Decoherence of quantum-nondemolition systems

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The density matrices of a generic quantum-nondemolition system coupled to the reservoir of an infinite number of harmonic oscillators and to the reservoir of an infinite number of two-level systems are evaluated exactly. It is found that quantum decoherence is independent of temperature in the latter case.

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It is not exaggerating to say that dissipative systems are ubiquitous in the physical world since no systems, even the thought models, are immune to the couplings of the surroundings. We know that classical dynamics of dissipative systems confirms to the Langevin equation, which is a phenomenological equation of motion. Historically, the first microscopic model describing dissipative effect was suggested by Ford, Kac, and Mazur, in which the system is assumed to be coupled to an infinite number of harmonic oscillators [1]. Interest in the quantum dynamics of dissipative systems was aroused in the last decade due to the pioneering work of Leggett, Zurek, and coworkers [2,3]. One believes that these studies may reveal the mystery of macroscopic quantum phenomena and solve some puzzles of quantum measurement. To have a reasonable Hamiltonian, Caldeira and Leggett have argued that the harmonic oscillator or bosonic reservoir can be viewed as a general form of surroundings [4]. The two-level-system (TLS) reservoir has also been proposed and shown to be equivalent to the bosonic one in some cases [5,6]. In general one cannot predict the exact quantum behavior if the couplings are not quantum nondemolition, i.e., the interaction term and the Hamiltonian of the system are not commutative. Although there are a few papers concerning the systems with nondemolition couplings to the bosonic reservoir [7–10], these authors did not recognize that these systems are generally exactly solvable. In this paper we will derive exactly the reduced density matrix (RDM) of two generic systems with nondemolition couplings to the bosonic and the TLS reservoirs.

Let us consider the bosonic reservoir first. The total Hamiltonian is

$$\hat{H}_T=\hat{H}+\hat{H}_R+V_I+f^2(\hat{H})\sum_k c_k^2/\omega_k \ , \tag{1}$$

where $\hat{H}$ is the Hamiltonian of the system, $\hat{H}|n\rangle=E_n|n\rangle$, $\hat{H}_R=\sum_k \omega_k b_k^\dagger b_k$ is the reservoir consisting of an infinite number of harmonic oscillators, and the interaction term is assumed to be general: $V_I=f(\hat{H})\sum_k c_k(b_k^\dagger+b_k)$, with c_k being the coupling constants. We should point out that the fourth term is a renormalization one; it can be removed by appropriate transformations. Here and in the following $\hbar$ is set to unity. Note that Tameshtit and Sipe have recently studied a special case of Eq. (1), i.e., $f(\hat{H})=\hat{H}$, and derived the master equation [10]. Their results were based on the Born and Markov approximation and are valid only at high temperatures.

Using the unitary transformation

$$U=\exp\left[f(\hat{H})\sum_k \frac{c_k}{\omega_k}(b_k^\dagger-b_k)\right] , \tag{2}$$

we have the transformed Hamiltonian,

$$\tilde{H}_T=U\hat{H}_TU^{-1}=\hat{H}+\sum_k \omega_k b_k^\dagger b_k \ , \tag{3}$$

which is a decoupled one. Thus the corresponding density matrix reads

$$\begin{aligned}\tilde{\rho}(t)&=\exp(-i\tilde{H}_Tt)\tilde{\rho}(0)\exp(i\tilde{H}_Tt)\\&=\exp(-i\hat{H}t)\exp\left[-it\sum_k\omega_k b_k^\dagger b_k\right]U\rho(0)U^{-1}\\&\quad\times\exp\left[it\sum_k\omega_k b_k^\dagger b_k\right]\exp(i\hat{H}t) \ .\end{aligned}$$

The converse transformation allows us to obtain

$$\begin{aligned}\rho(t)&=U^{-1}\tilde{\rho}(t)U\\&=\exp(-i\hat{H}t)U^{-1}\exp\left[-it\sum_k\omega_k b_k^\dagger b_k\right]U\rho(0)U^{-1}\\&\quad\times\exp\left[it\sum_k\omega_k b_k^\dagger b_k\right]U\exp(i\hat{H}t) \ .\end{aligned} \tag{4}$$

Denote the RDM as $\rho^S(t)$, $\rho^S(t)\equiv\mathrm{Tr}_R\rho(t)$. We assume that $\rho(0)=\rho^S(0)\prod\rho^k(0)$, where $\rho^k(0)$ is the density matrix of the kth harmonic oscillator in thermal equilibrium. To evade confusion, we should emphasize that $|m\rangle$ or $|n\rangle$ always stands for the eigenstate of the system. From Eq. (4), we find the RDM element $\langle m|\rho^S(t)|n\rangle$ as

$$\rho_{mn}^S(t)=\exp[-i(E_m-E_n)t]\langle m|\rho^S(0)|n\rangle\prod_k \mathrm{Tr}_k W_k \ , \tag{5}$$

where

$$W_k = U(E_m)^{-1}\exp(-it\omega_k b_k^\dagger b_k)U(E_m)\rho^k(0)U(E_n)^{-1} \times\exp(it\omega_k b_k^\dagger b_k)U(E_n)\ , \qquad (6)$$

with $U(x)=\exp[f(x)(c_k/\omega_k)(b_k^\dagger - b_k)]$. Employing the following Baker-Campbell-Hausdorff identities,

$$e^{\gamma(b_k^\dagger - b_k)} = e^{\gamma b_k^\dagger}e^{-\gamma b_k}e^{-\gamma^2/2} = e^{-\gamma b_k}e^{\gamma b_k^\dagger}e^{\gamma^2/2}\ ,$$

$$e^{\gamma b_k}e^{\alpha b_k^\dagger b_k} = e^{\alpha b_k^\dagger b_k}e^{\gamma e^{\alpha} b_k}\ ,$$

$$e^{\gamma b_k^\dagger}e^{\alpha b_k^\dagger b_k} = e^{\alpha b_k^\dagger b_k}e^{-\gamma e^{\alpha} b_k^\dagger}\ ,$$

where α and γ are c numbers, we obtain

$$\mathrm{Tr}_k W_k = \exp[-g_k(t)]\mathrm{Tr}_k \times\{\exp[p_k(t)b_k^\dagger]\exp[q_k(t)b_k]\rho^k(0)\}\ , \qquad (7)$$

where

$$g_k(t) = -[f(E_m)-f(E_n)]^2\frac{c_k^2}{\omega_k^2}(1-\cos\omega_k t) - i[f^2(E_m)-f^2(E_n)]\frac{c_k^2}{\omega_k^2}\sin\omega_k t\ ,$$

$$p_k(t) = [f(E_m)-f(E_n)]\frac{c_k}{\omega_k}(1-e^{it\omega_k})\ ,$$

$$q_k(t) = [f(E_m)-f(E_n)]\frac{c_k}{\omega_k}(e^{-it\omega_k}-1)\ .$$

It is a trivial task to evaluate the trace in the right-hand side of Eq. (7); the result is

$$\exp\left\{\frac{e^{-\beta\omega_k}p_k(t)q_k(t)}{1-e^{-\beta\omega_k}}\right\}.$$

Inserting these expressions into Eq. (7), we thus obtain

$$\mathrm{Tr}_k W_k = \exp[-ia_{mn}^{(1k)}(t) - a_{mn}^{(2k)}(t)]\ ,$$

where

$$a_{mn}^{(1k)}(t) = [f^2(E_m)-f^2(E_n)](c_k^2/\omega_k^2)\sin\omega_k t\ ,$$

$$a_{mn}^{(2k)}(t) = [f(E_m)-f(E_n)]^2(c_k^2/\omega_k^2) \times(1-\cos\omega_k t)\coth(\beta\omega_k/2)\ .$$

Therefore, we find an explicit form for the RDM element [see Eq. (5)],

$$\rho_{mn}^S(t) = \exp[-i(E_m-E_n)t]\langle m|\rho^S(0)|n\rangle \times\exp(-iA_{mn}^{(1)} - A_{mn}^{(2)})\ , \qquad (8)$$

where $A_{mn}^{(1)} = \sum_k a_{mn}^{(1k)}(t)$ and $A_{mn}^{(2)} = \sum_k b_{mn}^{(2k)}(t)$. Introducing the Ohmic spectral density function, i.e.,

$$\rho(\omega) = \eta\frac{\omega}{c^2(\omega)}e^{-\omega/\omega_c}\ ,$$

with ω_c being the high frequency cutoff, and manipulating elementary calculations (cf. [11]), we obtain

$$A_{mn}^{(1)} = \eta\arctan(\omega_c t)[f^2(E_m)-f^2(E_n)]\ ,$$

and

$$A_{mn}^{(2)} = \left\{\frac{\eta}{2}\ln(1+\omega_c^2t^2) + \eta\ln\prod_{k=1}^{\infty}\left[1+\left[\frac{\omega_c t}{1+k\beta\omega_c}\right]^2\right]\right\} \times[f(E_m)-f(E_n)]^2\ .$$

Define $R(t)\equiv\mathrm{Tr}[\rho^S(t)]^2$ as a measure of coherence [12]. $R(t)\leq 1$, and if and only if the system is in a pure state, $R(t)=1$. Now let us set

$$\rho^S(0) = \left[\sum_{n=1}^{N}p_n|n\rangle\right]\left[\sum_{m=1}^{N}p_m^*\langle m|\right].$$

In other words, the system starts from a pure state. Then we have

$$R_N(t) = \sum_{m,n=1}^{N}|p_m|^2|p_n|^2\exp(-A_{mn}^{(2)})\ .$$

Consider the behavior of $R_N(t)$. For $\omega_c\beta \gg 1$, the expression for $A_{mn}^{(2)}$ reduces to

$$A_{mn}^2 = \left[\frac{\eta}{2}\ln(1+\omega_k^2t^2) + \eta\ln\left[\frac{\sinh(t/\tau)}{t/\tau}\right]\right][f(E_m)-f(E_n)]^2\ ,$$

where $\tau=\beta/\pi$. Therefore, we obtain

$$R_N(t) = \sum_{m,n=1}^{N}|p_m|^2|p_n|^2(1+\omega_c^2t^2)^{-\eta[f(E_m)-f(E_n)]^2/2} \times\left[\frac{\sinh(t/\tau)}{t/\tau}\right]^{-\eta[f(E_m)-f(E_n)]^2}.$$

In the experimentally accessible domain of time, we have $\omega_c t \gg 1$. As a consequence, the loss of coherence at zero temperature is

$$R_N(t)|_{\beta\to\infty} = \sum_{m,n=1}^{N}|p_m|^2|p_n|^2(\omega_c t)^{-\eta[f(E_m)-f(E_n)]^2}\ . \qquad (9)$$

Then, the nondiagonal elements of the RDM evolve according to the power law. At finite temperatures, we obtain

$$R_N(t) = \sum_{m,n=1}^{N}|p_m|^2|p_n|^2(\omega_c\tau)^{-\eta[f(E_m)-f(E_n)]^2} \times\exp\{-t\eta[f(E_m)-f(E_n)]^2/\tau\}\ . \qquad (10)$$

Now, the relaxation of nondiagonal elements follow the exponential law. Compare Eq. (10) with the approximate result derived from the master equation in Ref. [10]:

$$R_N(t) = \sum_{m,n=1}^{N}|p_m|^2|p_n|^2\exp[-t\eta(E_m-E_n)^2/\tau]\ , \qquad (11)$$

which is written in the notation adopted here. [Note that the classical friction coefficient defined in [10], γ is equal to $\eta/2\pi$ and that $f(x)=x$]. It is clear that only if $\omega_c\tau=1$ the two formulas are identical. Thus Eq. (11) does not yield accurate results in general.

If the system is immersed in the TLS reservoir with nondemolition couplings, the total Hamiltonian has the form

$$\hat{H}_T=\hat{H}+\hat{H}_R+V_I \ , \tag{12}$$

where $\hat{H}$ is the same as before, but $\hat{H}_R=\sum_k \omega_k\sigma_{z_k}$ and $V_I=f(\hat{H})\sum_k c_k\sigma_{x_k}$. We still assume that $\rho(0)=\rho^S(0)\prod_k \rho^k(0)$. Note that the density matrix becomes

$$\begin{aligned}\rho(t)&=\exp(-i\hat{H}_T t)\rho(0)\exp(i\hat{H}_T t)\\&=\exp(-i\hat{H}t)\prod_k \exp[-i\hat{O}_k(H)t]\rho(0)\\&\quad\times\prod_k \exp[i\hat{O}_k(H)t]\exp(i\hat{H}t) \ ,\end{aligned} \tag{13}$$

where $\hat{O}_k(x)=\omega_k\sigma_{z_k}+f(x)c_k\sigma_{x_k}$. Then, the RDM element reads

$$\rho^S_{mn}(t)=e^{-i(E_m-E_n)t}\rho^S_{mn}(0)\prod_k \mathrm{Tr}_k W_k \ , \tag{14}$$

where $W_k=e^{-i\hat{O}(E_m)t}\rho^k(0)e^{i\hat{O}(E_n)}$. Making use of the identity

$$\exp[i(\xi\sigma_z+\mu\sigma_x)]=\cos\sqrt{\xi^2+\mu^2}+\frac{i\sin\sqrt{\xi^2+\mu^2}}{\sqrt{\xi^2+\mu^2}}(\xi\sigma_z+\mu\sigma_x) \ ,$$

we obtain

$$\begin{aligned}\mathrm{Tr}_k[e^{-i\hat{O}(E_m)t}\rho^k(0)e^{i\hat{O}(E_n)}]&=\cos[\omega'_k(E_m)t]\cos[\omega'_k(E_n)t]\\&+\frac{\sin[\omega'_k(E_m)t]\sin[\omega'_k(E_n)t]}{\omega'_k(E_m)\omega'_k(E_n)}\\&\quad\times[\omega_k^2+f(E_m)f(E_n)c_k^2] \ ,\end{aligned}$$

where $\omega'_k(x)=\sqrt{\omega_k^2+f^2(x)c_k^2}$. Obviously, $\mathrm{Tr}_k W_k$ is independent of temperature. Thus, the RDM elements, or the RDM, as a result, is independent of temperature. This observation has nothing to do with properties of the system, which represent the very difference between the harmonic-oscillator reservoir and the TLS reservoir. To get an explicit expression for the RDM element we should assume that the couplings are very weak, i.e., $c_k\approx 0$. Thus $\mathrm{Tr}_k W_k$ can be simplified as

$$\mathrm{Tr}_k W_k\approx 1-[f(E_m)-f(E_n)]^2\frac{c_k^2}{\omega_k^2}\sin^2\omega_k t \ .$$

Inserting into Eq. (14), we obtain

$$\begin{aligned}\rho^S_{mn}(t)&\approx e^{-i(E_m-E_n)t}\rho^S_{mn}(0)\\&\quad\times\prod_k\left\{1-[f(E_m)-f(E_n)]^2\frac{c_k^2}{\omega_k^2}\sin^2\omega_k t\right\}\\&=e^{-i(E_m-E_n)t}\rho^S_{mn}(0)\\&\quad\times(1+4\omega_c^2t^2)^{-\eta[f(E_m)-f(E_n)]^2/4} \ .\end{aligned} \tag{15}$$

Here we have used the Ohmic spectral density distribution in the derivation. The loss of coherence is according to power law.

We would like to give a remark to end the discussion. As shown above, a prominent consequence of the quantum-nondemolition couplings is that the diagonal elements in the RDM remain unchanged during evolution. That is, *the population distribution does not vary according to time.* (Note that the traditional (demolition) couplings will enforce the system to reach thermodynamic equilibrium (see, e.g., Ref. [12]). Then the population distribution is that of the canonical ensemble. In this sense we can call the traditional coupling the *canonical* one.

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[1] G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965).
[2] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987), and references therein.
[3] W. H. Zurek, Phys. Today **44**, (10), 36 (1991); Prog. Theor. Phys. **89**, 281 (1993), and references therein.
[4] A. O. Caldeira and A. J. Leggett, Ann. Phys. **149**, 374 (1983).
[5] A. O. Caldeira, A. H. C. Neto, and T. O. de Carvalho, Phys. Rev. B **48**, 13 974 (1993).
[6] Jiushu Shao (to be published).
[7] C. Caves, K. Thorne, R. Drever, V. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[8] G. Milburn and D. Walls, Phys. Rev. A **28**, 2646 (1983).
[9] R. O'Connell, C. Savage, and D. Walls, Ann. N. Y. Acad. Sci. **480**, 267 (1986).
[10] A. Tameshtit and J. E. Sipe, Phys. Rev. A **45**, 8280 (1992); **47**, 1697 (1993).
[11] C. Aslangul, N. Pottier, and D. Saint-James, J. Phys. (Paris) **46**, 2301 (1985).
[12] K. Blum, Density Matrix Theory and Applications (Plenum, New York, 1981).