Statistical fractals with cutoffs, Shlesinger-Hughes renormalization, and the onset of an epidemic

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A method for constructing cutoff values of the probability densities attached to statistical fractals is introduced for which only the beginning of the tail of a probability density is slowly decaying and given by an inverse power law; the end of the tail, however, is short, decaying exponentially or faster. This method is illustrated by an example from the theory of epidemics. The probability density $\tilde{\psi}(t)dt$ of the time interval t within which an infected individual is able to spread an epidemic is evaluated based on the following assumptions. (1) The spreading of the epidemic depends on the size of the germ load carried by an infected individual; the total germ load is measured in large units containing at least 10^5–10^6 germs. It is assumed that for each unit there is a constant probability p of infecting other individuals. (2) The total number of germ units carried by an individual is a random variable; the corresponding distribution is evaluated by assuming that there is a constant probability α that an infected individual carries a unit load of germs. (3) The encounters of an infected individual with healthy individuals susceptible to receiving the illness is a random event occurring with an average time-independent contact frequency ν. The probability density $\tilde{\psi}(t)dt$ determined by these three assumptions has a long tail characteristic of an ideal statistical fractal behavior: $\tilde{\psi}(t)dt \sim (\nu t)^{-(H+1)}\Xi(\ln(\nu t))d(\nu t)$ as $t \gg 1/\nu$, where $H = \ln(\alpha)/\ln(1-p)$ is a positive fractal exponent and $\Xi(\ln(\nu t))$ is a periodic function of $\ln(\nu t)$ with a period $-\ln(1-p)$. In this case all positive moments $\langle t^q \rangle$ of the infection time with $q \geq H$ are infinite. A statistical fractal with a cutoff emerges if a fourth hypothesis is added: (4) Due to the healing process the bacterial load of an individual decreases exponentially in time with a rate coefficient b. If the healing process is slower than the encounter process of healthy individuals, $\nu > b$, then only the beginning of the tail of $\tilde{\psi}(t)dt$ obeys an inverse power law scaling $\tilde{\psi}(t)dt \sim (\nu t)^{-(H+1)}\Xi(\ln(\nu t))d(\nu t)$ for $1/b > t > 1/\nu$; the end of the tail, however, is exponential and determined by the healing process $\tilde{\psi}(t) \sim \text{const} \times \exp(-bt)$ as $t \gg 1/b$. Due to the cutoff the moments $\langle t^q \rangle$ of the infection time, although finite, have an intermittent behavior characterized by the scaling law $\langle t^q \rangle \sim \text{const} \times b^{-(q-H)}$ as $b \to 0$. A generalization of the epidemic model is given by assuming that the size of the germ load carried by an infected individual is an arbitrary random function of time with known stochastic properties and that the encounters with healthy individuals can be described by a correlated random point process. An analytical expression is derived for the probability density $\tilde{\psi}(t)dt$ of the infection time in terms of the characteristic functionals of the germ load and of the encounter process. A comparison is performed between the onset of an epidemic and the passage over a fluctuating energy barrier with dynamical disorder. Some implications of the cutoff of a statistical fractal for the physics of fractal time are also analyzed.

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I. INTRODUCTION

Both geometrical [1] and statistical [2] fractals are commonly used for describing a large class of natural phenomena from physics, chemistry, and biology [1–4]. A geometrical fractal is characterized by the deterministic or random self-similarity of the shapes of structures characteristics of different length scales [1]. For a statistical fractal the self-similar behavior is not related to geometry but to statistics: the tails of the probability densities of the random variables that describe the system are self-similar and obey scaling laws of the inverse power law type. Although for an ideal geometrical or statistical fractal the self-similarity acts up to infinity, the fractals found in nature are always imperfect; for them the self-similarity is always limited by an upper and a lower cutoff value of the characteristic variable. For a geometrical fractal the introduction of cutoff lengths does not generate special problems; the geometrical fractal description is simply limited to a finite length range. For a statistical fractal, however, a cutoff value of the random variable cannot be introduced in the probabilistic descrip-

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tion of the system because it would lead to a violation of the normalization conditions of the probability densities. This is probably the reason why the imperfect statistical fractals have not yet been analyzed in the literature.

The purpose of this paper is to investigate the way in which a cutoff of a statistical fractal may be generated. Although the problem may be investigated in an abstract way, we prefer to investigate a concrete biological problem, the initial stage of the onset of an epidemic generated by the encounters of an ill individual with healthy individuals that are susceptible to being infected. Although apparently there is barely a connection between this problem and the physical or chemical problems described in terms of fractal time statistics, the structure of the basic evolution equations is similar, though not identical. Anyway the analysis of the similarity in the mathematical structure between a problem in biological sciences and other problems in the physical sciences would be beneficial. A similar comparison has been recently presented in the physical literature [5]. In our case the corresponding physical problem would be the passage over a fluctuating energy barrier with dynamical disorder [6,7].

The mathematical study of epidemics started at the beginning of the century with the seminal papers by Ross and Hudson [8] and by Kermack and McKendrick [9]. Now it is an independent branch of mathematical biology with its own monographs [10]. Most papers dealing with the study of epidemics aim at giving an overall description of the process in terms of deterministic or stochastic models for the whole population within which an epidemic process takes place. The main concern is the study of the time behavior of the numbers of the different types of individuals (susceptible, infectious, immune, recovered, etc.) that make up an epidemic system. A straightforward generalization is the study of the spatial distribution of an epidemic described in terms of partial differential equations [11], stochastic field equations [12], or cellular automata [13]. In this paper our purpose is different: instead of studying the whole population, we are interested in the random behavior of an infected individual spreading an illness in a population of healthy individuals, more precisely, in the stochastic properties of the time interval within which the individual is able to spread the epidemic. Our approach stems from the method suggested by one of the present authors [14,15] for the study of the spatial distribution of an epidemic in terms of cellular automata. The analysis of this problem is of interest from the points of view of both the biological and physical communities. From the biological point of view, it paves the way for the development of a microscopic theory of epidemics by starting out from a model for the random behavior of an individual. Also, it proves the efficiency, in the study of biological problems, of some mathematical techniques commonly used by physicists (the renormalization group, the stochastic Liouville equations, and the theory of random point processes) but almost unknown to biologists. From the physical point of view, it is of interest because it shows how a cutoff of a statistical fractal may emerge in a complex system, leading to the possibility of identifying and studying other nonideal statistical fractals.

The structure of the paper is as follows. In Sec. II we suggest an ideal statistical fractal description of an epidemic by using the Shlesinger-Hughes stochastic renormalization technique [16]. Sections III–V deal with an imperfect statistical fractal description of an epidemic by taking into account the healing process of an infected individual. In Sec. VI a comparison is made between the onset of an epidemic and the passage over a fluctuating energy barrier with dynamical disorder. Finally some general implications of the imperfect fractal statistics for the physics of fractal time are investigated.

II. EPIDEMICS AND IDEAL STATISTICAL FRACTALS

We consider an infected individual who carries a random number of germs that are the cause of the illness. We make three different assumptions.

(i) The spreading of the epidemic depends on the number of germs carried by an individual. We measure the total germ load of an individual by large units made up of at least 10^5–10^6 germs. We consider that for each unit there is a constant probability p for the occurrence of an infection event. During an encounter between an infected individual carrying m germ units with a healthy individual susceptible to infection, the probability β of infection depends on the germ load m, $\beta=\beta(m)$. The probability of infection $\beta(m)$ can be evaluated as follows. The probability that none of the m germ units leads to an infection is $(1-p)^m$. An infection event occurs if at least one of the m germ units leads to infection, and thus

$$\beta(m)=1-(1-p)^m=1-\exp(-km)\ , \tag{1}$$

where

$$k=-\ln(1-p)\ . \tag{2}$$

Equation (1) is the main relation of our approach. A similar equation has been used in Refs. [14,15] for describing the spread of an epidemic by means of cellular automata; there, however, the parameter m has a different physical significance.

(ii) The total number m of germ units carried by an individual is a random variable selected from a probability law. We denote by $\chi(m)$ the probability that the number of germ units carried by an individual is m. We have

$$\sum_{m=0}^{\infty}\chi(m)=1\ . \tag{3}$$

We evaluate the probability $\chi(m)$ by assuming that for each germ unit there is a constant probability α of occurrence; it follows that $\chi(m)$ is given by a Pascal distribution,

$$\chi(m)=(1-\alpha)\alpha^m\ . \tag{4}$$

(iii) The encounters of an infected individual with healthy individuals are independent random events obeying Poissonian statistics. The time between two successive encounters is an exponentially distributed random variable with a probability density

$$\nu \exp(-\nu t)dt \ , \tag{5}$$

characterized by constant average frequency of encounters ν.

Our aim is to evaluate the probability density $\tilde{\psi}(t)dt$ with

$$\int_0^\infty \tilde{\psi}(t)dt = 1 \tag{6}$$

of the time interval t within which an infected individual is able to spread the infection. $\tilde{\psi}(t)dt$ can be expressed as

$$\tilde{\psi}(t)dt = \sum_{m=0}^{\infty} \chi(m)\psi(t|m)dt \ , \tag{7}$$

where $\psi(t|m)dt$ with

$$\int_0^\infty \psi(t|m)dt = 1 \tag{8}$$

being the corresponding conditional probability density of the infection time corresponding to an individual with a germ load m. $\psi(t|m)$ is given by

$$\psi(t|m) = \sum_{N=0}^{\infty} P(N|m)[\nu \exp(-\nu t) \otimes]^{(N+1)} \ , \tag{9}$$

where $P(N|m)$ with

$$\sum_{N=0}^{\infty} P(N|m) = 1 \tag{10}$$

is the probability that an individual with a germ load m spreads the infection during N encounters, $\otimes$ denotes the temporal convolution product, and

$$[\nu \exp(-\nu t) \otimes]^{(N+1)} = \nu(\nu t)^N (N!)^{-1} \exp(-\nu t) \tag{11}$$

is the $(N+1)$-fold convolution product of the probability density $\nu \exp(-\nu t)$ of the time between two successive encounters; that is, Eq. (11) gives the probability density of the time necessary for the occurrence of a succession of $N+1$ encounters.

The conditional probability $P(N|m)$ of the number of infections generated by an individual with a germ load m can be expressed by a Pascal law similar to Eq. (4),

$$P(N|m) = [1-\beta(m)][\beta(m)]^N = (1-p)^m[1-(1-p)^m]^N \ , \tag{12}$$

which can be derived by considering a succession of $N+1$ encounters of which the first N lead to N infection events but the $(N+1)$th does not lead to an infection. By combining Eqs. (9), (11), and (12) we get the following expression for the conditional probability density $\psi(t|m)dt$ of the active infectious period of an individual carrying m germ units:

$$\psi(t|m)dt = \nu(1-p)^m \exp[-\nu t(1-p)^m]dt \ . \tag{13}$$

By combining Eqs. (4), (7), and (13) we get a similar expression for the corresponding unconditional probability density $\tilde{\psi}(t)dt$:

$$\tilde{\psi}(t)dt = \nu(1-\alpha) \sum_{m=0}^{\infty} [\alpha(1-p)]^m \exp[-\nu t(1-p)^m]dt \ . \tag{14}$$

We start out by investigating the stochastic properties of the duration of the active infectious period of an individual with a germ load m. The moments $\langle t^q(m) \rangle$ and the cumulants $\langle\langle t^q(m) \rangle\rangle$ of the probability density $\psi(t|m)dt$ are given by

$$\langle t^q(m) \rangle = q!\nu^{-q}(1-p)^{-mq} = q!\nu^{-q}\exp(kqm), \quad q = 1,2,\ldots, \tag{15}$$

$$\langle\langle t^q(m) \rangle\rangle = (q-1)!\nu^{-q}(1-p)^{-mq} = (q-1)!\nu^{-q}\exp(kqm), \quad q = 1,2,\ldots . \tag{16}$$

We note that both the central moments and the cumulants increase exponentially with the increase of the germ load; in particular, for $m \to \infty$ the infection process carries on up to infinity. Note that the relative fluctuations of different orders $q = 2,3,\ldots,$

$$\rho_q^{(m)} = \langle\langle t^q(m) \rangle\rangle^{1/q} / \langle t(m) \rangle, \quad q = 2,3,\ldots \tag{17}$$

do not decrease with the increase of the germ load m but are constant:

$$\rho_q^{(m)} = [(q-1)!]^{1/q} = \text{independent of } m \text{ and } p \ . \tag{18}$$

This situation corresponds to an intermittent behavior of the fluctuations of the duration of the active infectious period for large values of the germ load m.

The moments of the duration of the unconditional value of the active infectious period are given by

$$\langle t^q \rangle = \begin{cases} (1-\alpha)q!\nu^{-q}\dfrac{(1-p)^q}{(1-p)^q-\alpha} & \text{for } q < H, \\ \infty & \text{for } q \geq H, \end{cases} \tag{19}$$

where

$$H = \ln(\alpha)/\ln(1-p) \tag{20}$$

is a positive fractal exponent. In particular, for $1 > H > 0$ all positive moments are infinite, a situation that suggests the possible existence of an ideal statistical fractal behavior for the probability density $\tilde{\psi}(t)dt$.

For investigating the possible self-similar behavior of $\tilde{\psi}(t)$ we use the stochastic renormalization technique of Shlesinger and Hughes [16]. We rewrite Eq. (14) in the form

$$\tilde{\psi}(t) = (1-\alpha)\nu \exp(-\nu t) + \alpha(1-p)\nu(1-\alpha) \times \sum_{m=0}^{\infty} [\alpha(1-p)]^m \exp[-\nu t^*(1-p)^m] \ , \tag{21}$$

where

$$t^* = (1-p)t \ . \tag{22}$$

By eliminating the series from Eqs. (14) and (22) we get a functional equation for $\tilde{\psi}(t)$ of the renormalization group type (RG):

$$\tilde{\psi}(t) = (1-\alpha)\nu \exp(-\nu t) + \alpha(1-p)\tilde{\psi}[(1-p)t] \ . \tag{23}$$

The solution of the RG equation (23) is made up of two different additive contributions: a slowly decaying nonanalytic term of the power law type and an exponentially decaying analytic term. Searching for nonanalytic contributions of the type

$$\tilde{\psi}(t)\sim\nu(\nu t)^{-\epsilon}\Xi(t) \quad \text{for } t\gg 1/\nu \tag{24}$$

and inserting Eq. (24) into Eq. (23) in the limit $t\gg 1/\nu$, we come to

$$\epsilon=1+H\ , \tag{25}$$

where H is the fractal exponent given by Eq. (20) and Ξ is a periodic function of $\ln(\nu t)$ with a period $k=-\ln(1-p)$:

$$\Xi(\ln(\nu t)+k)=\Xi(\ln(\nu t))\ . \tag{26}$$

The explicit form of the periodic function $\Xi(\ln(\nu t))$ may be evaluated by using the Poisson summation technique [17]. The infinite series (14) can be expressed in the equivalent form

$$\tilde{\psi}(t)=\tfrac{1}{2}(1-\alpha)\nu\exp(-\nu t)+\nu(\nu t)^{-(H+1)} \times\frac{1-\alpha}{k}\left[\gamma(H+1,\nu t)+2\sum_{l=1}^{\infty}\left(\mathcal{F}^{+}(1+H,2\pi l/k,\nu t)\cos\{2\pi l[\ln(\nu t)]/k\}\right.\right. \left.\left.+\mathcal{F}^{-}(1+H,2\pi l/k,\nu t)\sin\{2\pi l[\ln(\nu t)]/k\}\right)\right]\ , \tag{27}$$

where

$$\gamma(c,x)=\int_0^x t^{c-1}\exp(-t)dt,\quad c>0,\quad x\geq 0 \tag{28}$$

is the incomplete γ function and

$$\mathcal{F}^{\pm}(a,b,x)=\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\gamma(c=a+ib,x)=\int_0^x t^{a-1}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(b\ln t)\exp(-t)dt \tag{29}$$

are the real and imaginary parts of the incomplete γ function of complex argument, respectively; the parameters k and H are given by Eqs. (2) and (20), respectively. Equation (27) is exact; it includes both the analytic and the nonanalytic contributions to $\tilde{\psi}(t)$. By applying Eq. (27) in the limit $t\gg 1/\nu$ and by keeping only the dominant terms in t, we recover the nonanalytic scaling law (24) with $\epsilon=1+H$ and where $\Xi(\ln(\nu t))$ is given by

$$\Xi(\ln(\nu t))=\frac{1-\alpha}{k}\left[\Gamma(1+H)+2\sum_{l=1}^{\infty}\left(F^{+}(1+H,2\pi l/k)\cos\{2\pi l[\ln(\nu t)]/k\}+F^{-}(1+H,2\pi l/k)\sin\{2\pi l[\ln(\nu t)]/k\}\right)\right]\ , \tag{30}$$

where $\Gamma(1+H)=\gamma(1+H,\infty)$ is the complete gamma function and $F^{\pm}(a,b)=\mathcal{F}^{\pm}(a,b,\infty)$ are the real and imaginary parts of the complete gamma function of complex argument, respectively.

The logarithmic oscillations are typical for a renormalization group approach. Although in other physical [18] and biological [19] contexts they actually exist, in the case of the present epidemic model they may be spurious. Indeed, it is easy to show that their period of oscillation depends on the size of a unit of germs that is arbitrary. Considering two different values, C and C' of the germ unit and denoting by p,m and p',m' the corresponding infection probabilities and the numbers of germ units, respectively, we get

$$Cm=C'm'=M\ , \tag{31}$$

where M is the total number of germs. Since the dynamics of the process should be independent of the units used we have

$$\beta=1-(1-p)^m=1-(1-p')^{m'}\ . \tag{32}$$

From Eqs. (31) and (32) we get

$$(1-p')=(1-p)^{C'/C}\ , \tag{33}$$

and thus the corresponding periods of oscillation $k=-\ln(1-p)$, $k'=-\ln(1-p')$ are related to each other through the relationship

$$k'=kC'/C\ . \tag{34}$$

It follows that the period of oscillation is not constant; by changing the value of a unit germ load it changes according to Eq. (34). This result leads to a contradiction that can be solved only by assuming that the logarithmic oscillations do not actually exist but are spurious.

For circumventing the above mentioned contradiction we consider the limit

$$p\searrow 0 \quad\text{and}\quad \alpha\nearrow 1 \quad\text{with}\quad H=\ln(\alpha)/\ln(1-p)=\text{const}\ , \tag{35}$$

for which the logarithmic oscillations disappear even though the nonanalytic scaling law of the inverse power

law type is still present. In the limit (35) the RG equation (23) becomes a differential equation,

$$td\tilde{\psi}(t)/dt+(H+1)\tilde{\psi}(t)=H\nu\exp(-\nu t)\ . \tag{36}$$

The normalized solution of Eq. (36) is

$$\tilde{\psi}(t)=H\nu(\nu t)^{-(H+1)}\gamma(1+H,\nu t)\ . \tag{37}$$

As expected, in the limit $t\gg 1/\nu$ Eq. (37) leads to a long tail, but the logarithmic oscillations are missing,

$$\tilde{\psi}(t)\sim H\Gamma(1+H)\nu(\nu t)^{-(H+1)},\quad t\gg 1/\nu\ . \tag{38}$$

In the limit (35) both the period of oscillation $k=-\ln(1-p)$ and the parameter

$$\mu=\ln(1/\alpha)\ , \tag{39}$$

tend to zero,

$$k,\mu\to 0\ . \tag{40}$$

For small values of k and μ, close to but different from zero, the probability $\chi(m)$ of the germ load can be approximated by a probability density

$$\chi(m)dm\quad\text{with}\quad\int_0^\infty\chi(m)dm=1\ , \tag{41}$$

where

$$\chi(m)=\mu\exp(-\mu m)=(\overline{m}_0)^{-1}\exp(-m/\overline{m}_0)\ , \tag{42}$$

and

$$\overline{m}_0=1/\mu \tag{43}$$

is the average germ load size. By using this continuous approximation, the sum in Eq. (14) is replaced by an integral, resulting in

$$\tilde{\psi}(t)=\int_0^\infty\chi(m)dm\ \psi(t|m)=\int_0^\infty\mu\exp(-\mu m)\nu\exp(-km)\exp[-\nu t\exp(-km)]dm\ . \tag{44}$$

By taking into account that from Eqs. (2), (20), and (39) we have

$$H=\mu/k \tag{45}$$

and expressing the integral in Eq. (44) in terms of the incomplete γ function, we recover Eq. (37).

III. LIOUVILLE EQUATION APPROACH TO HEALING

The model presented in the preceding section is not very realistic from the biological point of view because it ignores the possible time variation of the germ load m. We have assumed that the germ load m is a randomly selected variable from a discrete probability $\chi(m)$ [Eq. (4)] or from a continuous probability density $\chi(m)dm$ [Eq. (42)]. Once chosen according to a given probability law, the germ load m has been assumed to be frozen: it remains constant forever. In the real world, however, the germ load of an infected individual varies in time: it either decreases because of the healing process or increases because of the progress of the illness, eventually leading to the death of the individual.

In this section we improve the continuous version of the model presented in Sec. II by assuming that the germ load m is initially selected from a known probability density $\chi(m_0)dm_0$ with $m(t=0)=m_0$, which does not necessarily have the exponential shape given by Eq. (42), and then it evolves according to a deterministic evolution equation

$$dm(t)/dt=F(m,t)\ , \tag{46}$$

with the initial condition

$$m(t=0)=m_0\ , \tag{47}$$

where $F(m,t)$ is the rate of variation of the germ load. In this section we do not consider any particular form for the function $F(m,t)$; we assume only that the solution of the differential equation (46) with the initial condition (47) is known,

$$m(t)=\varphi(t;m_0)\ , \tag{48}$$

and that the Jacobian derivative

$$\gamma(t;m_0)=\partial\varphi(t;m_0)/\partial m_0 \tag{49}$$

is finite and different from zero; thus according to the existence theorem of the implicit functions, we can express, at least in principle, the initial germ load m_0 in terms of the germ load m at time t,

$$m_0=\varphi^{(-1)}(t;m)\ , \tag{50}$$

where the inverse function $\varphi^{(-1)}(t;m)$ fulfills the conditions

$$\varphi(t;\varphi^{(-1)}(t;m))=m;\quad\varphi^{(-1)}(t;\varphi(t;m_0))=m_0\ . \tag{51}$$

Now we introduce the instantaneous value $l(t)$ of the survival function of the infection process at time t. $l(t)$ is the probability that the period of active spreading of infection by a given individual ends up at a time t' bigger than the current time t ($t'>t$). The probability that an infection event does not take place at time t can be expressed in terms of the bacterial load $m(t)$ as

$$(1-p)^{m(t)}=\exp[-km(t)]\ . \tag{52}$$

Since the average frequency of encounters is ν, the rate of loss of the survival function $l(t)$ is

$$\nu(1-p)^{m(t)}=\nu\exp[-km(t)]\ . \tag{53}$$

It follows that the instantaneous value $l(t)$ of the survival function obeys the evolution equation

$$dl(t)/dt=-l(t)\nu\exp[-km(t)]\ , \tag{54}$$

with the initial condition

$$l(t=0)=1\ . \tag{55}$$

We introduce the joint probability density

$B(l,m;t)dldm$ with $\int_0^1\int_0^\infty B(l,m;t)dldm=1$ (56)

of the survival function and of the germ load at time t. By taking into account the evolution equations (46) and (54) for $l(t)$ and $m(t)$ and considering a Eulerian description of a statistical ensemble of trajectories in the (l,m) space starting from $(l_0=1,m_0)$, where m_0 is randomly selected from the initial probability density $\chi(m_0)dm_0$, we can derive an evolution equation for the joint probability density $B(l,m;t)$ of the Liouville type,

$$\partial_t B(l,m;t)=\partial_l[l\nu\exp(-km)B(l,m;t)] -\partial_m[F(m,t)B(l,m;t)] , \quad (57)$$

with the initial condition

$$B(l,m;t=0)=\delta(l-1)\chi(m) . \quad (58)$$

In the model introduced in this section and described by the Liouville equation (57) with the initial condition (58), there are two sources of stochasticity: the first one is due to the Poissonian nature of the encounters and is implicitly taken into account in Eq. (54), and the second one is due to the initial random distribution of the germ load and is described by Eqs. (46) and (47), where m_0 is a random number selected from the probability density $\chi(m)dm$.

The Liouville equation (57) can be solved explicitly by means of a transformation of the random variable l followed by the application of the method of characteristics combined with the Laplace transform technique. The calculations are lengthy and tedious; the main steps are presented in Appendix A. Here we give only the resulting expression for the positive moments of the survival function $l(t)$:

$$\langle l^q(t)\rangle=\int_0^1\int_0^\infty l^q B(l,m;t)dl\,dm =\int_0^\infty dy\,\chi(y)\exp\left\{-q\nu\int_0^t\exp[-k\varphi(t';y)]dt'\right\} , \quad q>0 , \quad (59)$$

where q is a positive number, not necessarily an integer.

A connection between the average survival function $\langle l(t)\rangle$ and the probability density $\tilde{\psi}(t)dt$ of the active infection period can be made by computing the probability that the active infection period ends up at a time t' bigger than the current time t. We obtain

$$\langle l(t)\rangle=\int_t^\infty\tilde{\psi}(t')dt' . \quad (60)$$

By differentiating Eq. (60) with respect to the current time t and making use of Eq. (59) applied for $q=1$, we come to

$$\tilde{\psi}(t)=-d\langle l(t)\rangle/dt=\int_0^\infty dy\,\chi(y)\nu\exp[-k\varphi(t;y)]\exp\left\{-\nu\int_0^t\exp[-k\varphi(t';y)]dt'\right\} . \quad (61)$$

It is easy to check that the theory presented in this section is consistent with the ideal statistical fractal approach suggested in Sec. II. For a constant germ load we have

$$\varphi(t;y)=y \quad \text{independent of } t . \quad (62)$$

By inserting Eqs. (42) and (62) into Eq. (61) and evaluating the integral over y in terms of the incomplete γ function, we recover again Eq. (37).

The Liouville equation description of the healing process can be extended to the more general case when not only the initial value of the germ load is random but also its time evolution. We replace the deterministic equation (46) for the time evolution of the germ load by a stochastic Markovian equation for the probability density,

$$P(m;t)dm \quad \text{with} \int_0^\infty P(m;t)dm=1 , \quad (63)$$

of the germ load at time t. Denoting by

$$W(m'\to m;t)dt\,dm \quad (64)$$

the rate of transition from a germ load m' to a germ load between m and $m+dm$ in a time interval between t and $t+dt$, we see that the Markovian master evolution equation for $P(m;t)$ is

$$\partial_t P(m;t)=\int_0^\infty[W(m'\to m;t)P(m';t) -W(m\to m';t)P(m;t)]dm' , \quad (65)$$

with the initial condition

$$P(m;t=0)=\chi(m) . \quad (66)$$

If the time evolution of the germ load $m(t)$ is Markovian and described by the master equation (65), then the joint probability density $B(l,m;t)dl\,dm$ obeys a compound stochastic Liouville-master equation of the Van Kampen type [20]:

$$\partial_t B(l,m;t)=\partial_l[l\nu\exp(-km)B(l,m;t)] +\int_0^\infty[W(m'\to m;t)B(l,m';t) -W(m\to m';t)B(l,m;t)]dm' , \quad (67)$$

with the initial condition (58). A method for solving the evolution equation (67) is presented in Appendix B. The corresponding expressions for the moments $\langle l^q(t)\rangle$ and for the probability density $\tilde{\psi}(t)dt$ of the duration of the active infection period are

$$\langle l^q(t)\rangle=\int_0^\infty dm\int_0^\infty dy\,\chi(y)g_q^*(m;t|y;0) , \quad (68)$$

$$\tilde{\psi}(t)=\int_0^\infty dm\int_0^\infty dy\,\chi(y)\nu\exp(-km)g_1^*(m,t|y;0) , \quad (69)$$

where $g_q^*(m;t|y;0)$ is a generalized Green's function obeying a nonconservative master equation

$$\partial_t g_q^*(m;t|y;0)=\int_0^\infty [W(m'\to m;t)g_q^*(m';t|y;0)-W(m\to m';t)g_q^*(m;t|y;0)]dm'-q\nu\exp(-km)g_q^*(m;t|y;0)\ ,$$
$$\text{with } g_q^*(m,t=0|y;0)=\delta(m-y)\ . \quad (70)$$

In Appendix B we show that Eqs. (37) and (61) can be recovered as particular cases of Eq. (69).

IV. HEALING AND RANDOM POINT PROCESSES

Although mathematically consistent, the Liouville equation approach presented in Sec. III does not have a clear physical significance. In this section we present a physical approach to the onset of the epidemic for a time-dependent germ load based on the analysis of the mechanism of the process. This approach is a direct generalization of the initial treatment presented in Sec. II. This mechanistic approach has the advantage that it can be extended for an arbitrary stochastic behavior of the germ load and of the statistics of the encounters. In contrast, the stochastic Liouville equation method is limited to the case of Poissonian statistics of the encounters and to a Markovian behavior of the germ load.

We start out by considering that the germ load $m(t)$ is a deterministic function of time with a random initial condition $m(t=0)=m_0$,

$$m(t)=\varphi(t;m_0)\ . \quad (71)$$

Unlike the case of the Liouville equation approach for the present treatment, it is not necessary that the function $\varphi(t;m_0)$ fulfill the conditions (49)–(51). If the encounters occur at different times $t_1,t_2,\ldots$, the corresponding probabilities of infection can be computed from Eqs. (1) and (71),

$$\beta_l=\beta(t_l;m_0)=1-\exp[-km(t_l)]$$
$$=1-\exp[-k\varphi(t_l;m_0)]\ . \quad (72)$$

Because of the time dependence of the germ load, the probabilities of infection are different for different encounters. The conditional probability $P(N|m)$ of the number of infections generated by an individual with a germ load m should be replaced by the conditional probability

$$P(N|t_1,\ldots,t_{N+1};m_0)\ ,$$
$$\text{with } \sum_{N=0}^{\infty} P(N|t_1,\ldots,t_{N+1};m_0)=1 \quad (73)$$

that there are N infection events generated by an individual with an initial germ load m_0 during $N+1$ encounters occurring at different times $t_1,\ldots,t_{N+1}$. This probability is given by a relationship similar to Eq. (12):

$$P(N|t_1,\ldots,t_{N+1};m_0)=\beta_1\cdots\beta_N(1-\beta_{N+1})$$
$$=\prod_{l=1}^{N}\{1-\exp[-k\varphi(t_l;m_0)]\}$$
$$\times\exp[-k\varphi(t_{N+1};m_0)]\ . \quad (74)$$

We introduce the time intervals $\tau_1,\tau_2,\ldots,\tau_{N+1}$ between two successive encounters

$$\tau_1=t_1,\quad \tau_2=t_2-t_1,\ldots,\tau_{N+1}=t_{N+1}-t_N\ . \quad (75)$$

For a Poissonian statistics of encounters, $\tau_1,\ldots,\tau_{N+1}$ are independent random variables selected from an exponential probability density

$$f(\tau)d\tau=\nu\exp(-\nu\tau)d\tau\ . \quad (76)$$

The total duration of the infection period $t=t_{N+1}$ can be expressed as

$$t=\tau_1+\tau_2+\cdots+\tau_{N+1}\ , \quad (77)$$

and thus the probability density $\tilde{\psi}(t)dt$ is an average of the Dirac's δ function

$$\delta(t-\tau_1-\tau_2-\cdots-\tau_{N+1})\ , \quad (78)$$

over the time intervals $\tau_1,\ldots,\tau_{N+1}$, over the number N of infection events and over the initial germ load $m_0=y$:

$$\tilde{\psi}(t)=\int_0^\infty dy\,\chi(y)\sum_{N=0}^{\infty}\int_0^\infty\cdots\int_0^\infty\delta(t-\tau_1-\cdots-\tau_{N+1})\prod_{l=1}^{N+1}[f(\tau_l)d\tau_l]P(N|t_1,\ldots,t_{N+1};y)\ . \quad (79)$$

By getting rid of the integral over τ_{N+1} by using the filtration property of the δ function, using the expressions (74) and (76) for $P(N|t_1,\ldots,t_{N+1};m_0)$ and expressing the remaining integrals in terms of the absolute times $t_1,\ldots,t_N$ we obtain

$$\tilde{\psi}(t)=\int_0^\infty dy\,\chi(y)\exp[-k\varphi(t;y)]\sum_{N=0}^{\infty}\nu^{N+1}\exp(-\nu t)\int_{t\ge t_N'\ge\cdots\ge t_1'\ge 0}\cdots\int dt_1'\cdots dt_N'\prod_{l=1}^{N}\{1-\exp[-k\varphi(t_l';y)]\}\ . \quad (80)$$

For evaluating the integrals in Eq. (80) we use the identity

$$\int_{t\ge t_N'\ge\cdots\ge t_1'\ge 0}\cdots\int I(t_1',\ldots,t_N')dt_1'\cdots dt_N'=\frac{1}{N!}\int_0^t\cdots\int_0^t I(t_1',\ldots,t_N')dt_1'\cdots dt_N'\ , \quad (81)$$

where $I(t'_1,\ldots,t'_N)$ is an arbitrary symmetric function of $t'_1,\ldots,t'_N$. By using Eq. (81) we can express the sum over N in Eq. (80) as the expansion of an exponential. We have

$$\tilde{\psi}(t)=\int_0^\infty dy\,\chi(y)\exp[-k\varphi(t;y)]\nu\exp(-\nu t)\sum_{N=0}^{\infty}\frac{1}{N!}\left\{\nu\int_0^t\{1-\exp[-k\varphi(t';y)]\}dt'\right\}^N$$

$$=\int_0^\infty dy\,\chi(y)\nu\exp[-k\varphi(t;y)]\exp\left\{-\nu\int_0^t\exp[-k\varphi(t';y)]dt'\right\}. \tag{82}$$

Equation (82) is the same as Eq. (61), derived in Sec. III by applying the method of the Liouville equation.

Now we can proceed to generalize Eq. (81) to the case of an arbitrary random behavior for the statistics of the encounters and for the time variation of the germ load. We describe the statistics of the encounters in terms of a correlated random point process [21] characterized by a set of grand canonical Janossy probability densities:

$$Q_0,Q_n(t_1,\ldots,t_N)dt_1\cdots dt_N,\quad N=1,2,\ldots. \tag{83}$$

$Q_N(t_1,\ldots,t_N)dt_1\cdots dt_N$ is the probability that there are N encounters occurring at times between t_1 and $t_1+dt_1,\ldots,$ and t_N and t_N+dt_N. We follow the usual convention [21], according to which there are no restrictions concerning the values of $t_1,\ldots,t_N$, and thus the normalization condition for the Janossy densities should include an $1/N!$ Gibbs factor:

$$Q_0+\sum_{N=1}^{\infty}\frac{1}{N!}\int_0^t\cdots\int_0^t Q_N(t_1,\ldots,t_N)dt_1\cdots dt_N=1\ . \tag{84}$$

In terms of the Janossy probability densities $Q_n(t_1,\ldots,t_N)dt_1\cdots dt_N$ we introduce the product densities [21]

$$\eta_N(t_1,\ldots,t_N)=\sum\frac{1}{S!}\int_0^t\cdots\int_0^t Q_{N+S}(t_1,\ldots,t_N,t_{N+1},\ldots,t_{N+S})dt_{N+1}\cdots dt_{N+S} \tag{85}$$

and the corresponding generating functional

$$\Xi[W(t');t\geq t'\geq 0]=1+\sum_{N=1}^{\infty}\frac{1}{N!}\int_0^t\cdots\int_0^t\eta_N(t'_1,\ldots,t'_N)W(t'_1)\cdots W(t'_N)dt'_1\cdots dt'_N\ , \tag{86}$$

where $W(t')$ is a suitable test function of time.

On the other hand we assume that the germ load $m(t)=\varphi(t;m_0)$ corresponding to a given initial value $m(0)=m_0$ is an arbitrary random function with known stochastic properties. The random behavior of $\varphi(t;m_0)$ is described in terms of the characteristic functional

$$G[K(t'');t\geq t''\geq 0;m_0]=\left\langle\exp\left[-\int_0^t K(t'')\varphi(t'';m_0)dt''\right]\right\rangle, \tag{87}$$

where $K(t'')$ is another suitable test function.

By generalizing the derivation of Eq. (82) we can express the probability density $\tilde{\psi}(t)dt$ of the duration of the infection period in terms of the functionals Ξ and G. The main steps of the derivation are presented in Appendix C. The final result is

$$\tilde{\psi}(t)=-\partial_t\int_0^\infty dm_0\chi(m_0)\Xi[W(t')=-G[K(t'')=k\delta(t''-t');t\geq t''\geq 0;m_0];t\geq t'\geq 0]\ . \tag{88}$$

V. EPIDEMICS AND NONIDEAL STATISTICAL FRACTALS

In this section we apply the general approaches developed before to a particular case, which can be described in terms of a statistical fractal with a cutoff. We assume that an infected individual is healing with a rate proportional to the germ load

$$F(m,t)=-bm\ , \tag{89}$$

where $b>0$ is a positive rate coefficient; thus during the healing process the germ load $m(t)$ decreases exponentially in time

$$m(t)=\varphi(t;m_0)=m_0\exp(-bt)\ . \tag{90}$$

We also assume that the initial germ load m_0 is randomly selected from the exponential probability density (42). By inserting Eqs. (42) and (90) into Eqs. (59) and (61) we get the following expressions for the moments $\langle l^q(t)\rangle$ of the survival function and for the probability density $\tilde{\psi}(t)dt$ of the duration of the active infection period:

$$\langle l^q(t)\rangle = \int_0^1 Hx^{H-1}\exp\left\{-q\nu\int_0^t \exp[(\ln x)\exp(-bt')]dt'\right\}dx \ , \tag{91}$$

$$\tilde{\psi}(t) = \int_0^1 Hx^{H-1}\nu\exp[(\ln x)\exp(-bt)]\exp\left\{-\nu\int_0^t \exp[(\ln x)\exp(-bt')]dt'\right\}dx \ , \tag{92}$$

where the integration variable x is given by

$$x = \exp(-ky) \ . \tag{93}$$

Unfortunately in Eqs. (91)–(92) the integrals over x and t' cannot be evaluated in a closed form in the general case; however, the asymptotic behavior of Eqs. (91) and (92) can be studied analytically in certain particular cases.

First we note that for $b=0$ we obtain

$$\langle l^q(t)\rangle = H\gamma(H,\nu t)(\nu t)^{-H} \ , \tag{94}$$

and $\tilde{\psi}(t)$ is given by Eq. (37). In this case the initial fluctuations of the germ load are completely frozen and the approach reduces to the particular situation analyzed in Sec. II.

For investigating other particular cases we define two characteristics time scales. The first is the mean time interval between two successive encounters

$$\langle \tau\rangle_e = \int_0^\infty \tau\nu\exp(-\nu\tau)d\tau = 1/\nu \ , \tag{95}$$

and the second one is the mean time necessary for healing

$$\langle t\rangle_h = \int_0^\infty t|\{[dm(t)/dt]/m_0\}|dt = \int_0^\infty tb\exp(-bt)dt = 1/b \ . \tag{96}$$

If the mean healing time $\langle t\rangle_h$ has the same order of magnitude as the average time interval between two successive encounters or is even smaller, then there is an overlapping between the healing and the encounter processes and Eqs. (91) and (92) cannot be simplified. In the opposite situation, when the healing is slower than the encounter process

$$\langle t\rangle_h > \langle \tau\rangle_e, \quad \text{i.e.,} \quad b > \nu \ , \tag{97}$$

then the integrals in Eqs. (91) and (92) can be approximately evaluated, provided that the average healing time is large enough, i.e., if

$$\langle t\rangle_h \gg 0, \quad b \sim 0 \ . \tag{98}$$

In Appendix D we show that in the limit (98), Eqs. (91) and (92) can be approximated by

$$\langle l^q(t)\rangle \cong \left\{\frac{\nu Hq}{b}[\exp(bt/H)-1]\right\}^{-H} \times\gamma\left[H, \frac{\nu Hq}{b}[\exp(bt/H)-1]\right], \quad \text{as } b\to 0 \ , \tag{99}$$

$$\tilde{\psi}(t) \cong H\nu\exp(bt/H)\left\{\frac{H\nu}{b}[\exp(bt/H)-1]\right\}^{-(H+1)} \times\gamma\left[H+1, \frac{H\nu}{b}[\exp(bt/H)-1]\right], \quad \text{as } b\to 0 \ . \tag{100}$$

By using Eqs. (99) and (100) the asymptotic behavior for large time of $\langle l^q(t)\rangle$ and of $\tilde{\psi}(t)$ can be easily evaluated in the limit $b\to 0$. We obtain

$$\langle l^q(t)\rangle \sim \begin{cases} \Gamma(H+1)(\nu qt)^{-H}, & \langle t\rangle_h > t > \langle \tau\rangle_e, \quad t\gg 0, \quad b\to 0, & \text{(101a)} \\ \Gamma(H+1)[b/(\nu Hq)]^H\exp(-bt), & t > \langle t\rangle_h, \quad t\gg 0, \quad b\to 0, & \text{(101b)} \end{cases}$$

$$\psi(t) \sim \begin{cases} H\Gamma(H+1)\nu(\nu t)^{-(H+1)}, & \langle t\rangle_h > t > \langle \tau\rangle_e, \quad t\gg 0, \quad b\to 0, & \text{(102a)} \\ H\Gamma(H+1)\nu[b/(\nu H)]^{H+1}\exp(-bt), & t > \langle t\rangle_h, \quad t\gg 0, \quad b\to 0 \ . & \text{(102b)} \end{cases}$$

From Eqs. (101) and (102) we notice that only the beginning of the tails ($\langle t\rangle_h > t > \langle \tau\rangle_e$) of the functions $\langle l^q(t)\rangle$ and $\tilde{\psi}(t)$ have a self-similar behavior of the inverse power law type; for larger times ($t > \langle t\rangle_h$) the ends of the tails fall off exponentially. The biological interpretation of these results is straightforward. In the time window $\langle t\rangle_h > t > \langle \tau\rangle_e$ the germ load is practically constant, the initial fluctuations of the germ load are frozen, and the model is equivalent to the fractal description presented in Sec. II. In this case the inverse power law behavior is generated by the equilibration between the contribution of very large germ loads, which are exponentially rare [see Eq. (42)], and the characteristic time of loss of the survival function [see Eq. (53)]

$$t_{\text{loss}} = 1/[\nu(1-p)^m] = \nu^{-1}\exp(km) \ , \tag{103}$$

which increases exponentially with the increase of the germ load. For larger times $t > \langle t\rangle_h$ the germ load decreases exponentially to zero [see Eq. (90)] and the initial fluctuations of the germ load decrease as time increases, in this time scale the spread of infection among healthy individuals is dominated by the healing process of the in-

fected individual considered, resulting in the exponential decay laws (101b) and (102b).

For the fractal epidemic studied in Sec. II all positive moments $\langle t^q \rangle$ of the infection time with $q \gtrsim H$ are infinite. For a fractal epidemic with a cutoff, however, the moments of the infection time, although possibly large, are instead finite because of the exponential decay of the tail of the function $\tilde{\psi}(t)$. In Appendix E we show that in the limit $b \to 0$, the moments $\langle t^q \rangle$ and the cumulants $\langle\langle t^q \rangle\rangle$ of the duration of the active infection period of an individual are given by

$$\langle t^q \rangle, \langle\langle t^q \rangle\rangle \sim H\Gamma(1+H)\nu^{-H}(H/b)^{q-H}J_q(H) , \quad q > H, \quad b \to 0, \quad 1 > H > 0 , \tag{104}$$

where

$$J_q(H) = \int_0^\infty y^q \exp(-Hy)[1-\exp(-y)]^{-(H+1)}dy = \Gamma(q+1)\sum_{l=0}^{\infty} \frac{\Gamma(H+l+1)}{\Gamma(H+1)l!}(H+l)^{-(q+1)} . \tag{105}$$

It is easy to check that for $q > H$ both the integral and the series in Eq. (105) are convergent.

Equations (104) show that in the limit $b \to 0$ the cumulants of the infection time are approximately equal to the corresponding moments. Although surprising at first sight, this result has a simple explanation: the expressions (104) for the moments and cumulants give only the dominant contribution in b as $b \to 0$, which both for moments and cumulants scales as $b^{-(q-H)} \sim (\langle \tau \rangle_e)^{q-H}$. Although the moments and the cumulants are generally different from each other in their expressions, the other contributions depend on lower powers of $\langle \tau \rangle_e = 1/b$, which in the limit $b \to 0$ are negligible.

An important consequence of the asymptotic expressions (104) for the cumulants of the duration of the infection period is that for $b \to 0$ the fluctuations of the infection time are intermittent. For proving the existence of intermittency we compute the relative fluctuation of order q ($q \geq 2$):

$$\rho_q(b) = \frac{\langle\langle t^q \rangle\rangle^{1/q}}{\langle\langle t \rangle\rangle} \sim [H\Gamma(1+H)]^{-(1-1/q)} \frac{[J_q(H)]^{1/q}}{J_1(H)} \left[\frac{\nu H}{b}\right]^{H(1-1/q)} , \quad \langle \tau \rangle_e = 1/b \gg 0, \quad q \geq 2 . \tag{106}$$

The relative fluctuation of order q increases with the increase of the mean healing time $\langle \tau \rangle_e = 1/b$ as $(\langle \tau \rangle_e)^{H(1-1/q)}$. As expected in the ideal fractal limit $\langle \tau \rangle_e \to \infty$, the relative fluctuations $\rho_q(b)$, $b \geq 2$ diverge to infinity.

Our analysis shows that the healing process slows down the spread of the epidemic by the germ carrier among healthy individuals susceptible to infection. Because of healing for a fractal epidemic with a cutoff, the contribution of very large germ loads is smaller than in the ideal fractal case $\langle \tau \rangle_e \to \infty$ for which the germ load is constant. This decrease of efficiency can be analyzed by evaluating the large time behavior of the effective hazard rate:

$$W_{\text{eff}}(t) = \tilde{\psi}(t)/\langle l(t) \rangle . \tag{107}$$

By using Eq. (60) the definition (107) of the effective hazard rate can be rewritten in the form

$$W_{\text{eff}}(t) = -[d\langle l(t) \rangle/dt]/\langle l(t) \rangle . \tag{108}$$

Equation (108) shows that the effective hazard rate $W_{\text{eff}}(t)$ is a measure of the relative differential decrease of the average survival function $\langle l(t) \rangle$. For the case of the ideal fractal epidemic we have

$$W_{\text{eff}}(t) = Ht^{-1}\{1+[(\nu t)^H \exp(-\nu t)]/\gamma(H+1,\nu t)\}^{-1} , \tag{109}$$

whereas in the case of the imperfect fractal epidemic with an exponential decrease of the germ load with a small rate $b \to 0$ we obtain

$$W_{\text{eff}}(t) \cong \frac{b\exp(bt/H)}{\exp(bt/H)-1}\left\{1+\frac{\{[\exp(bt/H)-1]\nu H/b\}^H \exp\{-[\exp(bt/H)-1]\nu H/b\}}{\gamma(H+1,[\exp(bt/H)-1]\nu H/b)}\right\}^{-1} \quad \text{as } b \to 0 . \tag{110}$$

By investigating the large time behavior of Eqs. (109) and (110) we notice that for an ideal fractal epidemic the effective hazard rate decreases hyperbolically to zero,

$$W_{\text{eff}} \sim H/t, \quad t \gg 0, \quad \langle \tau \rangle_e \to \infty , \tag{111}$$

whereas in the imperfect fractal case W_{eff} tends toward the healing rate b:

$$W_{\text{eff}} \sim b, \quad t \gg \langle \tau \rangle_e, \quad \langle \tau \rangle_e \gg 0, \quad \langle \tau \rangle_e = (\text{finite}) . \tag{112}$$

VI. COMPARISON WITH THE PASSAGE OVER A FLUCTUATING ENERGY BARRIER

In this section we make a comparison between the epidemic model and the passage over a fluctuating energy barrier with dynamical disorder [6,7]. We start out by considering the classical random activation energy model with static disorder (RAEM), which has been applied to many problems from condensed matter physics [22–24] and molecular biology [25,26]. Considering an exponential distribution of activation energies

$$\eta(E) = (k_B T_0)^{-1} \exp(-E/k_B T_0) , \tag{113}$$

which corresponds to a canonical distribution "frozen" at temperature T_0, we see that the probability density $\tilde{\psi}(t)dt$ of the passage time over the barrier is a weighted distribution of Poisson processes

$$\tilde{\psi}(t) = \int_0^\infty \eta(E) W \exp(-Wt) dE , \tag{114}$$

where the hopping frequency W is given by the usual Arrhenius expression

$$W=\nu\exp(-E/k_BT)\ , \tag{115}$$

ν is the maximum jump frequency corresponding to zero activation energy, and $T\leq T_0$ is the system temperature. From Eqs. (113)–(115) it follows that the probability density $\tilde{\psi}(t)dt$ of the passage time is given by a relationship that is isomorphic with Eq. (37) for the probability density of the duration of the active infection period derived by applying the continuous version of the ideal fractal epidemic model developed in Sec. II:

$$\tilde{\psi}(t)=H\nu(\nu t)^{-(1+H)}\gamma(1+H,\nu t)\ , \tag{116}$$

where the fractal exponent H is equal to

$$H=T/T_0\leq 1\ . \tag{117}$$

We notice that here the random activation energy E plays the role of the germ load m that is randomly selected from the exponential probability density $\chi(m)dm$ given by Eq. (42), which is the analog of the canonical distribution (113) of activation energies.

The main assumption of the RAEM approach is that a fluctuation of the height of the energy barrier lasts forever, which justifies the validity of the static ensemble average in Eq. (114). Although reasonable for some problems of condensed matter physics, the validity of this assumption is questionable in molecular biology. In the case of protein-ligand interactions [25] and of ion channel kinetics [26], the distribution of energy barriers is due to conformational fluctuations, which have a dynamical nature, and thus the fluctuations of the activation energy are continuously generated and destroyed by thermal agitation. The static RAEM approach has been recently generalized by two of the present authors by incorporating into it the possibility of existence of dynamical fluctuations of the height of the energy barrier. A detailed analysis of this generalized RAEM approach is presented elsewhere [27]. Here we outline only the main assumptions and results of the model and make a comparison with the theory of epidemics.

Unlike the static random activation energy model, the dynamical RAEM approach developed in Ref. [27] is not isomorphic with the imperfect fractal epidemic model developed in the present paper. However, despite the different form of the evolution equations, the two models still share some common features.

The dynamic fluctuations of the energy barrier are described by a dynamic Bloch-like equation

$$\partial_t\eta(E,t)=\mathbb{L}\eta(E,t)\ , \tag{118}$$

with the boundary condition $\eta=(E=0,t)=1/k_BT_0$, where $\eta(E,t)dE$ is a time-dependent probability density of the activation energy

$$\mathbb{L}\ldots=-\omega[\ldots+k_BT_0\partial_E\ldots] \tag{119}$$

is a linear evolution operator, and ω is the regression rate of the fluctuations of the barrier height. One assumes that the fluctuations of the activation energy are stationary and Markovian, and thus the one-time probability density of the activation energy is given by Eq. (113), which is the stationary solution of Eq. (118). Because of the Markovian character of fluctuations, the multitime joint probability densities $\eta_m(E_1,t_1;\ldots;E_m,t_m)dE_1\cdots dE_m$ are completely determined by the stationary one-time probability density (113) and by the Green's function $\eta_1(E_1,t_1|E_2,t_2)=\eta_1(E_1,t_1-t_2|E_2,0)$ of the evolution equation (118), which depends only on the time difference t_1-t_2 and not on the individual times t_1 and t_2. In particular, for $m=2$ we have

$$\eta_2(E_1,t_1;E_2,t_2)=h(t_1-t_2)\eta_1(E_2)\eta_1(E_1,t_1-t_2|E_2,0)+h(t_2-t_1)\eta_1(E_1)\eta_1(E_2,t_2-t_1|E_1,0)\ , \tag{120}$$

where $\eta_1(E_{1;2})$ is given by Eq. (113), the Green's function $\eta_1(E,t|E',0)$ is equal to

$$\eta_1(E,t|E',0)=(k_BT_0)^{-1}\exp(-E/k_BT_0)h(\omega tk_BT_0-E)+\delta(E-E'-\omega tk_BT_0)\exp(-\omega t)\ , \tag{121}$$

and $h(x)$ is the usual Heaviside function. The correlation function of the activation energy corresponding to Eqs. (120) and (121) is given by

$$\langle\Delta E(t)\Delta E(t')\rangle=\int_0^\infty\int_0^\infty(E_1-\langle E\rangle)(E_2-\langle E\rangle)\times\eta_2(E_1,t;E_2,t')dE_1dE_2=(k_BT_0)^{-2}\exp(-\omega|t-t'|)\ . \tag{122}$$

This equation is the analog of the relationship (90) for the time decrease of the germ load due to healing. There is, however, a difference. Even though the regression of fluctuations of the energy barrier is similar to the decrease of the germ load due to healing, Eq. (90) is a deterministic equation for the time dependence of the germ load, whereas the expression (122) for the correlation function is stochastic.

For dynamic fluctuations of the activation energy, the probability density of the passage time is given by a dynamic average

$$\tilde{\psi}(t)=\left\langle\nu\exp[-E(t)/k_BT]\exp\left\{-\nu\int_0^t\exp[-E(t')/k_BT]dt'\right\}\right\rangle\ , \tag{123}$$

which is the analog of the static equation (114). The average in Eq. (123) is given by a path integral, which takes into

account all possible random functions $E(t')$, $t \geq t' \geq 0$. The evaluation of the path average in Eq. (123) can be reduced to the solution of a Liouville equation, similar to but not identical with the Liouville equation (57) derived in Sec. III for the description of the healing process:

$$\partial_t B(l,E;t)=\partial_l\{l\nu \exp[-E/k_BT]B(l,E;t)\}+\mathrm{L}B(l,E;t) , \tag{124}$$

with the initial condition

$$B(l,E;t=0)=(k_BT_0)^{-1}\exp(-E/k_BT_0)\delta(l-1) , \tag{125}$$

where $B(l,E;t)dl\,dE$ is the joint probability density of the instantaneous survival function l attached to the passage process and of the activation energy E at time t. The mathematical structure of Eq. (124) is simpler than that of the similar equation (57) derived in the context of epidemic theory. Because of the simplicity of Eq. (124) for the passage problem, it is not necessary to solve it explicitly for the probability density $B(l,E;t)$. The tedious method of integration of Eq. (57) presented in Appendix A can be replaced by an indirect approach, which leads directly to closed expressions for the moments of survival function $\langle l^q(t)\rangle$ and of the probability density $\tilde{\psi}(t)$ of the passage time [27]:

$$\langle l^q(t)\rangle = H\{[\exp(\omega t/H)-1]\nu Hq/\omega\}^{-H}\gamma(H,[\exp(\omega t/H)-1]\nu Hq/\omega) , \tag{126}$$

$$\tilde{\psi}(t)=H\nu\exp(\omega t/H)\{[\exp(\omega t/H)-1]H\nu/\omega\}^{-(H+1)}\gamma(H+1,[\exp(\omega t/H)-1]H\nu/\omega) . \tag{127}$$

Incidentally Eqs. (126) and (127) have the same structure as Eqs. (99) and (100), where the healing rate coefficient b is replaced by the regression rate ω. Although the other symbols are the same in both sets of equations, they have a different biological and physical significance in each case. The main difference between these two sets of equations is that Eqs. (127) and (128) are exact and valid for any values of the regression frequency ω, low or high: in contrast, in the case of epidemics Eqs. (99) and (100) are only approximations for $b \to 0$ of the exact equations (91) and (92).

Because of the similarity of structure between Eqs. (99) and (100) and (126) and (127), the results concerning the imperfect statistical fractal behavior derived in Sec. V in the context of epidemics are also valid for the passage over a fluctuating energy barrier. In particular, Eqs. (101)–(112) remain valid, with the difference that the parameter b should be replaced by the frequency ω and the physical significance of the other symbols should be changed accordingly. It follows that the moments of the survival function and the probability density of the passage time have tails with long beginnings of the inverse power law type followed by a fast exponential decay. Similarly, the fluctuations of the passage time, although characterized by finite moments and cumulants, have an intermittent behavior as the regression rate of fluctuations ω tends to zero, $\omega \to 0$.

We emphasize that all these analogies are rather superficial and limited to the region of small rates b and $\omega(b,\omega \to 0)$. Despite the formal analogy of the evolution equations as $b,\omega \to 0$, the underlying mechanisms of the two phenomena are different. In the case of the epidemic models developed in this paper, the starting point was a discrete model inspired by a cellular automata description, which has been approximated by a continuous model in order to get rid of the logarithmic oscillations of the infection time that are spurious; no such discrete model can be imagined for the passage over a fluctuating energy barrier. In the theory of epidemics we have developed two different mathematical formalisms. The first formalism is biologically motivated and gives a direct description of the epidemic spreading because of the encounters of a germ carrier with healthy individuals. The second formalism is more abstract and based on the method of Liouville equations. In contrast, for the passage over a random activation barrier with dynamical disorder, there is no underlying physical mechanism similar to the encounter process, and therefore the unique mathematical formalism available is the one based on a Liouville equation description.

VII. CONCLUSIONS

In this paper we have suggested a model for the onset of an epidemic due to the encounters of an infective individual with other healthy individuals susceptible to infection. For the description of the process two types of mathematical techniques have been developed, based on a multiple convolution product representing the contributions of different encounters and on a conservation equation in the phase space of the Liouville type for the state probability density, respectively. Although the two formalisms are consistent with each other, their range of validity is different and lead to different generalizations.

The investigation of a particular epidemic model for which the process of healing of an infective obeys an exponential recovery law has led to a statistical fractal with a cutoff for which only the beginning of the tail of the probability density $\tilde{\psi}(t)dt$ of the duration of the active infection period obeys a self-similar scaling law of the inverse power law type. The cutoff of this statistical fractal distribution is generated by the healing process, which leads to an exponential shape of the end of the tail. This epidemic imperfect statistical fractal shares some features with another type of imperfect fractal generated by the passage of a particle or quasiparticle over a random activation energy barrier with dynamical disorder.

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APPENDIX A

We replace the survival function $l(t)$ by the new random variable

$$\epsilon(t) = -\ln l(t) \ . \tag{A1}$$

By analogy with the nomenclature used in nuclear physics, we call the parameter $\epsilon(t)$ "the lethargy." We introduce the joint probability density of the lethargy and the germ load at time t, respectively:

$$\phi(\epsilon,m;t)d\epsilon\,dm \quad \text{with} \quad \int_0^\infty \int \phi(\epsilon,m;t)d\epsilon\,dm = 1 \ . \tag{A2}$$

We have

$$B(l,m;t)d|l|dm = \phi(\epsilon,m;t)d|\epsilon|dm \ , \tag{A3}$$

from which we obtain

$$\phi(\epsilon,m;t) = \exp(-\epsilon)B(\exp(-\epsilon),m;t) \ ;$$
$$B(l,m;t) = \phi(-\ln l,m;t)/l \ . \tag{A4}$$

By expressing Eqs. (57) and (58) in terms of the probability density $\phi(\epsilon,m;t)$, we come to

$$\partial_t\phi(\epsilon,m;t) = -\partial_\epsilon[\nu\exp(-km)\phi(\epsilon,m;t)] - \partial_m[F(m,t)\phi(\epsilon,m;t)] \ , \tag{A5}$$

with the initial condition

$$\phi(\epsilon,m;t=0) = \delta(\epsilon)\chi(m) \ . \tag{A6}$$

Now we introduce the marginal characteristic function of the probability density $\phi(\epsilon,m;t)$ with respect to the lethargy variable ϵ as the Laplace transform

$$\bar{\phi}(\xi,m;t) = \int_0^\infty \exp(-\epsilon\xi)\phi(\epsilon,m;t)d\epsilon \ , \tag{A7}$$

where ξ is the Laplace variable conjugate to ϵ. Through Laplace transformation, Eqs. (A5) and (A6) become

$$\partial_t\bar{\phi}(\xi,m;t) = -\xi\nu\exp(-km)\bar{\phi}(\xi,m;t) - \partial_m[F(m,t)\bar{\phi}(\xi,m;t)] \ , \tag{A8}$$

$$\bar{\phi}(\xi,m;t=0) = \chi(m) \ . \tag{A9}$$

The characteristic system attached to the partial differential equation (A8) is

$$dm/dt = F(m,t) \ , \tag{A10}$$

$$d\ln\bar{\phi}/dt = -\partial_m F(m,t) - \xi\nu\exp(-km) \ . \tag{A11}$$

From Eqs. (46), (48), and (A10) it follows that the characteristic curves of Eq. (A8) in the (t,m) plane are given by

$$m = \varphi(t;C) \ , \tag{A12}$$

where C is a non-negative arbitrary constant. The general solution of Eq. (A8) can be obtained by integrating Eq. (A11) along the characteristics

$$\bar{\phi} = \Omega(C)\exp\left\{-\int_0^t \partial_\varphi F(\varphi(t;C),t')dt' - \xi\nu\int_0^t \exp[-k\varphi(t;C)]dt'\right\} \ , \tag{A13}$$

where $\Omega(C)$ is an arbitrary function of the integration constant which can be determined from the initial condition (A9). By applying Eq. (A13) for $t=0$ and comparing the result with Eq. (A9) we get

$$\Omega(C) = \chi(C) \ . \tag{A14}$$

By using Eq. (A14) and expressing the integration constant $C = m_0$ in terms of the germ load m at time t by means of Eq. (50), Eq. (A13) becomes

$$\bar{\phi}(\xi,m;t) = \chi(\varphi^{-1}(t;m))\exp\left\{-\int_0^t \partial_\varphi F(\varphi(t',\varphi^{-1}(t;m)),t')dt'\right\}\exp\left\{-\xi\nu\int_0^t \exp[-k\varphi(t';\varphi^{-1}(t;m))]dt'\right\} \ . \tag{A15}$$

For evaluating the F-dependent factor in Eq. (A15) we write the differential equation (46) as

$$d\varphi(t;m_0)/dt = F(\varphi(t;m_0),t) \tag{A16}$$

and differentiate both terms of Eq. (A16) with respect to m_0, resulting in

$$d\gamma(t;m_0)/dt = [\partial_\varphi F(\varphi,t)]\gamma(t;m_0) \ . \tag{A17}$$

As $\varphi(t=0,m_0) = m_0$ we have

$$\gamma(t=0;m_0) = 1 \ . \tag{A18}$$

By integrating Eq. (A17) with the initial condition (A18) we obtain

$$\gamma(t;m_0) = \exp\left\{\int_0^t \partial_\varphi F(\varphi(t';m_0),t')dt'\right\} \ . \tag{A19}$$

By combining Eqs. (A15) and (A18) we come to

$$\bar{\phi}(\xi,m;t) = \chi(\varphi^{-1}(t;m))\gamma^{-1}(t;\varphi^{(-1)}(t;m))\exp\left\{-\xi\nu\int_0^t \exp[-k\varphi(t';\varphi^{-1}(t;m))]dt'\right\} \ . \tag{A20}$$

For computing the moments of the survival function $\langle l^q(t)\rangle$ with $q>0$ we note that

$$\langle l^q(t)\rangle = \int_0^1 \int_0^\infty l^q B(l,m;t)\,dl\,dm = \int_0^\infty \int_0^\infty \exp(-\epsilon q)\phi(\epsilon,m;t)\,d\epsilon\,dm = \int_0^\infty \bar{\phi}(\xi=q;m;t)\,dm \ . \quad \text{(A21)}$$

By inserting Eq. (A20) into Eq. (A21) we get Eq. (59).

APPENDIX B

By following the same steps as in Appendix A and starting out from Eqs. (67) and (58), we can derive the following evolution equation for the marginal characteristic function $\bar{\phi}(\xi,m;t)$:

$$\partial_t \bar{\phi}(\xi,m;t) = -\nu\xi \exp(-km)\bar{\phi}(\xi,m;t) + \mathbb{M}\bar{\phi}(\xi,m;t) \ , \quad \text{(B1)}$$

with the initial condition (A9), where the Markovian evolution operator $\mathbb{M}$ is given by

$$\mathbb{M}f(m) = \int_0^\infty [W(m'\to m;t)f(m') - W(m\to m';t)f(m)]\,dm' \ . \quad \text{(B2)}$$

The marginal characteristic function $\bar{\phi}(\xi,m;t)$ can be expressed in terms of the initial condition $\bar{\phi}(\xi,m;t=0)=\chi(m)$ by noticing that the solution $g_\xi^*(m;t|y;0)$ of Eqs. (70) is the Green's function of Eq. (B1). We come to

$$\bar{\phi}(\xi,m;t) = \int_0^\infty dy\ \chi(y) g_\xi^*(m;t|y;0) \ . \quad \text{(B3)}$$

By inserting Eq. (B3) into Eq. (A21) we come to Eqs. (68). By differentiating the first of Eqs. (68) corresponding to $q=1$ and inserting in the resulting equation the expression of the time derivative $\partial_t g_\xi^*(m;t|y;0)$ given by the master equation (70), we obtain

$$\begin{aligned}\tilde{\psi}(t) &= -\partial_t \langle l(t)\rangle \\ &= -\int_0^\infty dm \int_0^\infty dm' \int_0^\infty dy\ \chi(y) W(m'\to m;t) g_1^*(m';t|y;0) \\ &\quad + \int_0^\infty dm \int_0^\infty dm' \int_0^\infty dy\,\chi(y) W(m\to m';t) g_1^*(m;t|y;0) + \int_0^\infty dm \int_0^\infty dy\ \chi(y)\nu \exp(-km) g_1^*(m;t|y;0) \ . \end{aligned} \quad \text{(B4)}$$

In Eq. (B4) the first two integral terms from the right-hand side cancel each other; this can be checked by replacing the integration variables m,m' by m',m. The result of this operation is Eq. (69).

For checking that the Markovian approach includes the Liouville equation description as a particular case, we express the operator $\mathbb{M}$ by its Kramers-Moyal expansion

$$\mathbb{M}\cdots = \sum_{j=1}^\infty (-1)^j \frac{\partial^j}{\partial m^j}[D_j(m,t)\ldots] \ , \quad \text{(B5)}$$

where

$$D_j(m;t) = \frac{1}{j!}\int_0^\infty W(m\to m';t)(m'-m)^j dm' \ , \quad \text{(B6)}$$

and keep only the first term. In this case Eq. (B1) reduces to the Liouville equation (A8), where the healing rate $F(m,t)$ is given by

$$F(m,t) = \int_0^\infty W(m\to m';t)(m-m')\,dm' \ . \quad \text{(B7)}$$

APPENDIX C

For a correlated random point process it is more advantageous to compute the average survival function $\langle l(t)\rangle$ and then to evaluate the probability density $\tilde{\psi}(t)dt$ by differentiating Eq. (60) with respect to t. The survival function $\langle l(t)\rangle$ can be expressed as an average of the product

$$\beta(t_1';m_0)\cdots\beta(t_N';m_0) \quad \text{(C1)}$$

over the number N and over the times $t_1',\ldots,t_N'$ of the encounters with healthy individuals, as well as over the initial germ load m_0 and over all possible germ loads at different times represented by the random functions $\varphi(t_1'';m_0),\ldots,\varphi(t_N'';m_0)$. For evaluating these averages we introduce the probability density functional of the function $\varphi(t'';m_0)$:

$$\mathcal{B}[\varphi(t'';m_0);t\ge t''\ge 0;m_0]\,D[\varphi(t'';m_0);t\ge t''\ge 0;m_0] \ , \quad \text{(C2)}$$

with the normalization condition

$$\overline{\int\int}\,\mathcal{B}[\varphi(t'';m_0]D[\varphi t'';m_0)]=1 \ , \quad \text{(C3)}$$

where $\overline{\int\int}$ stands for the operation of path integration and $D[\varphi(t'';m_0)]=D[\varphi(t'';m_0);\ t\ge t''\ge 0;m_0]$ is a suitable integration measure over the space of functions $\varphi(t'';m_0)$. Now a difficulty arises because the integration measure $D[\varphi(t'';m_0)]$ can be properly defined only if the random functions $\varphi(t'';m_0)$ are Gaussian. This is not a major difficulty because the average of the product (C1) can be expressed in terms of the functional integral of the type

$$\overline{\int\int}\,\mathcal{B}[\varphi(t'';m_0)]D[\varphi(t'';m_0)]\exp\left\{-k\int_0^t \varphi(t'';m_0)\delta(t''-t')dt'\right\} = G[K(t'')=k\delta(t''-t')] \ , \quad \text{(C4)}$$

which depends on the characteristic functional $G[K(t'')]$ given by Eq. (87). If the moments $\langle \varphi(t_1'';m_0)\cdots\varphi(t_q'';m_0)\rangle$ or the cumulants $\langle\langle \varphi(t_1'';m_0)\cdots\varphi(t_q'';m_0)\rangle\rangle$, $q=1,2,\ldots$, of the random function $\varphi(t'';m_0)$ exist and are finite, then the characteristic functional $G(K(t''))$ can be computed by means of the moment and cumulant expansions

$$G[K(t'')]=1+\sum_{q=1}^{\infty}\frac{(-1)^q}{q!}\int_0^t\cdots\int_0^t\langle\varphi(t_1'';m_0)\cdots\varphi(t_q'';m_0)\rangle K(t_1'')\cdots K(t_q'')dt_1''\cdots dt_N''$$
$$=\exp\left\{\sum_{q=1}^{\infty}\frac{(-1)^q}{q!}\int_0^t\cdots\int_0^t\langle\langle\varphi(t_1'';m_0)\cdots\varphi(t_N'';m_0)\rangle\rangle K(t_1'')\cdots K(t_q'')dt_1''\cdots dt_N''\right\}, \tag{C5}$$

which are independent of the integration measure $D(\varphi(t'';m_0))$.

The average of Eq. (C1) can be expressed as

$$\langle l(t)\rangle=\int_0^{\infty}dm_0\chi(m_0)\left\{1+\sum_{N=1}^{\infty}\frac{1}{N!}\overline{\int\int}\mathcal{B}[\varphi(t_1'';m_0)]D[\varphi(t_1'';m_0)]\cdots\right.$$
$$\left.\times\overline{\int\int}\mathcal{B}[\varphi(t_N'';m_0)]D[\varphi(t_N'';m_0)]\int_0^t\cdots\int_0^t dt_1'\cdots dt_N'Q_N(t_1',\ldots,t_N')\beta(t_1';m_0)\cdots\beta(t_N';m_0)\right\}. \tag{C6}$$

We express the Janossy probability densities $Q_N(t_1',\ldots,t_N')$ in terms of the product densities $\eta_m(t_1',\ldots,t_M')$ by using the equations [21]

$$Q_N(t_1',\ldots,t_N')=\sum_{S=0}^{\infty}\frac{(-1)^S}{S!}\int_0^t\cdots\int_0^t\eta_{N+S}(t_1',\ldots,t_N',t_{N+1}',\ldots,t_{N+S}')dt_{N+1}'\cdots dt_{N+S}'. \tag{C7}$$

By inserting Eqs. (C7) into Eq. (C6), using the new summation variables $N'=N$, $M=N+S$, changing the order of summation and evaluating the sum over N', we obtain

$$\langle l(t)\rangle=\int_0^{\infty}dm_0\chi(m_0)\left\{1+\sum_{M=1}^{\infty}\frac{1}{M!}\overline{\int\int}\mathcal{B}[\varphi(t_1'';m_0)]D[\varphi(t_1'';m_0)]\cdots\right.$$
$$\times\overline{\int\int}\mathcal{B}[\varphi(t_M'';m_0)]D[\varphi(t_M'';m_0)]\int_0^t\cdots\int_0^t dt_1'\cdots dt_M'\eta_N(t_1',\ldots,t_N')$$
$$\left.\times[\beta(t_1';m_0)-1]\cdots[\beta(t_M';m_0)-1]\right\}. \tag{C8}$$

Since

$$\beta(t';m_0)-1=-\exp\left[-\int_0^t k\,\delta(t''-t')\varphi(t'';m_0)dt''\right]. \tag{C9}$$

Eq. (C8) can be expressed in terms of the generating functional $\Xi[W(t')]$ of the product densities

$$\langle l(t)\rangle=\int_0^{\infty}dm_0\chi(m_0)\left\{1+\sum_{M=1}^{\infty}\frac{1}{M!}\int_0^t\cdots\int_0^t\eta_M(t_1',\ldots,t_M')W^*(t_1')\cdots W^*(t_M')dt_1'\cdots dt_M'\right\}$$
$$=\Xi[W(t')=W^*(t'):t\geq t'\geq 0], \tag{C10}$$

where

$$W^*(t')=-G[K(t')=k\delta(t''-t'),t\geq t''\geq 0;m_0] \tag{C11}$$

and therefore

$$\langle l(t)\rangle=\int_0^{\infty}dm_0\chi(m_0)\Xi[W(t')=-G[K(t'')=k\delta(t''-t');m_0];t\geq t'\geq 0]. \tag{C12}$$

By differentiating Eq. (60) with respect to t and inserting in the resulting equation the expression (C12) for $\langle l(t)\rangle$, we get Eq. (88).

APPENDIX D

By introducing the intrinsic time scales

$$\theta(t)=[\exp(bt/H)-1]H/b,\quad \theta'(t')=[\exp(bt'/H)-1]H/b, \tag{D1}$$

Eqs. (99) become

$$\langle l^q(t)\rangle = \int_0^1 Hx^{H-1}dx \exp\left\{-q\nu\int_0^{\theta(t)} \exp[(\ln x)(1+b\theta'/H)^{-H}](1+b\theta'/H)^{-1}d\theta'\right\} . \tag{D2}$$

By keeping the dominant terms in b as $b\to 0$, we get the following approximation for Eqs. (D2):

$$\langle l^q(t)\rangle \cong \int_0^1 Hx^{H-1}dx \exp[-qx\nu\theta(t)] = H\gamma(H, q\nu\theta(t))[q\nu\theta(t)]^{-H} . \tag{D3}$$

By substituting the definition (D1) of the intrinsic time scale $\theta(t)$ into Eqs. (D3) we obtain Eqs. (99). For computing the probability density $\tilde{\psi}(t)$ we differentiate Eqs. (D3) for $q=1$ with respect to t, resulting in

$$\tilde{\psi}(t) = -d\langle l(t)\rangle/dt = -[d\langle l(\theta(t))\rangle/d\theta(t)][d\theta(t)/dt] = \frac{H\gamma(H+1,\nu\theta(t))}{[\nu\theta(t)]^{H+1}}\nu\frac{d\theta(t)}{dt} . \tag{D4}$$

From Eqs. (D1) and (D4) we come to Eq. (100). We note that the approximate probability density $\tilde{\psi}(t)$ obtained by applying this method is properly normalized to unity.

APPENDIX E

The positive moments $\langle t^q\rangle$ of the duration of the active infectious period can be computed in the limit $b\to 0$ by using the approximate expression (100) for $\tilde{\psi}(t)$:

$$\langle t^q\rangle = \int_0^\infty t^q\tilde{\psi}(t)dt$$
$$= H(H/b)^{q-H}\nu^{-H}\int_0^\infty \frac{\exp(-Hy)\gamma(H+1,[\exp(y)-1]H\nu/b)}{[1-\exp(-y)]^{H+1}}y^q dy, \quad \text{as } b\to 0 . \tag{E1}$$

In the integral in Eq. (E1) the only b-dependent factor is the one depending on the incomplete γ function. In the limit $b\to 0$, the incomplete γ function can be approximated by the complete Γ function $\Gamma(H+1)$, leading to

$$\langle t^q\rangle \sim H\Gamma(1+H)(H/b)^{q-H}\nu^{-H}J_q(H), \quad q>H, \quad b\to 0, \quad 1>H>0 , \tag{E2}$$

which is the first of the two sets of Eqs. (104).

For computing the cumulants $\langle\langle t^q\rangle\rangle$ we evaluate the generating function of the probability density $\tilde{\psi}(t)$ in the limit $b\to 0$. Equation (100) leads to

$$\bar{\tilde{\psi}}(s) = \int_0^\infty \exp(-st)\tilde{\psi}(t)dt = \nu H\int_0^\infty \frac{\exp(-st)\exp(bt/H)\gamma(H+1,[\exp(bt/H)-1]H\nu/b)}{\{[\exp(bt/H)-1]H\nu/b\}^{H+1}}dt . \tag{E3}$$

In Eq. (E3) the incomplete γ function cannot be replaced by the complete gamma function $\Gamma(H+1)$ because this operation generates a divergence of the integral for $t=0$. We use an integral representation of the incomplete γ function and introduce the integration variable

$$z = \nu\theta(t) , \tag{E4}$$

resulting in

$$\bar{\tilde{\psi}}(s) = 1 - \int_0^1 Hx^H dx\int_0^\infty [1-(1+zb/H\nu)^{-sH/b}]\exp(-zx)dz . \tag{E5}$$

We expand the integrand in Eq. (E5) in a double series, keep the dominant terms in b as $b\to 0$, and try to regroup the remaining terms in the expansion of an exponential by using the method of cumulant expansion [21]. By summing the resulting series and using the integration variable

$$y = \ln(1+zb/H\nu) , \tag{E6}$$

we get

$$\bar{\tilde{\psi}}(s) \sim \exp\left\{-H\Gamma(1+H)\left[\frac{b}{H\nu}\right]^H\int_0^\infty dy\frac{\exp(-Hy)[1-\exp(-sHy/b)]}{[1-\exp(-y)]^{H+1}}\right\}, \quad b\to 0 . \tag{E7}$$

Now the cumulants can be easily evaluated from Eq. (E7) and from their definition

$$\langle\langle t^q\rangle\rangle = (-1)^q\partial^q \ln[\bar{\tilde{\psi}}(s=0)]/\partial s^q \sim H\Gamma(1+H)(H/b)^{q-H}\nu^{-H}J_q(H), \quad q>H, \quad b\to 0, \quad 1>H>0 , \tag{E8}$$

and thus in the limit $b\to 0$ the cumulants $\langle\langle t^q\rangle\rangle$ obey the same scaling law as the moments $\langle t^q\rangle$.

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