,\frac{\mathcal{V}'''(q)}{\mathcal{V}''(q)} = \frac{\gamma+1}{2}\,\frac{q}{\theta+q} < 1. \tag{7.12}$$

The dynamical phase diagram is thus identical to the statics.

The case $\gamma \to 0$

If we take the limit $\gamma \to 0$ the low-temperature phase disappears ($\mu_{\rm cr} \to 0$). Indeed, in this limit $\mathcal{V}' = 1/2$, $\mathcal{V}'' = 0$, and the dynamical equations for R and B can be solved exactly. The solutions are

$$\begin{aligned} R(t,t') &= \exp[-\mu(t-t')], \\ B(t,t') &= \frac{2T}{\mu}\,\{1-\exp[-\mu(t-t')]\} \\ &\quad + \left(C(0,0) - \frac{T}{\mu} + \frac{1}{\mu^2}\right) \\ &\quad \times\{\exp(-2\mu t) + \exp(-2\mu t') \\ &\quad -2\,\exp[-\mu(t+t')]\}\,. \end{aligned} \tag{7.13}$$

The response is time homogeneous for all times. It is interesting to note that both the response and the displacement have exponential decays for all times t, t' and hence some of the assumptions we have made in this paper are not fully justified in this case (for example, one is not allowed to throw the time derivative for large and separated times). This is most clearly seen by keeping the mass finite and taking the limit of large t' (independently of its relation to t); $B(t,t')$ also becomes a TTI function

$$\lim_{t'\to\infty} B(t,t') = \frac{2T}{\mu}\,\{1-\exp[-\mu(t-t')]\}\,. \tag{7.14}$$

The FDT relation holds. The large time dynamics is like the relaxation inside a well and equilibrium dynamics holds for all large times $t, t' > t_{\rm eq}$ related to the mass μ and there is no aging regime. Though our approximated analysis is not fully justified in this case it does not disagree with what we find from the exact calculation. The glassy region of the μ-T plane with nonequilibrium dynamics disappears in this limit. [If the limit of zero mass is taken before the large time limit there is difussion, e.g., $C(t,t) \to C(0,0) + 2T\,t$ when $\mu \to 0$.]

The case $\gamma = 0$ has also been investigated by Parisi [43] from the static point of view using the replica approach. The solution was replica symmetric and hence a sort of trivial dynamical behavior was to be expected.

VIII. SUMMARY AND CONCLUSIONS

We have studied analytically the nonequilibrium dynamics of a particle moving in a general random potential. We have described in more detail three particular potential correlations associated with three well-known models: a power law correlation, the random sine-Gordon model, and the p-spin spherical spin-glass model.

The model with power-type long-range correlation [model (1.1)] is solved by an ultrametric ansatz for the triangular relations and with a nontrivial function of the displacement B as the measure of the violation of FDT in the aging regime

$$b(t,t') = \max\left(b(t,t''), b(t'',t)\right), \quad t' < t'' < t\,,$$
$$X[b] = \frac{q}{4T}\,\frac{\mathcal{V}'''(b)}{\mathcal{V}''(b)}\,\sqrt{\frac{\mathcal{V}''(q)}{\mathcal{V}''(b)}}\,. \tag{8.1}$$

In this case the asymptotic values of the one-time parameters such as the energy density, the equivalent of the Edwards-Anderson parameter q, and the maximum displacement b_0 coincide with the equilibrium values obtained with the full replica symmetry breaking scheme. Though the system never reaches equilibrium, these quantities approach the equilibrium values asymptotically. This has been also shown to hold for the Sherrington-Kirkpatrick model and is expected to hold in models without a threshold in the Thouless-Anderson-Palmer energy landscape [10–12]. This seems to be a characteristic of models that are solved statically with a full replica symmetric scheme. Our results also agree with the numerical observations of Franz and Mézard [12].

Model (1.1) with short-range correlations is solved with a one-blob ansatz for the triangular relations. This means that in the aging regime the function $X[b]$ measuring the departure from the FDT is simply a constant and the dynamical equation determines the function $\overline{f}$ relating any three correlation functions. We have found that in most of the models we consider the inverse function $\overline{f}$ presents an irregularity at the beginning of the aging regime. Explicitly we find

$$b(t,t') - q \propto \left(\frac{h(t)}{h(t')} - 1\right)^{\alpha}\,, \tag{8.2}$$

where α is a continuously variable exponent. This irregularity is an interesting feature that, although generic, has not been previously observed, as far as we know, in the off-equilibrium dynamics of mean-field models (e.g., it does not appear in the p-spin spherical model). A related irregularity, at the beginning of the out of equilibrium relaxations, has been predicted to occur in a model of traps with a broad distribution of trapping times by Bouchaud [23]. This allows for a comparison of the mean-field glassy dynamics with the trap description of aging.

The one-step trap model for the nonequilibrium relaxation of glassy systems [23] predicts that the integrated response of a system to a perturbation applied during the interval $[0, t_w]$ (the thermoremanent magnetization for spin systems) has a very sharp decay to q_{EA} in a microscopic characteristic time. Afterward the aging regime sets on and the waiting time t_w is its only characteristic time. The decay is then described by $m(\tau, t_w) \sim (1 - q_{EA}) g(\tau) + q_{EA} f(\tau/t_w)$, where the first term is the FDT relaxation and the second one is the aging relaxation. The aging part is characterized by two exponents

$$f\left(\frac{\tau}{t_w}\right) \sim 1 - c\left(\frac{\tau}{t_w}\right)^{1-x_1} \quad \text{for } \tau \sim t_w\,,$$
$$f\left(\frac{\tau}{t_w}\right) \sim \left(\frac{\tau}{t_w}\right)^{-x_2} \quad \text{for } \tau \gg t_w\,. \tag{8.3}$$

τ represents the extra time elapsed after t_w. Our results for short-range correlated potentials show that the mean-field dynamics of a particle in a random potential may also be represented by a law of this type, at least in the beginning of the aging regime. Indeed when times are such that the displacement is larger than q we find $m(t_w + \tau, t_w) = x[b(t_w + \tau, t_w) - b_0]$, which at the beginning of this regime reads

$$\begin{aligned} m(t_w + \tau, t_w) &\sim x\left(\frac{h(t_w + \tau)}{h(t_w)} - 1\right)^{\alpha} + x(q - b_0) \\ &= x\left(\frac{\partial \ln h(t_w)}{\partial \ln t_w}\right)^{\alpha}\left(\frac{\tau}{t_w}\right)^{\alpha} + x(q - b_0) \end{aligned} \tag{8.4}$$

[see Eq. (6.32)]. Thus the behavior (8.3) is recovered with $1 - x_1 = \alpha$ provided that $h(t_w)$ is a power law at large t_w. If this is not the case a t_w-dependent amplitude may spoil the form (8.3).

At the end of the aging regime, i.e., when $t'/t \sim 0$, we can compare with the behavior obtained in the limit of a massless particle:

$$b(t_w + \tau, t_w) \sim q + \ln\left(\frac{h(t_w + \tau)}{h(t_w)}\right)\,, \tag{8.5}$$

which naturally leads to logarithmic behavior rather than power law at variance with (8.3).

The RSGM with nonzero mass belongs to the family of models solved by a one-blob ansatz with irregularities. The RSGM in the limit of a vanishing mass as well as the p-spin spherical spin glass are particularly simple; they can be studied in an independent way and they serve as checks to the method to obtain the exponent characterizing the decay at the beginning of the aging regime. As regards the case of a general correlation $\mathcal{V}$ we have obtained the conditions that $\mathcal{V}$ must satisfy in order to be solved by each particular ansatz.

On many occasions it has been pointed out [44,35] that the mean-field spin-glass dynamical equations, with the assumption of the relaxation being at equilibrium, are very similar to Götze's mode-coupling equations for a phenomenological description of the glass transition [45]. The mode-coupling equations for glasses involve only the density-density correlation function and are time homogeneous by construction. Recently, a very interesting first generalization of the mode-coupling approach to account for nonequilibrium phenomena has been proposed [46]. The mean-field dynamical equations for the correlation and response function (without any time homogenous assumption) of the p-spin spherical spin glass are indeed the dynamical equations arising from the mode-coupling treatment of a model without explicit quench disorder—

the Amit-Roginsky ϕ^3 model [46].

The equations we have studied in this paper constitute an enlargement of the class of possible extensions of the mode-coupling equations for glasses. Indeed, one can show that some of the mode-coupling equations of Götze are those associated with the high-temperature phases of the models we consider here with particular choices of the random potential. Though in these models there is explicit quench disorder, it is, however, interesting that the same equations appear. The further advantage of these equations is that they imply a more complicated behavior for their solutions as compared to the solution of the p-spin spherical spin glass. As we have shown in this paper they have nontrivial singularities at the beginning of the aging regime. These singularities may be relevant experimentally for glasses [47,48], an issue which will be discussed in Ref. [48].

Finally, this work opens the way to study other disordered models in finite dimension $D > 0$. We expect some features found here to extend to higher dimensions, such as aging effects with irregularity in the scaling function. Recent results on instabilities of renormalization group flows in several models, such as the sine-Gordon model [49], suggest that aging effects may be relevant also at finite N. We shall report on the finite-dimensional case $D > 0$ elsewhere [16].

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APPENDIX A

In this appendix we extend the Gaussian variational method (GVM) to the dynamics of disordered systems. This method was previously used to study the statics [5]. Here we extend it by applying a Gaussian decoupling to the exact equations of motion obtained from the Martin-Siggia-Rose generating functional. We derive in this way a set of dynamical equations associated with the model defined by the Hamiltonian (1.1) with a general Gaussian random potential V.

We start by defining

$$-\delta(\boldsymbol{x}-\boldsymbol{x}')\ \Delta(\boldsymbol{\phi}-\boldsymbol{\phi}') \equiv \langle V(\boldsymbol{\phi},\boldsymbol{x})\ V(\boldsymbol{\phi}',\boldsymbol{x}')\rangle\ . \tag{A1}$$

The standard Martin-Siggia-Rose action entering the dynamical partition function $\mathcal{Z}_{\rm dyn} = \int \mathcal{D}\boldsymbol{\phi}\ \mathcal{D}\hat{\boldsymbol{\phi}} \exp(-S)$ associated with a Langevin process is

$$\begin{aligned} S = \int d\boldsymbol{x}dt\ &\sum_{\alpha}^{N}\{-T\ [i\hat{\phi}^\alpha(\boldsymbol{x},t)]^2 \\ &+i\hat{\phi}^\alpha(\boldsymbol{x},t)\ \left(\partial_t - \nabla^2+\mu\right)\ \phi^\alpha(\boldsymbol{x},t)\ \} \\ &+\int d\boldsymbol{x}dtdt'\ \sum_{\alpha,\beta}^{N}\left[-\frac{1}{2}\ i\hat{\phi}^\alpha(\boldsymbol{x},t)\right. \\ &\left.\times\Delta''_{\alpha\beta}[\boldsymbol{\phi}(\boldsymbol{x},t)-\boldsymbol{\phi}(\boldsymbol{x},t')]\ i\hat{\phi}^\beta(\boldsymbol{x},t')\right]\ , \end{aligned} \tag{A2}$$

with $\Delta''_{\alpha\beta}(\boldsymbol{\phi}-\boldsymbol{\phi}') \equiv -(\delta^2/\delta\phi^\alpha\delta\phi'^\beta)\Delta(\boldsymbol{\phi}-\boldsymbol{\phi}')$. The indices α,β label the components of the N-dimensional vectors $\boldsymbol{\phi}$, $i\hat{\boldsymbol{\phi}}$.

The dynamical equations for $R^{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') \equiv \langle\phi^\alpha(\boldsymbol{x},t)\ \ i\hat{\phi}^\beta(\boldsymbol{x}',t')\rangle$, $t\geq t'$, and $C^{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') \equiv \langle\phi^\alpha(\boldsymbol{x},t)\ \phi^\beta(\boldsymbol{x}',t')\rangle$ follow from

$$\left\langle\ i\hat{\phi}^\beta(\boldsymbol{x}',t')\ \frac{\delta S}{\delta i\hat{\phi}^\alpha(\boldsymbol{x},t)}\ \right\rangle = \delta_{\alpha\beta}\ \delta^D(\boldsymbol{x}-\boldsymbol{x}')\ \delta(t-t')\ , \tag{A3}$$

$$\left\langle\ \phi^\alpha(\boldsymbol{x}',t')\ \frac{\delta S}{\delta i\hat{\phi}^\beta(\boldsymbol{x},t)}\ \right\rangle = 0\ , \tag{A4}$$

respectively. These are exact equations of motion that can be derived by standard methods [50]. The angular brackets here and in what follows denote a mean over the fields weighted with the dynamical effective action.

The first equation reads

$$\begin{aligned} &\delta_{\alpha\beta}\ \delta^D(\boldsymbol{x}-\boldsymbol{x}')\ \delta(t-t') \\ &\quad= \left(\partial_t-\nabla^2+\mu\right)\ R^{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') \\ &\qquad-\sum_\gamma\int dt''\langle i\hat{\phi}^\beta(\boldsymbol{x}',t')\ \Delta''_{\alpha\gamma}[\boldsymbol{\phi}(\boldsymbol{x},t)-\boldsymbol{\phi}(\boldsymbol{x},t'')] \\ &\qquad\times i\hat{\phi}^\gamma(\boldsymbol{x},t'')\ \rangle, \end{aligned} \tag{A5}$$

where we have used the fact that $\langle i\hat{\boldsymbol{\phi}} i\hat{\boldsymbol{\phi}}\rangle$ must be zero to preserve causality. The equation for C reads

$$\begin{aligned} &2T\ R^{\alpha\beta}(\boldsymbol{x}',t';\boldsymbol{x},t) \\ &\quad= \left(\partial_t-\nabla^2+\mu\right)\ C^{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') \\ &\qquad-\sum_\gamma\int dt''\langle\phi^\alpha(\boldsymbol{x}',t')\ \Delta''_{\beta\gamma}[\boldsymbol{\phi}(\boldsymbol{x},t)-\boldsymbol{\phi}(\boldsymbol{x},t'')] \\ &\qquad\times i\hat{\phi}^\gamma(\boldsymbol{x},t'')\ \rangle\ . \end{aligned} \tag{A6}$$

The last term on the right-hand side of Eqs. (A5) and (A6) can be computed using a Gaussian approximation, i.e., assuming that the fields $\boldsymbol{\phi}$ and $i\hat{\boldsymbol{\phi}}$ have a Gaussian distribution. Using the Ito prescription, which states $R(\boldsymbol{x},t;\boldsymbol{x},t) = 0$, and the following rule for averaging any set of Gaussian variables $\{\varphi_\imath\}$: $\langle \varphi_\imath F(\varphi)\rangle = \sum_j \langle \varphi_i \varphi_j\rangle \, \langle F'_j(\varphi)\rangle$, the equation for R becomes

$$
\begin{aligned}
\delta_{\alpha\beta}\, \delta^D(\boldsymbol{x}-\boldsymbol{x}')\, \delta(t-t') &= \left(\partial_t - \nabla^2 + \mu\right)\, R^{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') \\
&\quad - \sum_{\gamma\delta\epsilon} \int dt'' R^{\delta\gamma}(\boldsymbol{x},t;\boldsymbol{x},t'')[R^{\epsilon\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') \\
&\quad - R^{\epsilon\beta}(\boldsymbol{x},t'';\boldsymbol{x}',t')]\langle \Delta^{(4)}_{\alpha\gamma\delta\epsilon}(\boldsymbol{\phi}-\boldsymbol{\phi}')\rangle\, . \qquad \text{(A7)}
\end{aligned}
$$

Similarly, the equation for C reads

$$
\begin{aligned}
2T\, R^{\alpha\beta}(\boldsymbol{x}',t';\boldsymbol{x},t) &= \left(\partial_t - \nabla^2 + \mu\right)\, C^{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{x}',t') - \sum_\gamma \int dt'' R^{\alpha\gamma}(\boldsymbol{x}',t';\boldsymbol{x},t'')\, \langle \Delta''_{\beta\gamma}(\boldsymbol{\phi}-\boldsymbol{\phi}')\rangle \\
&\quad - \sum_{\gamma,\delta,\epsilon} \int dt'' R^{\delta\gamma}(\boldsymbol{x},t;\boldsymbol{x},t'')[C^{\epsilon\alpha}(\boldsymbol{x},t;\boldsymbol{x}',t') - C^{\epsilon\alpha}(\boldsymbol{x},t'';\boldsymbol{x}',t')]\langle \Delta^{(4)}_{\beta\gamma\delta\epsilon}(\boldsymbol{\phi}-\boldsymbol{\phi}')\,\rangle\, , \qquad \text{(A8)}
\end{aligned}
$$

where $\langle\,\rangle$ denotes an average over a Gaussian distribution. These are the general dynamical equations obtained from the GVM. We now specialize to the case of O(N) symmetry, i.e., isotropy, which also includes $N=1$ (RSGM) as a trivial case, and show that with the definition of $\mathcal{V}$ appropriate for studying the large-N limit, the *same* function $\hat{\mathcal{V}}$ has to be used in the statics and the dynamics.

Define the two functions

$$
-N\, \mathcal{V}\left(\frac{\boldsymbol{\phi}^2}{N}\right) \equiv \Delta(\boldsymbol{\phi}), \quad -N\, \hat{\mathcal{V}}\left(\frac{\boldsymbol{\phi}^2}{N}\right) \equiv \langle \Delta(\boldsymbol{\phi})\rangle. \qquad \text{(A9)}
$$

Using isotropy, Eqs. (A7) and (A8) contain only $\Delta''_{\alpha\beta}$ and $\Delta^{(4)}_{\gamma\gamma\alpha\beta}$, which are related to $\hat{\mathcal{V}}$ through

$$
\begin{aligned}
\langle \Delta''_{\alpha\beta}(\boldsymbol{\varphi})\rangle &= \frac{1}{N}\delta_{\alpha\beta} \int \frac{d^N q}{(2\pi)^N}\, \Delta(q) q^2 \exp\left(-\frac{q^2 \boldsymbol{\varphi}^2}{2N}\right) \\
&= 2\delta_{\alpha\beta}\hat{\mathcal{V}}'\left(\frac{\boldsymbol{\varphi}^2}{N}\right), \\
\left\langle \sum_\gamma \Delta^{(4)}_{\gamma\gamma\alpha\beta}(\boldsymbol{\varphi})\right\rangle &= 4\delta_{\alpha\beta}\hat{\mathcal{V}}''\left(\frac{\boldsymbol{\varphi}^2}{N}\right)\, . \qquad \text{(A10)}
\end{aligned}
$$

Replacing these expressions in (A7) and (A8), one recovers the dynamical equations (2.1) and (2.2) with $\hat{\mathcal{V}}$ substituted for $\mathcal{V}$. In the limit of $N\to\infty$ these two functions become identical [5].

One can also apply the Gaussian decoupling in the exact equations of motion for the static replica theory. This provides an alternative way of obtaining the approximated saddle point equations of Ref. [5]. The exact saddle point equation is

$$
\begin{aligned}
&(\nabla^2+\mu)\langle \phi_a(x)\phi_b(0)\rangle \\
&\quad - \left\langle \phi_a(x) \sum_{c,d=1}^{n} \frac{\delta}{\delta\phi_b(0)} \left(\sum_{c,d=1}^{n} \mathcal{V}(\phi_c(0)-\phi_d(0))\right)\right\rangle \\
&\quad = T\, \delta^D(x)\, \delta_{ab}. \qquad \text{(A11)}
\end{aligned}
$$

Applying now the Gaussian decoupling, one recovers the saddle point equations of Ref. [5].

APPENDIX B

In this appendix we describe the derivation of the long-time dynamical equations, i.e., for times such that $t > t' \gg 1$ and $B(t,t') = b(t,t') > q$. Neglecting the time derivative, the r equation reads

$$
\begin{aligned}
\frac{\partial r(t,t')}{\partial t} &\sim 0 \sim r(t,t')\left(-\mu + 4\int_0^{t_-} ds\, \mathcal{V}''(B(t,s))\, R(t,s) + 4\int_{t_-}^{t} ds\, \mathcal{V}''(b_F(t-s)) r_F(t-s)\right) \\
&\quad -4\int_{t'}^{t'_+} ds\, \mathcal{V}''(b(t,s))\, r(t,s)\, r_F(s-t') - 4\int_{t'_+}^{t_-} ds\, \mathcal{V}''(b(t,s))\, r(t,s)\, r(s,t') \\
&\quad -4\int_{t_-}^{t} ds\, \mathcal{V}''(b_F(t-s))\, r_F(t-s)\, r(s,t')\, . \qquad \text{(B1)}
\end{aligned}
$$

We have explicitly separated in the integrals the FDT regimes (integrals symbolically denoted $\int_{t_-}^{t} ds$) from the widely separated time regimes. In the last term of Eq. (B1) s varies from t_- to t and it is very far away from t'. The assumption is that in that case the functions vary very slowly (we have already neglected the time derivative in the left-hand side). Hence $r(s,t')$ is almost constant in this interval and it can be approximated by $r(t,t')$. The last term then cancels the third one. In addition, using the same approximation and the FDT relations (3.5) one computes explicitly the integrals involving the FDT parts

$$-4\int_{t'}^{t'_+} ds\ \mathcal{V}''(b(t,s))\ r(t,s)\ r_F(s-t')$$
$$\sim -4\mathcal{V}''(b(t,t'))r(t,t')\int_0^{\infty} d\tau\ r_F(\tau)$$
$$\sim \frac{-2q}{T}\mathcal{V}''(b(t,t'))r(t,t'),$$

$$4\int_{t_-}^{t} ds\ \mathcal{V}''(b_f(t-s))\ r_F(t-s) = -\frac{2}{T}\ [\mathcal{V}'(0)-\mathcal{V}'(q)] \tag{B2}$$

using $b_F(\infty)=q$ and $b_F(0)=0$. Finally, the second term inside the set of large parentheses involves both short and long times since the integral goes from the initial time, which is strictly zero, to t_-, which is large and belongs to the asymptotic regime. One should then separate the contribution from finite times to that of long times

$$\int_0^{t_-} ds\ \mathcal{V}''(B(t,s))\ R(t,s)$$
$$\sim \int_0^{0_+} ds\ \mathcal{V}''(b_s(t,s))\ r_s(t,s)$$
$$+\int_{0_+}^{t_-} ds\ \mathcal{V}''(b(t,s))\ r(t,s)\ . \tag{B3}$$

We recall the weak long-term memory hypothesis and assume that the finite times do not contribute to the long-time dynamics and that the system "forgets" at large times what happened in the very short times after the initial time. Then the first term on the right-hand side of Eq. (B3) is neglected and one finally obtains the equation for the slow part of the response function

$$0 = r(t,t')\big(-\mu + 4\int_0^{t} ds\ \mathcal{V}''(b(t,s))r(t,s)$$
$$-\frac{2q}{T}\mathcal{V}''(b(t,t'))\big) - 4\int_{t'}^{t} ds\ \mathcal{V}''(b(t,s))\ r(t,s)r(s,t'). \tag{B4}$$

The same procedure can be carried through for $b(t,t')$. First, one can use that at large t,t', $C(t,t)\simeq C(t',t')\simeq c_F(0)=\tilde{q}$. Then the integrals that appear in the b equation and contain FDT pieces are

$$2\int_{t_-}^{t} ds\ \mathcal{V}'(b_F(t-s))\ r_F(t-s) = \frac{-1}{T}\ [\mathcal{V}(0)-\mathcal{V}(q)],$$

$$-2\int_{t'_-}^{t'} ds\ \mathcal{V}'(b(t,t'))\ r_F(t'-s) = \frac{-q}{T}\ \mathcal{V}'(b(t,t')),$$

$$2\int_{t_-}^{t} ds\ \mathcal{V}''(b_F(t-s))\ r_F(t-s)\ b_F(t-s)$$
$$= \frac{q}{T}\ \mathcal{V}'(q) + \frac{1}{T}\ [\mathcal{V}(0)-\mathcal{V}(q)],$$

$$2\int_{t_-}^{t} ds\ \mathcal{V}''(b_F(t-s)(\ r_F(t-s)\ [b(t,t')-b(s,t')] \sim 0. \tag{B5}$$

One also needs to compute

$$-2\int_{t'_-}^{t'+} ds\ \mathcal{V}''(b(t,s))\ r(t,s)b_F(s,t')$$
$$\sim -2\mathcal{V}''(b(t,t'))\ r(t,t')\int_{t'_-}^{t'+} ds\ b_F(s,t')\ . \tag{B6}$$

One can see that this integral is subdominant and gives a vanishing contribution to the $b(t,t')$ equation in the large-t limit. Since $r(t,t')$ itself is of order $1/t$ (see below) it results that the FDT part of this integral is of order $q(t'_+ - t'_-)/t$. The rapidly varying part of the total integral is thus vanishingly small compared to the total integral, which is dominated by the slowly varying part.

Again, this equation could have contributions from the initial times $0\le s<0_+$. We assume they are subleading, i.e.,

$$0 \sim \int_0^{0_+} ds\ \mathcal{V}'(B(t,s))\ [R(t,s)-R(t',s)],$$

$$0 \sim \int_0^{0_+} ds\ \mathcal{V}''(B(t,s))$$
$$\times R(t,s)\ [B(t,s)+B(t,t')-B(s,t')]\ . \tag{B7}$$

One finally obtains the equation for the slow part of the displacement correlation function

$$0=\left(-\frac{\mu}{2}+2\int_0^t ds\,\mathcal{V}''(b(t,s))\,r(t,s)\right)b(t,t')+T+\frac{q}{T}[\mathcal{V}'(q)-\mathcal{V}'(b(t,t'))]$$
$$+2\int_0^t ds\,\mathcal{V}'(b(t,s))\,r(t,s)-2\int_0^{t'} ds\,\mathcal{V}'(b(t,s))\,r(t',s)$$
$$+2\int_0^t ds\,\mathcal{V}''(b(t,s))\,r(t,s)b(t,s)-2\int_0^{t'} ds\,\mathcal{V}''(b(t,s))\,r(t,s)b(t',s)-2\int_{t'}^t ds\,\mathcal{V}''(b(t,s))\,r(t,s)b(s,t')\,. \quad \text{(B8)}$$

From (2.3) one finds, using the same decomposition of the integrals and formulas (B5) to evaluate the FDT parts, the large time dynamical equation for $C(t,t)$ and $C(t,t')$.

APPENDIX C

Let us also indicate the derivation of another (equivalent) form for the equation for $R(t,t')$ and $r(t,t')$. Starting from (2.1) one defines

$$\mathcal{F}(t,t')=-\int_{t'}^t dsR(t,s) \quad \text{(C1)}$$

such that $R(t,t')=\frac{\partial\mathcal{F}(t,t')}{\partial t'}$. One obtains

$$\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\mathcal{F}(t,t')=-\mu\frac{\partial\mathcal{F}(t,t')}{\partial t'}$$
$$+4\int_0^t ds\,\mathcal{V}''(B(t,s))\,\frac{\partial\mathcal{F}(t,s)}{\partial s}\frac{\partial\mathcal{F}(t,t')}{\partial t'}$$
$$-4\int_{t'}^t ds\,\mathcal{V}''(B(t,s))\,\frac{\partial\mathcal{F}(s,t')}{\partial t'}\frac{\partial\mathcal{F}(t,s)}{\partial s}. \quad \text{(C2)}$$

Integrating over t' and using $\mathcal{F}(t',t')=0$, one obtains

$$\frac{\partial\mathcal{F}(t,t')}{\partial t}$$
$$=-1+\left(-\mu+4\int_0^t ds\,\mathcal{V}''(B(t,s))\,\frac{\partial\mathcal{F}(t,s)}{\partial s}\right)\mathcal{F}(t,t')$$
$$-4\int_{t'}^t ds\,\mathcal{V}''(B(t,s))\,\mathcal{F}(s,t')\frac{\partial\mathcal{F}(t,s)}{\partial s}. \quad \text{(C3)}$$

The integration constant has been fixed using the limit $t\to t'$ and

$$\lim_{t'\to t}\frac{\partial\mathcal{F}(t,t')}{\partial t}=-\lim_{t'\to t}\left(R(t,t)+\int_{t'}^t ds\frac{\partial R(t,s)}{\partial s}\right)$$
$$=\lim_{\epsilon\to 0}[r_F(0)-r_F(\epsilon)]=-1. \quad \text{(C4)}$$

Note that at this stage equation (C3) is exact and provides another convenient form of Eq. (2.1).

APPENDIX D

In this appendix we obtain the recursion relations for the function $B(u)$ suitable for a numerical solution. Defining $v=u^\alpha$ and $B(u)=vg(v)$, one introduces $g(v)^n=\sum_{m=0}^\infty g(n,m)v^m$ and $g(n,0)=1$. Equation (6.24) then leads to

$$0=\sum_{n=2}^\infty c_n v^n g(v)^n+Xc_{n-1}v^n\,H(n,v), \quad \text{(D1)}$$

where $c_n\equiv\mathcal{V}^{(n+1)}(q)/n!$, $X=-2Tx/q$, and

$$H(n,v)=\sum_{m=0}^\infty h(n,m)v^m$$
$$=\frac{1}{v^n}\frac{d}{du}\left[uv^n\int_0^1 dx x^{(n-1)/a}(1-x)^\alpha\right.$$
$$\left.\times g(vx^\alpha)^{n-1}g(v(1-x)^\alpha)\right]. \quad \text{(D2)}$$

After some algebra, we obtain two recursion relations for the coefficients $g(n,m)$ and $h(n,m)$

$$g(n,m)=\sum_{j=0}^m g(n-1,j)\,g(1,m-j), \quad \text{(D3)}$$
$$h(n,m)=\sum_{j=0}^m g(n-1,j)\,g(1,m-j)\left(1+\frac{n+m}{a}\right)$$
$$\times\beta\left(1+\frac{n-1+j}{a},1+\frac{m-j+1}{a}\right), \quad \text{(D4)}$$

where $\beta(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$. At the iteration n the coefficients $g(i,j)$ and $h(i,j)$, with $i=3,\ldots,n$ and $j=n-i=0,\ldots,n-3$, are first directly obtained from the recursion relations (D3) and (D4). The coefficients $g(1,n-2)$ and $g(2,n-2)$ [and $h(1,n-2)$ $h(2,n-2)$] are then obtained from the equation

$$2g(1,0)g(1,n-2)[c(2)+Xc(1)\tilde\beta(1,n-1)]$$
$$=-\sum_{k=3}^n[c(k)g(k,n-k)+Xc(k-1)h(k,n-k)]$$
$$-\sum_{k=1}^{n-3}g(1,k)g(1,n-2-k)$$
$$\times[c(2)+Xc(1)\tilde\beta(1+k,n-1-k)]\,, \quad \text{(D5)}$$

with

$$g(2,n-2) = 2g(1,0)g(1,n-2) + \sum_{1}^{n-3} g(1,k)g(1,n-2-k), \qquad \text{(D6)}$$

$$h(2,n-2) = 2g(1,0)g(1,n-2)\tilde{\beta}(1,n-1) + \sum_{1}^{n-3} g(1,k)g(1,n-2-k) \times\tilde{\beta}(1+k,n-1-k)\ , \qquad \text{(D7)}$$

where $\tilde{\beta}(x,y) = [1+(x+y)/a]\beta(1+x/a,1+y/a)$.

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