# Discrete lattice effects on breathers in a spatially linear potential

David Cai, A.R. Bishop, and Niels Grønbech-Jensen
*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
(Received 17 August 1995)

In the presence of a spatially linear, time dependent potential, we study discrete lattice effects on a nonlinear Schrödinger breather in the form of a composite excitation comprising two soliton components. We obtain an exact breather solution by generalizing the Hirota method to include the external potential. The solution is a discrete generalization of the two-soliton continuum solution with the initial condition as a superposition of two identical solitons. Unlike the continuum breather in the presence of a static ramp, the discrete breather will break up into two spatially separate, coherent structures undergoing bounded individual motions. We show that this breakup is a general discrete effect for breathers in an external potential.

PACS number(s): 03.40.Kf, 63.20.Pw, 46.10.+z, 42.81.Dp

Discrete nonlinear Schrödinger (NLS) equations play a fundamental role in the study of lattice dynamics in fiber optics, condensed matter physics, biology, etc. [1] in that they capture the discrete nonlinear dynamics in the low amplitude limit and enable us to gain rich analytical insight into these dynamical systems by virtue of their simple mathematical structures. In this work, in an attempt to go beyond the single-soliton dynamics, we will study the dynamics of breathers comprised of two soliton components and discuss discrete effects on breathers in the presence of a spatially linear, time dependent external potential. To obtain an exact breather solution, we will invoke the Hirota method [2] and generalize it to systems with an external potential. We will compare the discrete breather with its continuum counterpart and point out qualitative differences in their respective dynamics.

First, we will briefly discuss the breather solution in the continuum NLS in the presence of a static ramp potential. The governing equation is

$$i\psi_t + \psi_{xx} + \left[V(x) + 2|\psi|^2\right]\psi = 0 \ , \tag{1}$$

where

$$V(x) = -\mathcal{E}x. \tag{2}$$

The following transformations [3],

$$\begin{aligned} \psi(x,t) &= \psi'(x',t')\exp\left[-i\mathcal{E}xt - \frac{1}{3}i\mathcal{E}^2t^3\right], \\ x &= x' - \mathcal{E}t'^2, \\ t &= t', \end{aligned} \tag{3}$$

bring Eq. (1) to

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0. \tag{4}$$

For the continuum NLS, starting with the initial condition,

$$\psi(x, t=0) = 2\eta\,\text{sech}(\eta x), \tag{5}$$

the solution is a breather with two solitons riding on top of each other with a coinciding center [4] (the two-pole solution of inverse scattering transform theory). Using the two-soliton solution from Ref. [4], which we shall refer to as the Satsuma-Yajima (SY) solution in the following, we obtain the two-soliton solution with the initial condition (5) in the presence of the linear potential (2),

$$\psi(x,t) = 4\exp\left[i\eta^2t - i\mathcal{E}xt - \frac{1}{3}i\mathcal{E}^2t^3\right]\frac{\cosh\left[3\eta\left(x+\mathcal{E}t^2\right)\right] + 3\exp(8i\eta^2t)\cosh\left[\eta\left(x+\mathcal{E}t^2\right)\right]}{\cosh\left[4\eta\left(x+\mathcal{E}t^2\right)\right] + 4\cosh\left[2\eta\left(x+\mathcal{E}t^2\right)\right] + 3\cos(8\eta^2t)}. \tag{6}$$

This is a breathing soliton with the same temporal evolution of the envelope as that of the SY solution [4]. However, the center of this breather translates in a parabolic trajectory in time. This is expected since the breather can be viewed as a particle with an internal degree of freedom and it will accelerate down a ramp just as a Newtonian particle will do.

Next, we discuss the discrete NLS dynamics. The Ablowitz-Ladik discretization [5] of the NLS in the presence of an external potential is

$$i\dot{\psi}_n = -(\psi_{n+1} + \psi_{n-1}) - (\psi_{n+1} + \psi_{n-1})|\psi_n|^2 + V_n\psi_n, \tag{7}$$

where $-2\psi_n$ in the finite difference Laplacian has been removed by a trivial gauge transformation. We will discuss a potential of the following form,

$$V_n(t) = \mathcal{E}(t)\,n, \tag{8}$$

where $\mathcal{E}(t)$ is any function of time. This potential corresponds to a time-dependent, spatially uniform force along the lattice. Equation (7) with this potential (8) is integrable and the general aspects of its dynamics have been studied by a time dependent spectral theory in the inverse scattering transform (IST) framework [6]. Here we study special solutions, namely, two-soliton solutions. As is well known, Hirota's method is very powerful in the study of multiple soliton solutions [2]. We will generalize the Hirota method to the case that includes the external potential $V_n$. To obtain a bilinear form of Eq. (7), we set

$$\psi_n = \frac{G_n}{F_n} \tag{9}$$

with $F_n$ being real, and separate Eq. (7) into the following two equations,

$$(iD_t + 2\cosh D_n - V_n)\, G_n \cdot F_n = 0, \tag{10}$$

$$(\cosh D_n - 1)\, F_n \cdot F_n = G_n G_n^*, \tag{11}$$

where the operators $D_n$ and $D_t$ satisfy

$$2\cosh D_n G_n \cdot F_n = G_{n+1}F_{n-1} + G_{n-1}F_{n+1}, \tag{12}$$

$$D_t^m G_n \cdot F_n = [(\partial_t - \partial_{t'})^m\, G_n(t)F_n(t')]\,|_{t=t'}. \tag{13}$$

For the asymptotic expansion:

$$F_n = 1 + \epsilon^2 F_{2,n} + \epsilon^4 F_{4,n} + \cdots, \tag{14}$$

$$G_n = \epsilon G_{1,n} + \epsilon^3 G_{3,n} + \cdots, \tag{15}$$

we derive from Eqs. (10) and (11)

$$(i\partial_t + \Delta + 2 - V_n)\, G_{1,n} = 0, \tag{16}$$

$$\Delta F_{2,n} = G_{1,n}G_{1,n}^*, \tag{17}$$

$$(i\partial_t + \Delta + 2 - V_n)\, G_{3,n} = -(iD_t + 2\cosh D_n - V_n)G_{1,n}\cdot F_{2,n}, \tag{18}$$

$$\Delta F_{4,n} + (\cosh D_n - 1)F_{2,n}\cdot F_{2,n} = G_{1,n}G_{3,n}^* + G_{1,n}^*G_{3,n}, \tag{19}$$

where $\Delta G_{1,n} = G_{1,n+1} + G_{1,n-1} - 2G_{1,n}$, etc. For the potential (8), after Fourier transforming Eq. (16) into $k$ space to arrive at

$$\frac{\partial}{\partial t}\tilde{G}_1 - \mathcal{E}(t)\frac{\partial}{\partial k}\tilde{G}_1 = 2i(\cos k)\tilde{G}_1, \tag{20}$$

where $\tilde{G}_1(k,t)$ is the Fourier transform of $G_{1,n}(t)$, and then applying the method of characteristics to Eq. (20), we obtain the general solution for $G_{1,n}$. This is any linear combination of the form

$$G_{1,n} \sim \exp\left\{ i[k_0 - E(t)]n + 2i\int_0^t \cos[k_0 - E(\tau)]d\tau \right\} \tag{21}$$

for any $k_0$. Here $E(t) = \int_0^t \mathcal{E}(\tau)\,d\tau$. Using a special solution of $G_{1,n}$, namely, a linear combination for $k_0 = -\beta i$ and $k_0 = -3\beta i$:

$$G_{1,n} = a\exp[\Theta(\beta)] + b\exp[\Theta(3\beta)]\,, \tag{22}$$

where

$$a = 4\cosh\beta\,\sinh\beta\exp(-\beta x_0), \tag{23}$$

$$b = 4\cosh\beta\,\sinh 3\beta\exp(-3\beta x_0), \tag{24}$$

with $x_0$ being an arbitrary real number, and

$$\Theta(\beta) = \beta n + 2v(t)\sinh\beta - iE(t)n + 2iu(t)\cosh\beta, \tag{25}$$

$$u(t) = \int_0^t \cos[E(\tau)]\,d\tau, \tag{26}$$

$$v(t) = \int_0^t \sin[E(\tau)]\,d\tau, \tag{27}$$

we can show that the asymptotic expansion truncates at $O(\epsilon^5)$. The algebra involved in the proof is rather lengthy. It suffices to comment that the choice of $G_{1,n}$ [Eq. (22)] is a key step to ensure this truncation and to achieve a specific initial condition [Eq. (29), see below] for the wave function $\psi_n$. In what follows, we will omit the intermediate steps and quote the final exact result for the wave function:

$$\psi_n = 2\sinh(2\beta)\exp[-iE(t)n + 2i\cosh\beta\, u(t)] \times \frac{\cosh[3\beta(n-x_3)] + (4\cosh^2\beta - 1)\exp[i\Omega u(t)]\cosh[\beta(n-x_1)]}{\cosh[4\beta(n-x_4)] + 4\cosh^2\beta\cosh[2\beta(n-x_2)] + (4\cosh^2\beta - 1)\cos[\Omega u(t)]}, \tag{28}$$

where $\Omega = 8\cosh\beta\sinh^2\beta$, and

$$x_1 = -\frac{2\sinh\beta}{\beta}v(t) + x_0,$$

$$x_2 = -\frac{\sinh(3\beta) - \sinh\beta}{\beta}v(t) + x_0,$$

$$x_3 = -\frac{2\sinh(3\beta)}{3\beta}v(t) + x_0,$$

$$x_4 = -\frac{2\cosh^2\beta\sinh\beta}{\beta}v(t) + x_0.$$

For $t = 0$, Eq. (28) is

$$\psi_n(t=0) = \sinh(2\beta)\,\mathrm{sech}\,[\beta(n-x_0)]\,. \tag{29}$$

It is evident that the above initial condition is the discrete counterpart of the initial condition (5) for the continuum case and Eq. (28) is a discrete generalization of the breather solution (6) in the presence of a spatially linear potential (8). This breather (28) is comprised of two poles at $\exp(-3\beta/2)$ and $\exp(-\beta/2)$ from the perspective of IST. Note that this lattice solution has a *continuous* translational symmetry on account of the arbitrariness of $x_0$. Furthermore, since Eq. (7) is invariant under the transformations

$$\begin{aligned} \psi_n &\longrightarrow (-1)^n\psi_n, \\ \mathcal{E}(t) &\longrightarrow -\mathcal{E}(-t), \\ t &\longrightarrow -t, \end{aligned} \tag{30}$$

for any odd function $\mathcal{E}(t)$ of time $t$ or the potential-free case, there are "staggered" breathers (i.e., deriving the parentage from the upper edge of the linear phonon zone rather than the center of the zone) whose solutions are obtained from Eq. (28) under the above transformations (30). These staggered breathers have initial conditions of the form

$$\psi_n(t=0) = (-1)^n \sinh 2\beta\,\mathrm{sech}[\beta(n-x_0)]. \tag{31}$$

The translational symmetry and the existence of staggered breather solutions are, of course, special properties of the Ablowitz-Ladik discretization [7].

For $\mathcal{E}(t)=0$, i.e., the potential-free case, the evolution of the discrete breather resembles that of the continuum SY solution [4]. An example of such a breather is shown in Fig. 1. Plotted here is the modulus of $\psi_n$. In addition to a carrier-wave frequency $\Omega_c = 2\cosh\beta$, the coherent structure has a breathing shape mode that has the frequency $\Omega_s = 8\cosh\beta\sinh^2\beta$. This breathing mode can be clearly seen in Fig. 1. Although the evolution of the envelope, i.e., $|\psi_n(t)|$, is periodic, the wave function $\psi_n(t)$ is, in general, *not* periodic in time since the frequencies $\Omega_s$ and $\Omega_c$ are generally incommensurate. However, since $\lim_{\beta\to 0}\Omega_s/(\Omega_c-2)=8$, the continuum breather is *always* a periodic solution. Note that here we used $\Omega_c-2$ rather than $\Omega_c$ because, for the continuum case, the gauge transformation mentioned above has to be invoked to restore the term $-2\psi_n$, giving rise to an additional phase $\exp(-2it)$. Currently, we are investigating the resonance issue of the system (7) when these two frequencies become commensurate.

For the general case, i.e., in the presence of an external field, the discrete breather (28) behaves quite differently from the continuum one. As shown in Fig. 2, where the potential is a static ramp, $V_n = \mathcal{E}_0\,n$, $\mathcal{E}_0$ being a constant, the breather no longer evolves as a single coherent entity as in the continuum case (6). Rather, it breaks up into two spatially separate, coherent structures, one with a stable envelope and the other with a time dependent envelope. Clearly, this is a discreteness effect and the separation of the two lumps becomes more prominent with the increase of the value of $\beta$. It can be shown that only under the conditions $\beta^2 \ll 1$ and $t \ll 1/\mathcal{E}_0$ can we recover the continuum solution (6) from Eq. (28) for the spatially linear, static potential. Of course, the evolution of the discrete breather is still periodic with a period, $2\pi/\mathcal{E}_0$, as all scattering data in the spatially linear, static potential are periodic with the period $2\pi/\mathcal{E}_0$ [6]. For other temporal dependences of $\mathcal{E}(t)$, e.g., $\mathcal{E}(t) = A\cos\omega t$, we also found the breakup of the breather into two lumps. From

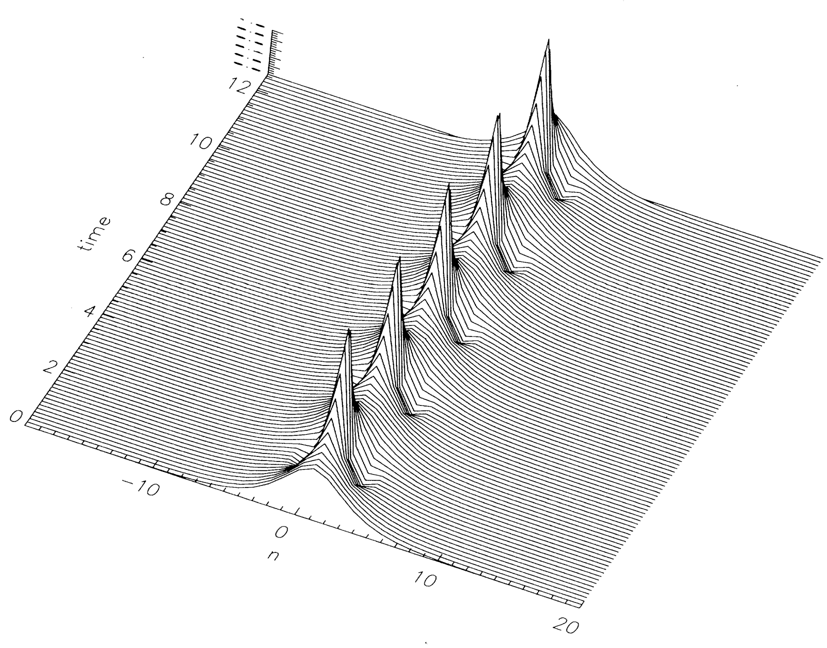


FIG. 1. Time evolution of a discrete breather (28) in the absence of an external potential. The initial condition is $\psi_n = \sinh(2\beta)\,\mathrm{sech}(\beta n)$, $\beta = 0.5$. Plotted here is $|\psi_n(t)|$.

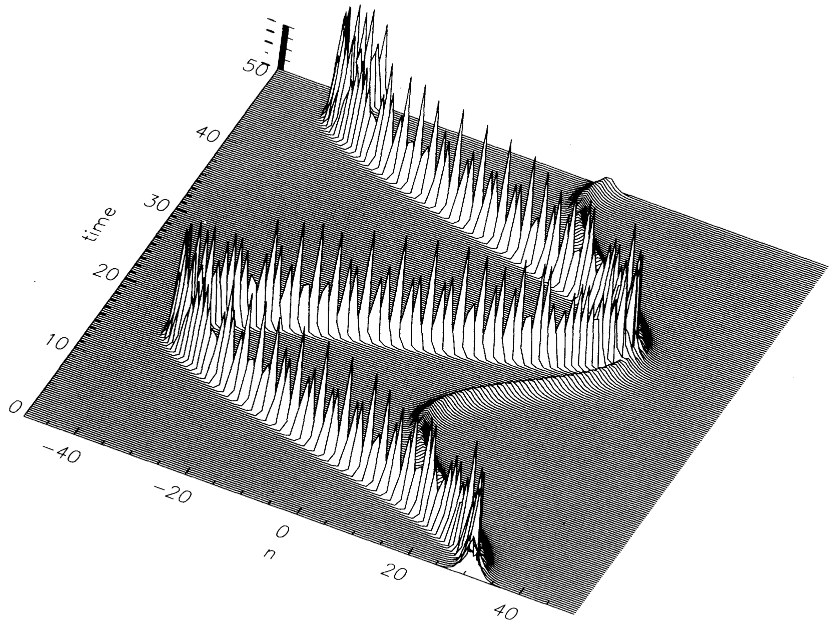


FIG. 2. Time evolution of a discrete breather (28) in the presence of a static linear potential $V_n = \mathcal{E}\,n$ with $\mathcal{E} = 0.2$. The initial wave function $\sinh(2\beta)\,\mathrm{sech}[\beta(n-x_0)]$, $\beta = 1$, evolves into two spatially separate, coherent structures with bounded periodic motions, cf. Fig. 1.

the structure of $x_i$, $i = 1, 2, 3, 4$ above, we can conclude that this breakup is a general discrete effect for this two-pole solution (28). Obviously, as expected, the motion is then no longer, in general, periodic. We point out that for a one-soliton solution, the envelope is always a hyperbolic secant for both the continuum and the discrete cases, and only the motion of the center is different. For instance, in the presence of the static, linear potential, the continuum one is unbounded and the discrete one is bounded and periodic [6,8].

In summary, we have generalized the Hirota method to include a spatially linear, time dependent external field. We have presented an exact breather solution, which is, in the potential-free case, a discrete generalization of the Satsuma-Yajima continuum solution starting with two identical solitons riding on each other with a coinciding center. Unlike the continuum breather in a spatially linear, static potential, which remains the same coherent structure as that in the potential-free case, but with its center executing a parabolic motion in time, the discrete breather will evolve into two spatially separate, coherent structures whose motions are individually bounded. We point out that this breakup is a general discrete effect on breathers in the presence of the external potential.

[1] For example, *Future Directions of Nonlinear Dynamics in Physical & Biological Systems,* special issue of Physica **68D** (1993), and also in Vol. 312 of *NATO ASI Series B*, edited by P.L. Christiansen, J.C. Eilbeck, and R.D. Parmentier (Plenum, New York, 1993).
[2] For example, R. Hirota, in *Solitons*, edited by R.K. Bullough and P.J. Caudrey (Springer-Verlag, New York, 1980).
[3] H. Chen and C. Liu, Phys. Rev. Lett. **37**, 693 (1976).
[4] J. Satsuma and N. Yajima, Suppl. Prog. Theor. Phys. **55**, 284 (1974).
[5] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, New York, 1991).
[6] D. Cai, A.R. Bishop, N. Grønbech-Jensen, and M. Salerno, Phys. Rev. Lett. **74**, 1186 (1995).
[7] D. Cai, A.R. Bishop, and N. Grønbech-Jensen, Phys. Rev. Lett. **72**, 591 (1994).
[8] R. Scharf and A.R. Bishop, Phys. Rev. A **43**, 6535 (1991).