Scaling laws for river networks

Amos Maritan,[1] Andrea Rinaldo,[2] Riccardo Rigon,[3,4]
Achille Giacometti,[5] and Ignacio Rodríguez-Iturbe[4]
[1] Istituto Nazionale di Fisica della Materia, International School for Advanced Studies and Istituto Nazionale di Fisica Nucleare INFN, via Beirut 2-4, 34014 Trieste, Italy
[2] Istituto di Idraulica "G.Poleni," Università di Padova, via Loredan 20, I-35131 Padova, Italy
[3] Dipartimento di Ingegneria Civile e Ambientale, Università di Trento, Mesiano di Povo, Trento, I-38050, Italy
[4] Department of Civil Engineering, Texas A & M University, College Station, Texas 77843-3136
[5] Institut für Festkörperforschung der Kernforschungsanlange, Postfach 1913, D-52425, Jülich, Germany
(Received 4 May 1995)

Seemingly unrelated empirical hydrologic laws and several experimental facts related to the fractal geometry of the river basin are shown to find a natural explanation into a simple finite-size scaling ansatz for the power laws exhibited by cumulative distributions of river basin areas. Our theoretical predictions suggest that the exponent of the power law is directly related to a suitable fractal dimension of the boundaries, to the elongation of the basin, and to the scaling exponent of mainstream lengths. Observational evidence from digital elevation maps of natural basins and numerical simulations for optimal channel networks are found to be in good agreement with the theoretical predictions. Analytical results for Scheidegger's trees are exactly reproduced.

PACS number(s): 64.60.Ht, 68.70.+w, 92.40.Gc, 92.40.Fb

I. INTRODUCTION

Power laws, which are the signature of fractal behavior [1], have been experimentally observed over a huge range of scales in probability distributions describing river basin morphology [2–6]. As a part of the river basin, fluvial networks display a rich variety of fractal and multifractal structures [1,7] whose dynamic origin and geometrical description are of greatest importance both from the practical point of view and for a deeper understanding of how natural events occur.

The starting point for this work is the seminal work of Hack [8]. He demonstrated the applicability and the far-reaching implications of a power function by relating the area of a drainage basin with the length of the principal river of the basin, for rivers of the Shenandoah Valley and adjacent mountains in Virginia. This empirical relation is widely accepted nowadays although its spatial range of validity [9–11] and the inference of the complex role of unchanneled valleys [5] are still debated. Many other empirical laws describing structural properties in natural basins through power functions have been likewise discovered [10,12]. All of them support the ubiquitous presence of fractal forms in the river basin. Furthermore, the independence of the characters of the scale-free spatial organization of a river network from, say, geology, climate, vegetation, or exposed lithology has suggested [13,14] that principles of self-organized criticality [15] are at work in the development of a river basin.

Although recently significant steps forward have been made in the understanding of the evolutional properties of river networks [13–19], it is only currently that a common theoretical basis where all hydrological empirical laws can find a natural explanation is being developed. In this work we shall show how a simple finite scaling ansatz [20] leads to such a natural explanation for many empirical and experimental facts, as well as for the scaling properties exhibited by real natural basins and their suitable numerical simulations.

A river basin is an anisotropic system defined by a *longitudinal* typical length $L_\parallel = L$ (which we will identify with the linear size of the system) and a typical perpendicular length $L_\perp \le L$ (see Fig. 1). We shall assume that the latter scales as

$$L_\perp = L^H, \quad 0 \le H \le 1 \tag{1}$$

and call the basin *self-affine* if $H < 1$ and self-similar if $H = 1$. Here H is called the *Hurst* exponent due to the similarity to the analog in the fractional Brownian motion context [1]. On the other hand the Hurst exponent can be regarded as a *wandering* exponent by mapping the two-dimensional substrate into a $1 + 1$ tracer motion, where the time plays the role of the longitudinal distance [21]. Equation (1) postulates that basin boundaries are self-affine *curves* for which $L_\parallel, L_\perp$ can be seen, respectively, as diameter and width [7]. Interestingly, it is a fairly general property of self-affine boundaries that their embedded area, say a, is related to the diameter and width via

$$a = L^{1+H} , \tag{2}$$

valid whatever the self-affine scalings of the boundaries [22].

Hack's law [8] relates the length of the longest stream l in the drainage region measured from any site to the divide, with the drainage area of the basin, say a (the number of sites connected to the actual site through drainage directions; that is, the area of land that collects precipitations contributing to the network):

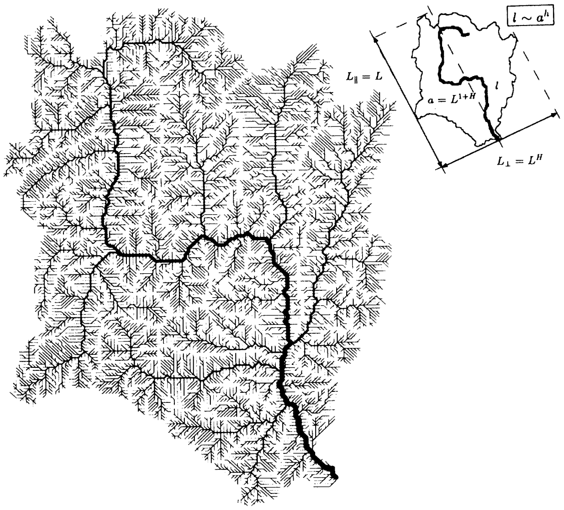

FIG. 1. Fella river network (Northern Italy). The solid lines are drainage directions indicating local gradients of elevation. In the inset we show a sketch of the river basin where L is the longitudinal length (and also the linear size of the system), $L_\perp$ is the transverse length, and l is the length of the mainstream.

$$l \sim a^h , \tag{3}$$

where the accepted values for the exponent are in the range $h = 0.57 - 0.6$ [9]. The significant departure of h from the Euclidean value 0.5 led [1] to early speculations on the fractal nature of river networks.

Many observations of real basins [3,4,23] reveal that the length l is related to the linear size of the system through

$$l \sim L^{d_l}, \quad 1 \leq d_l \leq 1 + H , \tag{4}$$

where $d_l \sim 1.1$. Here the bounds mean that the mainstream ranges from straight ($d_l = 1$) to space filling ($d_l = 1 + H$).

Another well-known hydrological law states that the density of sources in a basin is constant, i.e., its number is *independent* of the size of the system [24]. The fact that $h > 0.5$ is an indication of anisotropy in the basin shape. Indeed Hack noted that if geometrical similarity is to be preserved as a drainage basin increases in area downstream, meaning that there is no change in shape, then the exponent in Eq. (3) should be 0.5. We shall define $1 - H$ as the *elongation exponent* [23] because

$$\frac{a}{L^2} \sim L^{-(1-H)}, \quad 1 - H > 0 . \tag{5}$$

The different experimental values [9,11] of Hack's exponent suggest that in real basins there is a tendency toward *elongation* of the larger catchments; that is, basins tend to become longer and narrower as they enlarge. In terms of the elongation exponent this means $(1 - H) > 0$, while if $(1 - H) < 0$ the basins experience contraction.

II. SCALING OF AREAS

Within this context, a river network is a spanning tree defined in a lattice of arbitrary size and shape. A network is therefore a set of drainage directions (Fig. 1). A river basin is a scalar field of elevations that is consistent with the mapping of drainage directions defined by the corresponding network. Drainage directions are usually identified by steepest descent, i.e., the local gradient of the elevation field.

In a river basin, the area of a point of the substrate is defined as the total contributing area draining into the point through drainage directions, or the number of sites upstream of the point connected by the network. From the computational point of view, it can also be regarded as a measure of the flow rate if a unit weight is assigned to each source—a unit rainfall (constant mass injection) is applied uniformly over the basin. In natural basins, total contributing areas are amenable to detailed experimental investigation through the experimental analysis of digital elevation maps (DEM's) [3] (see below).

Let then a_x be the area at a given site x (in the case of real basin the site is assumed to coincide with the pixel in the DEM). The equation for the area at site x, say a_x, reads

$$a_x = \sum_{y \in \mathrm{NN}(x)} W_{x,y} a_y + 1 , \tag{6}$$

where $W_{x,y}$ is a matrix that is 1 if x collects water from its nearest-neighbor (NN) site y and 0 otherwise.

Rodriguez-Iturbe *et al.* [4] showed that real river basins exhibit a power-law probability of exceedence of the total contributing area a, say $P(a)$, that scales like $P(a) \propto a^{-\beta}$ with $\beta = 0.43 \pm 0.02$. This was shown for several log scales, and the finite scale effect shown by all probability plots was not considered therein. We will now explore the inference of the finite-size effect yielding $P(a, L)$, where L is a given linear size of the system. Figure 2 shows $P(a, L)$ in four sub-basins of different size L within the river Fella region [25].

Let us define the probability density distribution $p(a, L) = -\partial P(a, L)/\partial a$ for a basin to have area a for a given linear size L. If the system is self-organized [15], then one might expect, in analogy to critical phenomena [20,26], that the finite-size distribution $p(a, L)$ obeys a scaling form of the type

$$p(a, L) = a^{-\tau} f\left(\frac{a}{a_c(L)}\right), \quad 1 \leq \tau \leq 2 , \tag{7}$$

where $a_c(L)$ is a characteristic area and $f(x)$ is a scaling function satisfying the following properties:

$$\lim_{x \to \infty} f(x) = 0 \quad \text{sufficiently fast} , \tag{8a}$$

$$\lim_{x \to 0} f(x) = c , \tag{8b}$$

where c is a suitable constant. We observe that τ cannot be smaller than 1 because for $L \to \infty$ we expect a pure

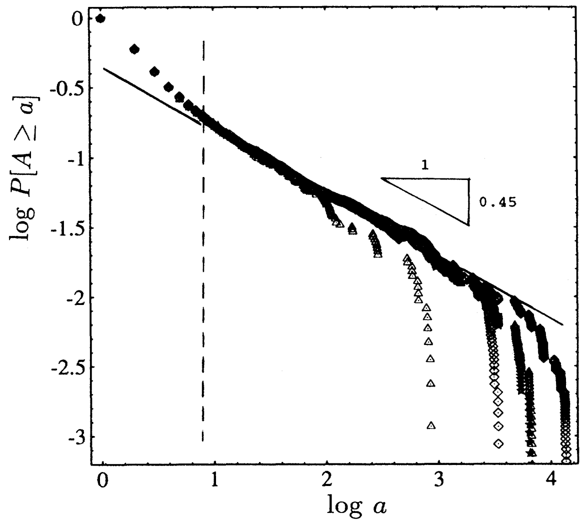

FIG. 2. Probability of exceedence $P(a, L)$ of total contributing area of any site in the basin of size L for four real sub-basins of the River Fella of different size L. The largest basin covers 4 $\log_{10}$ scales of a. Here we employ pixels units for which a source has $a = 1$. Here the dashed line indicates a reasonable threshold for channelized areas [30] (i.e., the suitable minimum area supporting a channel head).

power-law behavior and $\int_1^\infty p(a, L)da$ must be 1. Similar arguments require that $\tau \leq 2$. Equation (8a) ensures the correct behavior at infinity, while Eq. (8b) gives a power-law behavior in the infinite-size limit. Due to Eq. (2) we have that the characteristic area $a_c(L)$ obeys

$$a_c(L) \sim L^\phi \ , \tag{9}$$

with $\phi = 1 + H$. Since we require normalization,

$$\begin{aligned} 1 &= \int_1^\infty da \ a^{-\tau} f\left(\frac{a}{L^\phi}\right) \\ &= L^{\phi(1-\tau)} \int_{L^{-\phi}}^\infty dx \ x^{-\tau} f(x) = \frac{c}{\tau - 1} \ . \end{aligned} \tag{10}$$

The last equality follows by noting that since $1 \leq \tau \leq 2$, we cannot allow the lower cutoff to go to zero. Therefore we find

$$p(1, L \to \infty) = c = \tau - 1 \ ; \tag{11}$$

that is, the number of sources in a basin is *independent* of the size of the system. This property resembles the well-known geomorphological result known as the independence of drainage density on total basin area [24], where drainage density is a measure of the number of sources (i.e., the ratio of channelized area divided by total area). However, we recall that $p(1, L \to \infty)$ is an extrapolation of the power-law behavior for low values of a, a procedure that is impaired, in practice, by the lower cutoff imposed in natural river basins by channel initiation [2]. Hillslope, i.e., nonfluvial areas, have, in fact, a markedly different behavior, and the relationship Eq. (7) is characteristic of the fluvial network. Nevertheless, the result $p(1, \infty) = \text{const}$ is reminiscent of the observational properties of drainage density. Also, in topologically random networks [2] where $\tau = 1.5$ exactly, Eq. (11) applies since 50% of the links are indeed sources.

Often one is interested in the behavior of the distribution probability of the *total contributing area*; that is,

$$P(a, L) = \int_a^{+\infty} dA \ p(A, L) \ . \tag{12}$$

Since $\phi = 1 + H$ and using Eqs. (7) and (12) jointly with $a_c(L) \sim L^\phi$, one gets

$$P(a, L) = a^{-\beta} \mathcal{F}\left(\frac{a}{L^\phi}\right) \ , \tag{13}$$

with

$$\beta = \tau - 1 \tag{14}$$

and where we have defined the quantity

$$\mathcal{F}(x) = x^\beta \int_x^{+\infty} dy \ y^{-(1+\beta)} f(y) \ , \tag{15}$$

which is independent of L.

Many network models that have been studied in different contexts have a *directed* character due to the fact that they are typically grown on a slope that gives a preferred direction to the flow. Under this condition one expects [27,30] that

$$\langle a \rangle = \int_1^{+\infty} da \ a \ p(a, L) \sim L \ . \tag{16}$$

In this case, the lower cutoff can be allowed to go to zero, since the integral is convergent for $L \to \infty$, and thus one easily finds

$$\phi = (2 - \tau)^{-1} \ , \tag{17}$$

which means that there is only *one* independent exponent in Eq. (7). This result has already been observed [26,28]. Notice, however, that Eq. (16) holds only if statistically relevant river configurations are directed [27].

Using Eqs. (14), (17), and (9) we get

$$\beta = \frac{H}{H + 1} \ , \tag{18}$$

in substantial agreement with theoretical results [23] and experimental data [4,23,29] from which we expect H in the range $0.75 - 0.8$ and $\beta = 0.43 \pm 0.02$. Interestingly, the elongation exponent is $1 - H$ [see Eq. (5)], thereby showing elongation because $H < 1$. We will now show that the nondirect character yields a minor modification to (18).

In the general case (nondirected networks), from Eq. (7) one obtains $\langle a \rangle = L^{\phi(2-\tau)} \int_{1/L^\phi}^\infty dx x^{1-\tau} f(x)$, where the lower cutoff is irrelevant because the integral converges for $L \to \infty$. It thus follows [30] that

$$\langle a\rangle \propto L^{\phi(2-\tau)} \ . \tag{19}$$

The above results also allow one to relate the exponents τ and ϕ, thus linking the planar aggregation structure to the elongation imposed by the characteristic size, since $\langle a\rangle = \ell \propto L^{\varphi}$ [30]. Whatever the definition, we find the relationship linking the scaling coefficients to be

$$\phi = \frac{\varphi}{2-\tau} \ , \tag{20}$$

where $\varphi \approx 1.05$ is defined in [23,30]. Notice that in the case $\varphi = 1$ one recovers Eq. (18).

The above relationship is analytically verified, for instance, in the simple Scheidegger models of network development [31], where $\tau = 4/3$ and $\varphi = 1$ yields exactly $\phi = 3/2$. We also note that in the general case one obtains

$$\beta = \frac{1+H-\varphi}{H+1} \ . \tag{21}$$

We finally note that, from Eqs. (7) and (9), one obtains

$$\langle a^n\rangle \propto L^{(n-\tau+1)\phi} \ , \tag{22}$$

and thus we can provide an alternative definition of characteristic *size* of the contributing area, $\mathcal{A}$, as

$$\mathcal{A} = \frac{\langle a^2\rangle}{\langle a\rangle} \propto L^{\phi} \ , \tag{23}$$

which, once substituted in Eq. (7), yields the final scaling relationship in the form

$$p(a,\mathcal{A}) = a^{-\tau} f\left(\frac{a}{\mathcal{A}}\right) \ . \tag{24}$$

From the data shown in Fig. 2, one obtains the results shown in Fig. 3. for the river Fella basin, in domains ranging from 2200 to 140 km^2. Here the exponent of the power law is the experimental value $\tau = 1.45$ [25]. The collapse of the various curves is deemed excellent, and thus we conclude that the generalized scaling law (7) is supported by experimental and theoretical data.

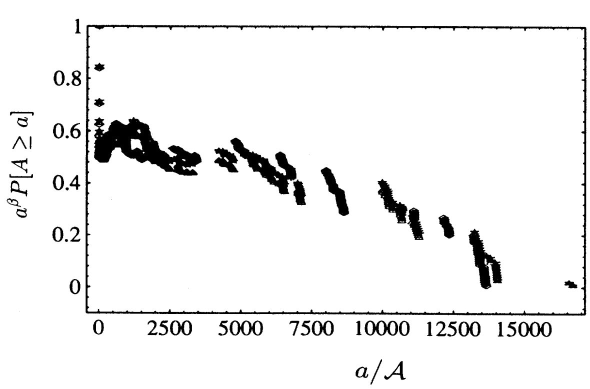

FIG. 3. Collapse of the scaling curves $a^{\beta}P(a,\mathcal{A})$ vs $a/\mathcal{A}$ for the four sub-basins shown in Fig. 1. The collapse is correctly obtained for $\beta = 0.45$.

III. SCALING OF LENGTHS

The stream length at any point is defined as the *main* distance measured through the network from the point to the boundary of the basin. Technically one defines the mainstream pattern upstream of any junction following the site having maximum area (in case of equal contributions one chooses at random) until a source is reached.

Let us now define the probability distribution $\pi(l,a)$ of the lengths defined in this way *for points with a given area a*. The latter constraint plays an important role in the definition of Hack's law (see below). From the convolution property, the length distribution is given by

$$\hat{\pi}(l,L) = \int_1^{+\infty} da\ \pi(l,a)\ p(a,L) \ . \tag{25}$$

We shall further denote with $\Pi(l,a)$ and $\hat{\Pi}(l,L)$ the corresponding probabilities of exceedence, e.g., $\Pi(l,a) = \int_l^{\infty} \pi(x,a)\ dx$. In general nothing can be said about the distribution $\hat{\pi}(l,L)$or $\hat{\Pi}(l,L)$ unless the distribution $\pi(l,a)$ is known. However, it is generally accepted that the function $\pi(l,a)$ is a sharply peaked function of one variable with respect to the other, thus leading to an effective constraint between areas and lengths. Thus one may assume that

$$\pi(l,a) = \delta(l - a^h) \ , \tag{26}$$

which is a mathematical statement of Hack's law when this law is assumed to hold without dispersion (see be-

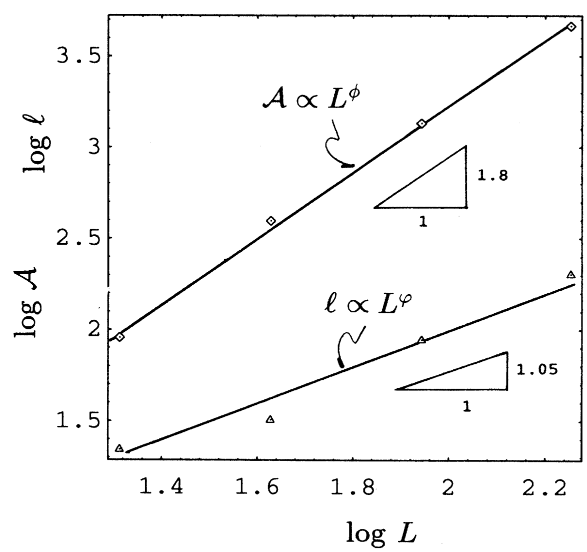

FIG. 4. Experimental scaling relationships for the river Fella basin: an independent verification of the relationship $\mathcal{A} \propto L^{\phi}$ and $\ell \propto L^{\varphi}$ with $\phi = 1.8 \pm 0.01$ and $\varphi = 1.05 \pm 0.02$. Here we employ pixel units, i.e., $\mathcal{A}$ is obtained from the second moment of total contributing area whose units are discussed in Fig. 2. Lengths L, ℓ, are multiples of the unit ruler identified by the linear size of the pixel. Logarithms are in base 10.

TABLE I. Summary of scaling coefficients defined in Eqs. (3), (9), (14), (17–20), (29), (32). Notice that only the results of *direct* measurements are shown. As an example, the value of $\phi = 1+H$ is measured through the scaling relationship of $\mathcal{A}$ vs L [Eq. (23)], while H has been measured [33] from the self-affine properties of the basin boundaries. The matching of the corresponding (and independent) values of ϕ and H can be appreciated, as well as the other related quantities.

Exponent	Scheidegger [21]	OCN [18,19,26,33]	Real basins [4,3,29,32]
$\tau = 1+\beta$	4/3	1.43 ± 0.02	1.43 ± 0.02
$\phi = 1+H$	3/2	$1.8 - 1.9$	1.8 ± 0.1
H	0.5	-	$0.75 - 0.80$
d_f	1	1.1	$1.05 - 1.12$
$h = d_l/\phi$	2/3	$0.57 - 0.58$	$0.57 - 0.60$
d_l	1	1.1	1.1 ± 0.01
φ	1	1.05	1.05 ± 0.01
$\gamma = 1+\beta/h$	1.5	1.8 ± 0.05	$1.8 - 1.9$

low). In this case it is easily derived that in the $L \to \infty$ limit

$$\hat{\Pi}(l, L \to \infty) \propto l^{-\beta/h} \tag{27}$$

and $\hat{\pi}(l,L) \propto l^{-\gamma}$, with $\gamma = 1 + \beta/h$.

The same result (27) can be derived if we assume for $\pi(l,a)$ a scaling form such as

$$\pi(l,a) = \frac{1}{l} g\left(\frac{l}{a^h}\right) \ , \tag{28}$$

which is a generalization of Eq. (26). We notice that Eq. (28) does not strictly presume Hack's law, since the original assumptions do not consider statistical fluctuations about the mean value. Our interpretation suggests that fluctuations are indeed scaling, and Hack's law holds for the mean; see below. The same data [25] used to produce Figs. 3 and 4 support our interpretation from direct measurements [23].

A summary of scaling exponents in real basins [3,4,8,29,32], optimal channels networks [18,19,26,33], and Scheidegger's trees [21] is shown in Table I. Notice that in the table we make the explicit use of a divider fractal dimension d_f [32] for basin boundaries, which cannot be directly related to the value of H for self-affine boundaries [23].

We will now show how the above scaling properties relate to Hack's law. It is important to stress the fact that *a priori* the areas a and the lengths l are *dependent* random variables. As we said above this means that the lengths of the streams must be measured with respect to a given value of the area a. Let us define

$$\overline{l_a} \equiv \langle l \rangle_a = \int_1^{+\infty} dl \ \pi(l,a) \ l \ , \tag{29}$$

where the distribution $\pi(l,a)$ has been assumed to be normalized to unity and is given by either Eqs. (26) or (28).

We shall identify the length of the maximum stream with the *typical* value of the distribution of the length that becomes equal to the *average* value if the distribution is *self-averaging.* Using Eqs. (1,4) and identifying $\overline{l_a}$ with the length l and $a_c(L)$ with the area a one indeed gets Hack's law (3), with

$$h = \frac{d_l}{1+H} \ . \tag{30}$$

Notice that it would be inconsistent to infer $d_l \neq 1$ without assuming $\varphi \neq 1$ in the determination of the proper value of H [Eqs. (18,21)]. In the important case where one assumes $\varphi \sim d_l$, we obtain the result

$$\beta = 1 - h \ , \tag{31}$$

which is an intriguing result. Notice that one cannot justify theoretically the equality of φ and d_l because the former is related to the scaling of the mean distance to the outlet and the latter to the scaling exponent of the mainstream. The observational coincidence $h \sim 0.57$ and $\beta \sim 0.43$ is noteworthy.

Finally, using the inequality $1 \leq d_l \leq 1 + H$ it immediately shows that the elongation exponent in Eq. (5), $1 - H$, is

$$1 - H = 2 - \frac{d_l}{h} > 0 \ , \tag{32}$$

which implies again that the basins elongate with the size. This result is also consistent with observational evidence [8,9,11] and recent numerical results [13–19,28,33].

[1] B.B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).

[2] I. Rodriguez-Iturbe and A. Rinaldo, *Fractal River Basins: Chance and Self-Organization* (Cambridge University Press, Cambridge, 1996).

[3] D.G. Tarboton, R.L. Bras, and I. Rodriguez-Iturbe, Water Resour. Res. **24**, 1317 (1988); **25**, 2037 (1989); **26**, 2243 (1990); P. La Barbera and R. Rosso, *ibid.* **25**, 735 (1989).

[4] I. Rodriguez-Iturbe, R.L. Bras, E. Ijjasz-Vasquez, and D.G. Tarboton, Water Resour. Res. **28**, 988 (1992).

[5] R.R. Montgomery and W.E. Dietrich, Nature **336**, 232

(1988); Science **255**, 826 (1992).

[6] M. Marani, A. Rinaldo, R. Rigon, and I. Rodriguez-Iturbe, Geophys. Res. Lett. **21**, 1213 (1994).

[7] J. Feder, *Fractals* (Plenum, New York, 1988); P. Meakin, Rev. Geophys. **29**, 335 (1991).

[8] J.T. Hack, U.S. Geol. Surv. Prof. Paper **294-B**, 1 (1957).

[9] D.M. Gray, J. Geophys. Res. **66**, 1215 (1961).

[10] W.B. Langbein, Geol. Surv. Prof. Paper (U.S.) **968-C**, 1 (1947).

[11] J.E. Muller, Geol. Soc. A Bull. **84**, 3127 (1973).

[12] R.E. Horton, Bull. Geol. Soc. Am. **56**, 275 (1939); R.L. Shreve, J. Geol. **74**, 17 (1966); **77**, 397 (1967).

[13] A. Rinaldo, I. Rodriguez-Iturbe, R. Rigon, E. Ijjasz-Vasquez, and R.L. Bras, Phys. Rev. Lett. **70**, 822 (1993); R. Rigon, Rodriguez-Iturbe, and A. Rinaldo, J. Geophys. Res. **99**, 11971 (1994).

[14] A. Rinaldo, W.E. Dietrich, R. Rigon, G.K. Vogel, and I. Rodriguez-Iturbe, Nature **374**, 632 (1995).

[15] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); P. Bak, K. Chen, and M. Creutz, Nature **342**, 780 (1989); P. Bak, K. Chen, and C. Tang, Phys. Lett. A **147**, 297 (1990); P. Bak and K. Chen, Sci. Am. **264** (1), 46 (1991); P. Bak and M. Paczuski, Phys. World **6** (12), 39 (1993).

[16] Models of river network evolution explicitly accounting for the interplay of fluvial and nonfluvial landforming processes originate from G.R. Willgoose, R.L. Bras, and I. Rodriguez-Iturbe, Water Resour. Res. **27**, 1671 (1991); **27**, 1685 (1991); Geomorphology **5**, 21 (1992); examples of lattice models of river basin development are in S. Kramer and M. Marder, Phys. Rev. Lett. **68**, 205 (1992); R.L. Leheny and S. R. Nagel, *ibid.* **71**, 1470 (1993).

[17] Detailed modeling of the erosional processes is, e.g., in A.D. Howard, Water Resour. Res. **30**, 2261 (1994); A.D. Howard, W.E. Dietrich, and M. Seidl, J. Geophys. Res. **99**, B7, 971 (1994).

[18] Optimal channel networks (OCN's) were proposed in I. Rodriguez-Iturbe, A. Rinaldo, R. Rigon, R.L. Bras, E. Ijjasz-Vasquez, and A. Marani, Water Resour. Res. **28**, 1095 (1992); Geophys. Res. Lett. **19**, 889 (1992).

[19] The implications of minimum energy dissipation on fractal growth were first explored in A. Rinaldo, I. Rodriguez-Iturbe, R. Rigon, R.L. Bras, E. Ijjasz-Vasquez, and A. Marani, Water Resour. Res. **28**, 2183 (1992); R. Rigon, A. Rinaldo, I. Rodriguez-Iturbe, E. Ijjasz-Vasquez, and R.L. Bras, *ibid.* **29**, 1635 (1993).

[20] M. E. Fisher, in *Critical Phenomena*, Proceedings of the International School of Physics Enrico Fermi, Course 51, Varenna, Italy, 1970, edited by M.S. Green (Academic, New York, 1971).

[21] H. Takayasu, M. Takayasu, A. Provata, and G. Huber, J. Stat. Phys. **65**, 725 (1991); A.E. Scheidegger, Bull. Assoc. Sci. Hydrol. **12**, 15 (1967).

[22] C. Tricot, *Curves and Fractal Dimensions* (Springer-Verlag, New York, 1995).

[23] R. Rigon, I. Rodriguez-Iturbe, A. Rinaldo, A. Maritan, and A. Giacometti (unpublished).

[24] J.S. Smart, Adv. Hydrosci. **8**, 305 (1972).

[25] I. Rodriguez-Iturbe, M. Marani, R. Rigon, and A. Rinaldo, Water Resour. Res. **30**, 3531 (1994).

[26] T. Sun, P. Meakin, and T. Jøssang, Phys. Rev. E **49**, 4865 (1994); P. Meakin, J. Feder, and T. Jøssang, Physica A **176**, 409 (1991). An equation similar to (7) has also been suggested by T. Nagatani, J. Phys. A **26**, L489 (1993). However, the implications of Eq. (7) were not fully exploited.

[27] Equation (16) can be easily obtained assuming that the spanning tree defining the river network is strictly direct, like, e.g., in the Scheidegger model [21]. This means that along the path from any site to the outlet the tangent always has a positive projection along a fixed direction. If the river network is not direct, then Eqs. (19) and (20) hold with $\varphi \geq 1$. Notice that in Ref. [26] it was implicitly assumed a direct character for river networks.

[28] A. Giacometti, A. Maritan, and J.R. Banavar, Phys. Rev. Lett. **75**, 577 (1995).

[29] E. Ijjasz-Vasquez, R.L. Bras, and I. Rodriguez-Iturbe, Physica A **209**, 288 (1994).

[30] In fact, if we characterize the sum of the total lengths to the outlet $\sum_i x_i$ (where i spans all sites from 1 to N, where N is the total number of sites and x_i is the length to the outlet from site i measured along the network), we have $\sum_i x_i = \sum_i \sum_{j\in\gamma(i)} \Delta x_{ij} = \sum_i a_i$, where j indexes the intermediate steps in the path $\gamma(i)$ from i to the outlet, Δx_{ij} is the lattice size, i.e., it is the unit in isotropic lattices. Thus M is the maximum distance from source to outlet, and reversing the order of summation we can write in general for loopless structures $\langle a\rangle = \sum_{x=1}^{M} xW(x) \propto \ell$, where $W(x)$ counts the relative proportion of sites at distance x from the outlet (i.e., the so-called geomorphological width function, e.g., [19]) and ℓ is the mean distance to the outlet of the network. The length ℓ is characteristic of the entire aggregation pattern of the area distribution. The relationship with Euclidean length L is readily established for self-organized critical networks as $\ell \propto L^{\varphi}$ with $\varphi = 1.05$ [23]. See also Fig. 4 for an experimental evaluation of φ. Notice that φ does not coincide, in principle, with d_l [see Eq. (4)] because ℓ is the mean distance to the outlet computed along the network of *all* sites [i.e., the centroid of the width function $W(x)$], whereas l is just the *mainstream* length.

[31] H. Takayasu and H. Inaoka, Phys. Rev. Lett. **68**, 966 (1992).

[32] S.P. Breyer and R.S. Snow, Geomorphology **5**, 143 (1992).

[33] E. Ijjasz-Vasquez, R.L. Bras, and I. Rodriguez-Iturbe, Geophys. Res. Lett. **20**, 1583 (1993).