Singularities and special soliton solutions of the cubic-quintic complex Ginzburg-Landau equation

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Soliton solutions of the one-dimensional (1D) complex Ginzburg-Landau equations (CGLE) are analyzed. We have developed a simple approach that applies equally to both the cubic and the quintic CGLE. This approach allows us to find an extensive list of soliton solutions of the CGLE, and to express all these solutions explicitly. In this way, we were able to classify them clearly. We have found and analyzed the class of solutions with fixed amplitude, revealed its singularities, and obtained a class of solitons with arbitrary amplitude, as well as some other special solutions. The stability of the solutions obtained is investigated numerically.

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I. INTRODUCTION

Many nonequilibrium phenomena, such as processes in lasers [1–3], binary fluid convection [4], phase transitions [5], and wave propagation in nonlinear optical fibers with gain and spectral filtering [6–8], can be described by the generalized complex Ginzburg-Landau equation (CGLE). We write it here in the form used in nonlinear optics,

$$i\psi_z+\tfrac{1}{2}\psi_{tt}+|\psi|^2\psi=i\delta\psi+i\epsilon|\psi|^2\psi+i\beta\psi_{tt}+i\mu|\psi|^4\psi-\nu|\psi|^4\psi, \qquad (1)$$

where t is the retarded time, z is the propagation distance, δ, β, ϵ, μ, and ν are real constants (we do not require them to be small), and ψ is a complex field. For the specific case of the optical fiber mentioned above, the physical meaning of these quantities is the following: ψ is the complex envelope of the electrical field, δ is the linear gain at the carrier frequency, β describes spectral filtering ($\beta>0$), ϵ accounts for nonlinear gain-absorption processes, μ represents a higher order correction to the nonlinear amplification-absorption, and ν is a higher order correction term to the nonlinear refractive index. Equation (1) has been written in such a way that if the right-hand side of it is set to zero we would obtain the standard nonlinear Schrödinger equation (NLSE).

If the coefficients δ, β, ϵ, and ν on the right-hand side are small and $\nu=0$, then solitonlike solutions of Eq. (1) can be studied by applying perturbative theory to the soliton solutions of the NLSE [9,10]. This approach, however, cannot give all the relevant properties of solitonlike pulses and the regions in the parameter space where they exist. Finding exact solutions is an important step for understanding the full range of properties of the complex CGLE, thus helping to predict the behavior resulting from an arbitrary initial condition. We consider both the cubic and the quintic CGLE, and derive all soliton solutions for both cases following the same procedure. In this way, we cover the solutions which were known before and obtain other solutions.

The case of the cubic CGLE has been studied extensively (see, e.g., [11–14]) and its general solution, i.e., pulse with fixed amplitude, is known. Nevertheless, we found that an important class of solitonlike solutions had been overlooked. As we investigate the solutions with fixed amplitude more carefully, we notice that it becomes singular at some values of the parameters, corresponding to special line on the (β,ϵ) plane. Although the solution with fixed amplitude does not apply in this case, a new class of solutions arises, namely, the class of arbitrary-amplitude solitons. We present an analytic expression for this class of solutions.

The case of the quintic CGLE has been considered in a number of publications using numerical simulations, perturbative analysis, and analytic solutions. Perturbative analysis of the solitons of the quintic CGLE in the NLSE limit has been developed by Malomed [10] and Hakim, Jakobsen, and Pomeau [15]. The existence of solitonlike solutions to the quintic CGLE in the case of subcritical bifurcations ($\epsilon>0$) has been shown numerically [16,17]. A qualitative analysis of the transformation of the regions of existence of the pulselike solutions when the coefficients on the right hand side change from zero to infinity has been done by Hakim, Jakobsen, and Pomeau [15]. An analytic approach based on the reduction of Eq. (1) to a three variable dynamical system which allows one to get exact solutions for the quintic equation has been developed by van Saarloos and Hohenberg [18,19], although solutions in explicit form have not been written.

The most comprehensive mathematical treatment of the exact solutions of the quintic CGLE using Painlevé analysis and symbolic computations is given in the recent work by Marcq, Chaté, and Conte [20]. The general approach, used in [20], is the reduction of the differential equation into a purely algebraic problem. However, the technique used in [20] assumes that analytical results can be obtained in a reasonable time only by using computers. More important, the final formulas for the pulselike solutions in [20] have parameters which are expressed implicitly through the coefficients of CGLE and still need some work to calculate the pulse shapes numerically. For this reason, the use of complicated technique and the aspiration to find all type of solutions (pulses, fronts, sources, and sinks) did not allow the authors of [20] to classify fully the pulselike solutions. In particular, the solutions with arbitrary amplitude, algebraic solutions, and flat-top pulses were missing in their analysis. Moreover, the range of existence and stability were not discussed, even briefly. The great diversity of possible types of solutions requires a careful analysis of each class of solutions separately. This is the reason we have concentrated our effort in this work only on pulselike solutions. In this way we are able to find in explicit form and classify all the solutions of this restricted class.

We propose a relatively simple method which allows us to obtain and classify the diversity of pulselike solutions described by our ansatz. Thus we obtain the class of solutions with fixed amplitude, then we reveal its singularities and isolate several special solutions, including a class of arbitrary-amplitude solitons, the family of flat-top solutions, a class of algebraic solutions, and the chirp-free solutions. Although these solutions do not cover the whole range of parameters due to the restrictions imposed by our ansatz they can serve as a basis for further generalizations.

Preliminary studies of the stability of our analytic solutions show that the majority of them are unstable relative to small perturbations. However, the whole class of solutions with arbitrary amplitude is stable in both cubic and quintic cases. This fact allows us to suppose that these solutions can have a variety of real applications. Another example of stable solitons is the class of flat-top pulses. We give in this work only a few numerical examples of stable propagation. A more detailed study on their stability will be presented elsewhere.

The paper is organized as follows. The general ansatz and the analytical procedure are described in Sec. II. Exact solutions of the cubic CGLE are described in Sec. III. The quintic CGLE solutions are obtained and analyzed in Sec. IV. We discuss the results obtained, with possible applications and generalizations, in Sec. V. Finally, we summarize in Sec. VI.

II. ANALYTICAL PROCEDURE

Let us consider first the stationary solutions of Eq. (1) with zero transverse velocity. This happens when $\beta \neq 0$. The case $\beta = 0$ is considered in a special section. Hence we look for a solution of the form

$$\psi(t,z) = A(t)\exp(-i\omega z), \tag{2}$$

where ω is a real constant. The complex function $A(t)$ can always be written in an explicit form as

$$A(t) = a(t)\exp[i\phi(t)], \tag{3}$$

where a and ϕ are real functions of t. By inserting Eqs. (2) and (3) into Eq. (1) and separating real and imaginary terms, we obtain

$$\begin{aligned}&(\omega - \tfrac{1}{2}\phi'^2 + \beta\phi'')a + 2\beta\phi' a' + \tfrac{1}{2}a'' + a^3 + \nu a^5 = 0,\\&(-\delta + \beta\phi'^2 + \tfrac{1}{2}\phi'')a + \phi' a' - \beta a'' - \epsilon a^3 - \mu a^5 = 0,\end{aligned} \tag{4}$$

where a prime stands for differentiation with respect to t.

Let us now assume that

$$\phi(t) = \phi_0 + d\ \ln[a(t)], \tag{5}$$

where d is the chirp parameter and ϕ_0 is an arbitrary phase. We suppose $\phi_0 = 0$ for simplicity. Equation (5) is, obviously, a restriction imposed on $\phi(t)$ because the chirp could have a more general functional dependence on t. However, this restriction allows us to find some families of solutions in analytical form. For the cubic case, our ansatz covers all pulselike solutions. In the quintic case the solutions reported in this paper are only those which can be represented in the form (3), (5). Equations (4) become then

$$\begin{aligned}&\omega a + \left(\frac{1}{2} + \beta d\right)a'' + \left(\beta d - \frac{d^2}{2}\right)\frac{a'^2}{a} + a^3 + \nu a^5 = 0,\\&-\delta a + \left(\frac{d}{2} - \beta\right)a'' + \left(\frac{d}{2} + \beta d^2\right)\frac{a'^2}{a} - \epsilon a^3 - \mu a^5 = 0.\end{aligned} \tag{6}$$

Now, we have two second order ordinary differential equations (ODE) relative to the same dependent variable, $a(t)$. To have a common solution, the two equations must be compatible. In general, this is not the case. However, for this particular system, they can be made compatible by a proper choice of the parameters.

To find the conditions of compatibility we apply the following procedure. We eliminate the first derivatives from the set of Eqs. (6) to get

$$\begin{aligned}&\frac{d}{4}(1+d^2)(1+4\beta^2)\frac{a''}{a} + \left(\frac{d}{2} + \beta d^2 + \epsilon\beta d - \frac{\epsilon d^2}{2}\right)a^2\\&\quad + \left[\nu\left(\frac{d}{2} + \beta d^2\right) + \mu\left(\beta d - \frac{d^2}{2}\right)\right]a^4 + \frac{\omega d}{2}(1+2\beta d)\\&\quad + \delta\left(\beta d - \frac{d^2}{2}\right) = 0.\end{aligned} \tag{7}$$

After integrating Eq. (7) we have

$$\begin{aligned}&\frac{d}{4}(1+d^2)(1+4\beta^2)\frac{a'^2}{a^2} + \frac{1}{2}\left(\frac{d}{2} + \beta d^2 + \epsilon\beta d - \frac{\epsilon d^2}{2}\right)a^2\\&\quad + \frac{1}{3}\left[\nu\left(\frac{d}{2} + \beta d^2\right) + \mu\left(\beta d - \frac{d^2}{2}\right)\right]a^4 + \frac{\omega d}{2}(1+2\beta d)\\&\quad + \delta\left(\beta d - \frac{d^2}{2}\right) = 0.\end{aligned} \tag{8}$$

The integration constant is zero for solutions decreasing to zero at infinity.

On the other hand, we can eliminate the second derivative from Eqs. (6), obtaining

$$\frac{d}{4}(1+d^2)(1+4\beta^2)\frac{a'^2}{a^2}+\left(\beta-\frac{d}{2}-\frac{\epsilon}{2}-\epsilon\beta d\right)a^2 -\left[\nu\left(\frac{d}{2}-\beta\right)+\mu\left(\beta d+\frac{1}{2}\right)\right]a^4 +\omega\beta-\frac{\omega d}{2}-\frac{\delta}{2}-\delta\beta d=0. \qquad (9)$$

These last two equations must coincide. Hence the following set of three algebraic equations must be satisfied:

$$\nu(4d+2\beta d^2-6\beta)+\mu(8\beta d-d^2+3)=0,$$
$$3d+2\beta d^2-4\beta+6\epsilon\beta d+2\epsilon-\epsilon d^2=0,$$
$$2\omega(d-\beta+\beta d^2)+\delta(1-d^2+4\beta d)=0. \qquad (10)$$

Equations (10) are the conditions of compatibility for Eqs. (6).

If both coefficients μ and ν are nonzero, then the first of these equations gives the relation between the four parameters ϵ, β, μ, and ν when the solution exists in the form (3), (5). The parameter d can be found from the second of Eqs. (10),

$$d=d_{\pm}=\frac{3(1+2\epsilon\beta)\pm\sqrt{9(1+2\epsilon\beta)^2+8(\epsilon-2\beta)^2}}{2(\epsilon-2\beta)}. \qquad (11)$$

This is an important result, which shows that (i) d can be found in terms of β and ϵ only, and (ii) the expression for d is the same for both the cubic and the quintic CGLE.

From the third equation in (10) we obtain for ω

$$\omega=-\frac{\delta(1-d^2+4\beta d)}{2(d-\beta+\beta d^2)}. \qquad (12)$$

Now taking into account Eqs. (10)–(12), and after some cumbersome transformations, we can rewrite Eq. (9) [or Eq. (8)]:

$$\frac{a'^2}{a^2}+\frac{2\nu}{8\beta d-d^2+3}a^4+\frac{2(2\beta-\epsilon)}{3d(1+4\beta^2)}a^2-\frac{\delta}{d-\beta+\beta d^2}=0. \qquad (13)$$

The coefficient in front of a^4 can equally be written in another way:

$$\frac{2\nu}{8\beta d-d^2+3}=\frac{\mu}{3\beta-2d-\beta d^2}. \qquad (14)$$

It is important to note that Eq. (13) is the consequence of the set (6) and its solutions are equivalent to the solutions of (6). Equation (13) is an elliptic equation and its solutions can be found relatively easily. The most important for us is the presentation of coefficients in Eq. (13). They are reduced to the simplest forms, which allows us to classify the solutions mainly in terms of ϵ and β.

In what follows, we consider the solitons of the cubic and the quintic CGLE separately. In each section we derive the analytical solution and then look for special cases and singularities.

An important issue is the stability of the exact solutions. As the system described by Eq. (1) is nonconservative, the stability can be analyzed only numerically. Such an analysis includes solution of the linearized problem, i.e., calculation of the perturbation eigenmodes and their growth rates. In this paper we are using the results of this analysis to present the region of parameters where the solutions are stable. In these cases we present the numerical results of the propagation of the exact solution with an added small perturbation. In the cases of solutions with arbitrary amplitude we carried out numerical solution of Eq. (1) with initial conditions in the form of exact solution with added small perturbation

$$\psi_0(t)=\psi_{\rm st}(t)+A\,\psi_{\rm pert}(t), \qquad (15)$$

where a small real constant A is the amplitude of the symmetric or antisymmetric perturbations, respectively. For simplicity, we chose as the perturbation function the solution itself, $\psi_{\rm st}(t)$, or its first derivative $\partial\psi_{\rm st}(t)/\partial t$. The exact perturbation function would necessarily grow out of one of these functions.

We conclude this section by analyzing the region of parameters, where we can expect stable pulses. The parameter β clearly must be positive, in order to stabilize the soliton in the frequency domain. We suppose $\delta\leq 0$ to provide the stability of the background. In this case, pulses can exist only for ϵ above the line given by Eq. (19) below. Finally, we choose $\mu<0$ to stabilize the pulse against the collapse.

III. SOLITONS OF THE CUBIC CGL EQUATION

A. Solitons with fixed amplitude

First we concentrate our efforts on the cubic CGLE, that is in Eq. (1) with $\nu=\mu=0$. Then Eq. (13) reduces to

$$\frac{a'^2}{a^2}+\frac{2(2\beta-\epsilon)}{3d(1+4\beta^2)}a^2-\frac{\delta}{d-\beta+\beta d^2}=0, \qquad (16)$$

which has the solution:

$$a(t)=BC\ \mathrm{sech}(Bt), \qquad (17)$$

where

$$C=\sqrt{\frac{3d(1+4\beta^2)}{2(2\beta-\epsilon)}},\quad B=\sqrt{\frac{\delta}{d-\beta+\beta d^2}}, \qquad (18)$$

and d is given by Eq. (11) after choosing the minus sign in front of the root. The second value of d leads to an unphysical solution, as the expression under the square root for C becomes negative. Solution (17) has been found by Pereira and Stenflo [12] (see also [11,13,14]). An important feature of the solution (17) is that its amplitude and width depend uniquely on the parameters of the equation. This is a common property of solutions in nonconservative systems. In other words, (17) is the solution with fixed amplitude.

To find the range of existence of the solution (17), note that on the plane (β,ϵ) the denominator in the expression for B is positive below the curve S given by

$$\epsilon_S=\beta\frac{3\sqrt{1+4\beta^2}-1}{4+18\beta^2}, \qquad (19)$$

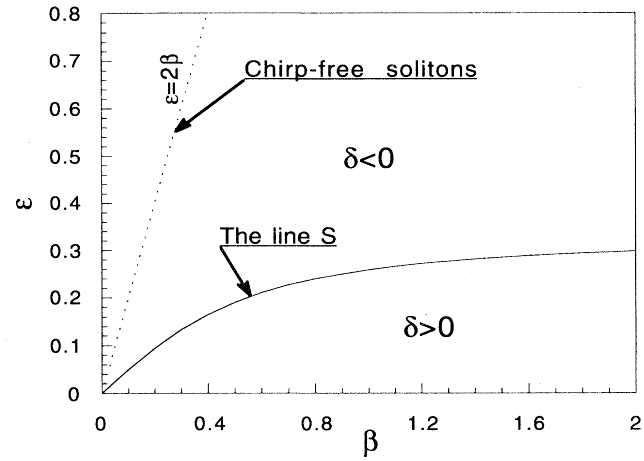

FIG. 1. Line (19) (the line S) on the plane ϵ,β where the solutions with fixed amplitude [(17),(35)] become singular and where the classes of special solutions with arbitrary amplitude [(21), (41)] exist. This plot applies for both the cubic and the quintic cases.

and negative above it (see Fig. 1). Hence, for the solution (17) to exist, the value δ must be positive below the curve S and negative above it. As this solution exists almost everywhere on the (β,ϵ) plane, we call it the general solution. The curve S itself is the line where this solution becomes singular, i.e., its amplitude BC tends to infinity, while the width $1/B$ vanishes.

To find the stability range of the solution (17), we recall from the perturbation theory that the solution is stable provided $\delta>0$ and $\epsilon<\beta/2$. However, the perturbation theory can be applied only for $|\delta|,|\beta|,|\epsilon|\ll 1$. On the other hand, the curve S separates the two regimes at any values of these parameters. It has two limits,

$$\epsilon\approx\beta/2 \text{ for } \beta\ll 1, \quad \epsilon\to 1/3 \text{ for } \beta\gg 1, \tag{20}$$

so at small β the curve S coincides with the stability threshold given by the perturbation theory. We suppose, using this observation, that the curve S separates the regions of stable and unstable solitons on the plane β,ϵ. Thus the solution (17) exists and is stable below the curve (19) for $\delta>0$. This conjecture has been checked in our numerical simulations. It is presented schematically in Fig. 1.

On the other hand, for positive linear amplification $(\delta>0)$, the background state $(\psi=0)$ becomes unstable. If the initial conditions are close to the exact solution (17) and $\delta\ll 1$, this instability develops slowly and the soliton can propagate distances up to $z_0\sim\delta^{-1}$. Beyond that, radiation waves growing linearly from the noise become appreciable and can distort the soliton itself. The distance z_0 can be large enough to observe soliton interactions [8]. This situation is of interest for soliton-based communication lines [6,7]. However, in other problems δ can be large, so z_0 is small. The general conclusion is that either the soliton itself or the background state is unstable at any point in the plane (ϵ,β). This means that the total solution is always unstable.

We have to emphasize the importance of the line S. For the solution with fixed amplitude it gives the range of existence, singularity, and stability. Moreover, as we will see in the next section, another significant class of solutions exists on this line.

B. Solution with arbitrary amplitude

It is easy to see that the solution (17) does not exist on the line (19). However, if we also impose the condition $\delta=0$, a new solution, valid only on the line (19), can be found:

$$a(t)=GF\,\mathrm{sech}(Gt), \tag{21}$$

where G is an arbitrary positive parameter, and d, ω, and F are given by

$$d=\frac{\sqrt{1+4\beta^2}-1}{2\beta}, \tag{22}$$

$$\omega=-\frac{(1+4\beta^2)(\sqrt{1+4\beta^2}-1)}{4\beta^2}G^2=-d\frac{1+4\beta^2}{2\beta}G^2, \tag{23}$$

$$F=\left(\frac{d\sqrt{1+4\beta^2}}{2\epsilon}\right)^{1/2}=\left[\frac{(2+9\beta^2)\sqrt{1+4\beta^2}(\sqrt{1+4\beta^2}-1)}{2\beta^2(3\sqrt{1+4\beta^2}-1)}\right]^{1/2}. \tag{24}$$

The solution (21) represents the arbitrary-amplitude soliton.

The reason for the existence of the arbitrary-amplitude solutions is that, when $\delta=0$, the cubic CGLE becomes invariant relative to the scaling transformation $\psi\to G\psi$, $t\to Gt$, $z\to G^2z$. Hence, if we know a particular solution of this equation, the whole family can be generated using this transformation. The singularity of the solution (17) and existence of the arbitrary-amplitude solutions were discovered in [21], although the analytical solution was not found. Note that all the parameters of the solution (21) (except G) and the coefficient ϵ are expressed in terms of β.

It can be shown that the arbitrary-amplitude solution (21) arises as the limit of the fixed-amplitude solution (17) for $\epsilon\to\epsilon_S$. To reveal this, we analyze the amplitude-width products, C for the solution with fixed amplitude and F for the arbitrary-amplitude solution. At the line of singularity S, the amplitude-width product C remains finite for the general solution (17) and has a finite limit on the line. This limit concides with the amplitude-width product F (see Fig. 2).

We have found, from numerical simulations, that the class of arbitrary-amplitude solutions is stable relative to small perturbations at any point of the line S. If we take an initial condition in the form of a superposition of the exact solution (21) and a small symmetric perturbation, eventually a stationary solution with some new value of G will be formed, and the shift in G will be proportional to the amplitude of the perturbation. For small β, the stationary solution (21) can be formed from the chirp-free initial condition $\psi_0(t)=\eta\,\mathrm{sech}(\eta t)$ as well. Figure 3 demonstrates the steady propagation of three well-separated solitons (21) with $G=0.75$, 1, and 1.5.

The most important feature of these solutions is that the background state $\psi=0$ is also stable because $\delta=0$. This

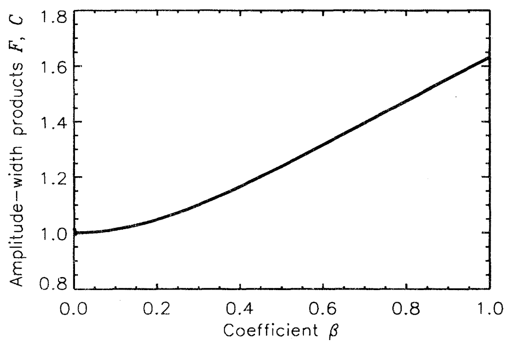

FIG. 2. Amplitude-width products C and F versus β for the solution with fixed amplitude of the cubic CGLE and for the solutions with arbitrary amplitude, respectively.

means that, by removing the linear gain from the system and applying a special relation between the coefficients ϵ and β, we can achieve the stable propagation of these solitons on the stable background. It is remarkable that this class of solutions is the only family of stable pulses in the cubic model.

C. Chirp-free soliton

Besides the singularity on the line (19), the solution (17) does not apply on the line $\epsilon=2\beta$, as C then becomes indeterminate ($d\rightarrow 0$ when $\epsilon\rightarrow 2\beta$). However, the soliton amplitude remains finite in the vicinity of this line on the (β,ϵ) plane for finite fixed δ. It follows from Eq. (11) that for $\epsilon=2\beta$ the chirp parameter $d=0$, and from Eq. (12) that $\omega=\delta/2\beta$. Equation (13) becomes

$$\frac{a'^2}{a^2}+a^2+\frac{\delta}{\beta}=0. \tag{25}$$

Its solution is

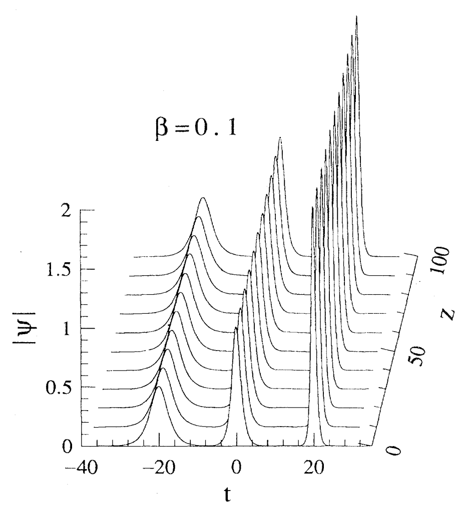

FIG. 3. Simultaneous propagation of three soliton solutions with different amplitudes of the cubic CGL.

$$a(t)=\sqrt{-\frac{\delta}{\beta}}\ \text{sech}\left(\sqrt{-\frac{\delta}{\beta}}\ t\right). \tag{26}$$

The coefficients δ and β must have opposite signs for this solution to exist. As $d=0$, the solution (26) does not have any phase chirp, in contrast to other soliton solutions of the CGL equation. This happens because of the special choice of the coefficients. In this case the complex constant $(1-i2\beta)$ can be factorized out from Eq. (1) when it is reduced to an ODE in terms of $a(t)$.

The pulse itself is unstable, as these solutions are located on the plane (ϵ,β) above the curve (19) (see Fig. 1). We have confirmed this instability with numerical simulations.

IV. SOLITONS OF THE QUINTIC CGL EQUATION

A. Relation between coefficients

The soliton solutions of the quintic CGLE exist for a wide range of values of the coefficients β, ϵ, μ, and ν. The ansatz (5) is the condition that restricts this range by imposing the relation [the first of Eqs. (10)] on them. Using Eq. (11), this relation can be rewritten as a linear equation in d:

$$\nu\left[\frac{12\epsilon\beta^2+4\epsilon-2\beta}{\epsilon-2\beta}d-2\beta\right]+\mu\left[\frac{2\epsilon\beta-16\beta^2-3}{\epsilon-2\beta}d+1\right]=0. \tag{27}$$

We can also eliminate d completely from the first two Eqs. (10) to obtain the following relation between the four coefficients β, ϵ, μ, and ν:

$$\frac{27(\mu-2\beta\nu)(1+2\epsilon\beta)^2}{(\epsilon-2\beta)^2}-\frac{32(\nu+2\beta\mu)^2}{(2\beta\nu-\mu)}-\frac{60(1+2\epsilon\beta)(\nu+2\beta\mu)}{\epsilon-2\beta}-\mu+2\beta\nu=0. \tag{28}$$

Solving (28) for ϵ, we obtain

$$\epsilon=\frac{4\beta\mu^2+30\mu\nu+120\beta^2\mu\nu+4\beta\nu^2\pm 3U}{-\mu^2+12\beta\mu\nu+32\nu^2+108\beta^2\nu^2}, \tag{29}$$

where

$$U=\sqrt{(\mu-2\beta\nu)^2(3\mu^2+16\beta^2\mu^2+4\beta\mu\nu+4\nu^2+12\beta^2\nu^2)}. \tag{30}$$

This expression is the relation between the coefficients in explicit form. Due to the existence of several branches, each of them must be analyzed separately. In contrast to the cubic equation, the general solution exists for both signs in the expression (11) for d. Equation (28) also applies to both cases. Hence four different cases have to be considered.

Now we consider zeroes of μ and ν in the (β,ϵ) plane which are results of the relation (29). If in the expression (11) for d we choose the negative sign ($d=d_-$), then μ has to be zero on the line [solid line in Fig. 4(a)]

$$\epsilon=\beta\frac{1-3\sqrt{1+3\beta^2}}{8+27\beta^2}. \tag{31}$$

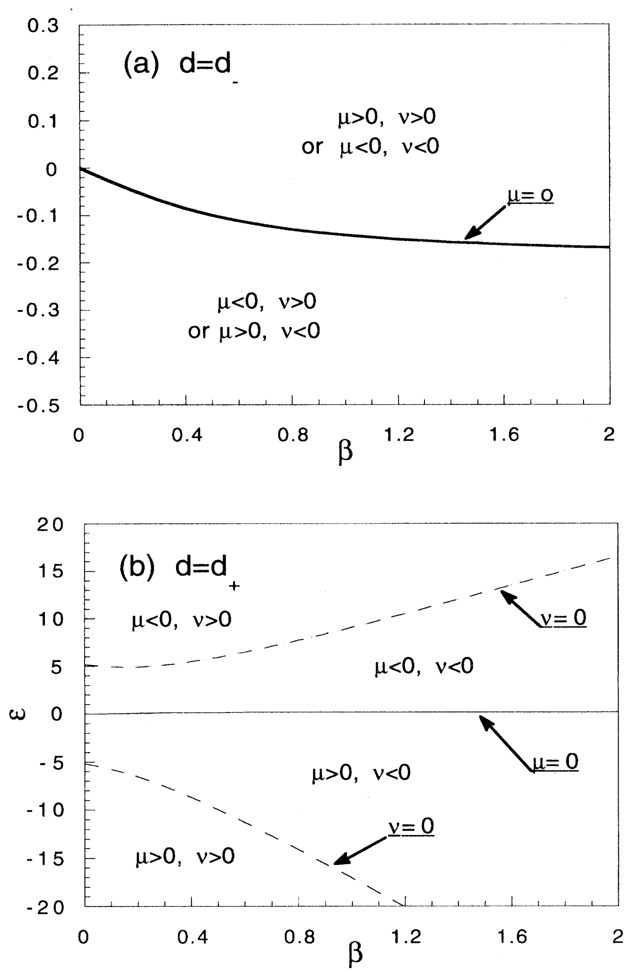

FIG. 4. Relation between the parameters μ and ν on the semiplane ϵ,β for which the quintic CGL has analytic solutions. (a) The case of negative sign in Eq. (11). (b) The case of positive sign in Eq. (11).

The values μ and ν have the same sign in the region above this line and opposite signs below it [see Fig. 4(a)].

If we choose the positive sign in the expression (11) for d ($d=d_+$), then μ becomes zero on the line [solid line in Fig. 4(b)]

$$\epsilon=\beta\frac{1+3\sqrt{1+3\beta^2}}{8+27\beta^2}, \tag{32}$$

and ν becomes zero on the two lines [dashed lines in Fig. 4(b)] defined by

$$\epsilon=\pm 3\sqrt{16\beta^2+3}-4\beta. \tag{33}$$

The value of ν changes sign on these lines [see Fig. 4(b)]. These conclusions can be made more specific when we consider regions of existence for solutions.

In what follows we consider solutions which exist when at least one of the coefficients μ or ν is nonzero, and express the solutions in terms of β, ϵ, and ν. Using Eq. (14), the solutions can alternatively be expressed in terms of β, ϵ, and μ.

B. Solutions with fixed amplitude

By using the substitution $f=a^2$ we can rewrite Eq. (13) in the form

$$\frac{f'^2}{f^2}+\frac{8\nu}{8\beta d-d^2+3}f^2+\frac{8(2\beta-\epsilon)}{3d(1+4\beta^2)}f-\frac{4\delta}{d-\beta+\beta d^2}=0. \tag{34}$$

This is again an elliptic-type differential equation. Bounded solitonlike solutions exist only if $4\delta/d-\beta+\beta d^2>0$. The positive solution of (34) is [22]

$$f(t)=\frac{2f_1f_2}{(f_1+f_2)-(f_1-f_2)\cosh(2\alpha\sqrt{f_1|f_2|}t)}, \tag{35}$$

where

$$\alpha=\sqrt{\left|\frac{2\nu}{8\beta d-d^2+3}\right|}=\sqrt{\left|\frac{\mu}{3\beta-2d-\beta d^2}\right|}, \tag{36}$$

and f_1 and f_2 are the roots of the equation:

$$\frac{2\nu}{8\beta d-d^2+3}f^2+\frac{2(2\beta-\epsilon)}{3d(1+4\beta^2)}f-\frac{\delta}{d-\beta+\beta d^2}=0, \tag{37}$$

namely,

$$f_{1,2}=\frac{-(2\beta-\epsilon)\pm\sqrt{(2\beta-\epsilon)^2+\dfrac{18\,\delta d^2\nu(1+4\beta^2)^2}{(8\beta d-d^2+3)(d-\beta+\beta d^2)}}}{6d\nu(1+4\beta^2)}(8\beta d-d^2+3). \tag{38}$$

On the line (33), this expression must be replaced by

$$f_{1,2}=\frac{-(2\beta-\epsilon)\pm\sqrt{(2\beta-\epsilon)^2+\dfrac{9\,\delta d^2\mu(1+4\beta^2)^2}{(3\beta-2d-\beta d^2)(d-\beta+\beta d^2)}}}{3d\mu(1+4\beta^2)}(3\beta-2d-\beta d^2). \tag{39}$$

We now discuss the conditions under which the soliton solution (35) exists. Clearly, one of the roots (we choose f_1) must be positive. The second one can have either sign. If it is also positive, we choose $f_1<f_2$. Two situations arise.

(1) $2\nu/8\beta d-d^2+3>0$. Then, f_1 is positive, f_2 is negative, but $(2\beta-\epsilon)/d$ can have either sign. Hence, both values of d are suitable. At any values of ϵ and at any $\beta>0$, the value $(8\beta d-d^2+3)$ is positive when we use $d=d_-$. If we use $d=d_+$, the value $(8\beta d-d^2+3)$ is negative in the area between the two dashed lines in Fig. 4(b) and positive otherwise. Therefore ν must be positive in the former case and its sign changes on the dashed lines as in Fig. 4(b) in the latter one.

(2) $2\nu/8\beta d-d^2+3<0$. Both roots f_1 and f_2 are positive, so $(2\beta-\epsilon)/d$ must be positive. Only $d=d_-$ satisfies this criterion. The value $(8\beta d-d^2+3)$ is always positive in this case, and ν can only be negative. This case is shown in Fig. 4(a).

In both cases the solution is defined by Eq. (35). The above analysis shows that, for a given set of parameters, ϵ, β, δ, and ν, in the area between the two dashed lines in Fig. 4(b), there are two solutions when ν is negative, but only one when ν is positive. Conversely, outside of this area, there are two solutions when ν is positive and only one when ν is negative. Besides, as $\delta/d-\beta+\beta d^2$ must always be positive, the restrictions on the sign of δ are the same as in the cubic case (see Fig. 1).

We can see that the solution (35) has two different branches for the same set of parameters. It can be shown that one of these branches coincides with the solution (24) of [20]. However, to show this, the solution (24) of [20] needs to be expressed explicitly in terms of the parameters of the CGLE. This task still needs some complicated algebraic transformations.

The solution (35) is unstable for arbitrary choice of parameters, i.e., for nonzero A in Eq. (15) the amplitude of the symmetric perturbation grows exponentially. However, in special cases of the flat-top soliton the perturbation does not grow.

C. Singularity at $\nu\rightarrow 0^-$

When ν is negative, one of the solutions has a singularity at $\nu\rightarrow 0^-$. The value $(2\beta-\epsilon)/d$ must be positive and finite. Then f_2 has the limit $3\delta d(1+4\beta^2)/2(d-\beta+\beta d^2)(2\beta-\epsilon)^2$ and f_1 goes to infinity as $(\epsilon-2\beta)(8\beta d-d^2+3)/3d\nu(1+4\beta^2]$ and so the soliton amplitude goes to infinity. The singularity does not occur when $\nu\rightarrow 0^+$. The second solution in the limit $\nu\rightarrow 0^-$ coincides with the solution (17) which applies in the case of the cubic CGLE. Clearly, this singularity is trivial and is not related to any new solution.

D. Solution with arbitrary amplitude

Another singularity appears at

$$d-\beta+\beta d^2=0. \tag{40}$$

This occurs on the same line S in the (β,ϵ) plane as in the cubic case [see Eq. (19)]. The singularity exists when the roots f_1 and f_2 have opposite signs. If β and ϵ satisfy (19) and we have $\delta=0$, the class of soliton solutions with arbitrary amplitude exists:

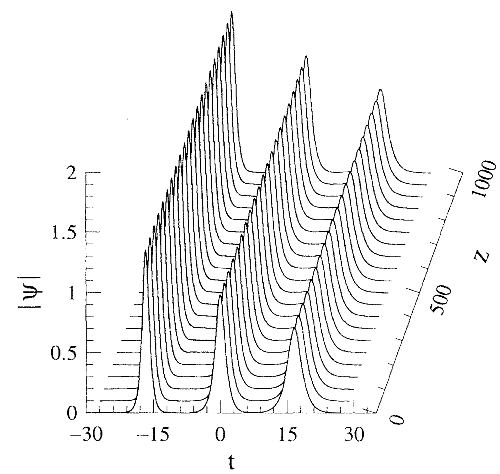

FIG. 5. Simultaneous propagation of three soliton solutions with different amplitudes of quintic CGL.

$$f(t)=\frac{3d(1+4\beta^2)P}{(2\beta-\epsilon)+S\cosh(2\sqrt{P}t)}, \tag{41}$$

where P is an arbitrary positive parameter and

$$S=\sqrt{(2\beta-\epsilon)^2+\frac{18\,d^2\nu(1+4\beta^2)^2}{(8\beta d-d^2+3)}P}. \tag{42}$$

The values d and ω are given by

$$d=\frac{\sqrt{1+4\beta^2}-1}{2\beta}, \tag{43}$$

$$\omega=-d\frac{1+4\beta^2}{2\beta}P. \tag{44}$$

We found from numerical simulations, that this class of solutions is stable at any point of the special line S and for any P in Eq. (41). The background state is also stable as $\delta=0$. Figure 5 shows the pulse profiles of three of these solutions, with $P=2$, 1, and 0.5, respectively, as they propagate along the medium. For this specific example we chose $\beta=\nu=0.1$. No changes on the profiles are observed after propagating a very long distance. These pulses are the only analytic solutions which are stable in the whole region of parameters where they exist.

E. Flat-top solitons

The soliton (35) becomes wider and flatter as the two positive roots approach each other. When $f_1=f_2$, the soliton splits into two fronts with zero velocity. Each of them can be written in the form (we ignore the translations along t)

$$f(t)=\frac{f_1}{1+\exp(\pm\alpha f_1 t)}, \tag{45}$$

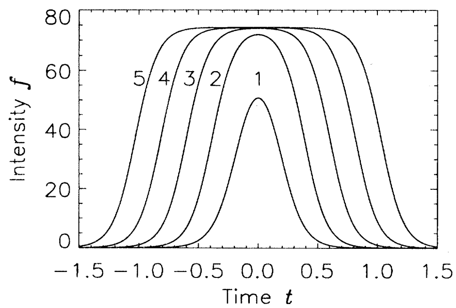

FIG. 6. Shapes of solutions (35) when the two roots f_1 and f_2 are close to each other. Separation into two fronts is the result of this proximity. Pulses marked 1, 2, 3, 4, 5 correspond to $m=1$, 3, 5, 7, 9, respectively.

where

$$f_1=\frac{(\epsilon-2\beta)(8\beta d-d^2+3)}{6d\nu(1+4\beta^2)}, \qquad (46)$$

and the sign in (45) determines the orientation of the front. The two roots f_1 and f_2 become identical when

$$(2\beta-\epsilon)^2=-\frac{18\,\delta\nu d^2(1+4\beta^2)^2}{(d-\beta+\beta d^2)(8\beta d-d^2+3)}. \qquad (47)$$

This condition involves all parameters of the equation. Depending on δ and ν it can exist at any point of the plane (ϵ,β).

The transition from general solution (35) to flat-top solution (45) for $f_1\to f_2$ is shown in Fig. 6. To plot this figure, we express $\delta=\delta_1$ from Eq. (47) and take $\delta=\delta_1(1-10^{-m})$, with m having the values 1, 3, 5, 7, and 9. The top of the soliton becomes flatter as the roots become close to each other.

If $f_1=f_2$ exactly, the width of the pulse goes to infinity and the pulse decomposes into two fronts. Note that in the region of nonzero intensity solution phase $\phi(t)$ tends to stationary value exponentially. So, if we combine the two fronts (45) with opposite orientation to form a wide, rectangular pulse of finite width, the influence of one front on another is exponentially small. In other words, the two fronts (45) can match each other without a domain boundary between them (cf. [19]).

Pulses and fronts have usually been considered as different solutions of the CGLE [18–20]. Our results show that they can be transformed to each other by changing parameters of the system. Moreover, our results give, at least partly, the range of parameters where we can expect smooth transition from solitons to fronts. Stable stationary flat-top pulses were observed experimentally in binary fluid convection [24].

F. Algebraic solution

If $\delta=0$ and (β,ϵ) is not located on the line (19), then $\omega=0$ and one of the roots of Eq. (37) (say f_2) becomes zero. The other root is

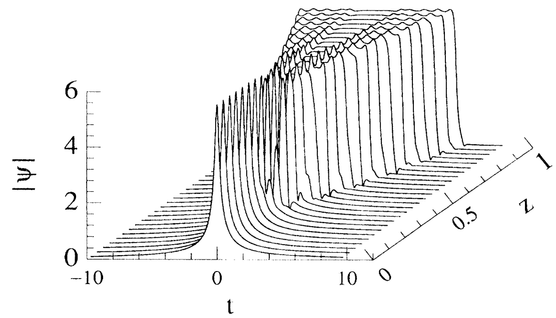

FIG. 7. Evolution of the algebraic soliton at $\beta=0.15$, $\nu=0$, and $\mu=-0.2$. The parameter d is chosen with plus sign in Eq. (11).

$$f_1=\frac{(\epsilon-2\beta)(8\beta d-d^2+3)}{3d\nu(1+4\beta^2)}. \qquad (48)$$

Equation (34) can then be written in the form

$$f'^2+4k_0[f-f_1]f^3=0, \qquad (49)$$

where $k_0=2\nu/(8\beta d-d^2+3)$. The solution to this equation is a Lorentz function:

$$f(t)=\frac{f_1}{1+k_0f_1^2t^2}. \qquad (50)$$

The values f_1 and k_0 must be positive, which restricts the allowed values of the coefficients of the equation for this solution to exist.

The algebraic soliton is unstable for the full range of the parameters where it exists. We have carefully studied the propagation dynamics of the corresponding solitons for $\nu=0$, and observed that it transforms into two fronts when μ is negative (Fig. 7).

The algebraic solution represents a special, weakly localized limit of the solution with fixed amplitude (35). Note that algebraic solitons exist and play an important role in other integrable and nonintegrable systems, including NLSE [22] and its generalizations [27].

G. Chirp-free soliton

Another degenerate case occurs when $\epsilon=2\beta$ and $\mu=2\beta\nu$. Equation (34) then reduces to

$$f'^2+\frac{8}{3}\nu f^4+4f^3+\frac{4\delta}{\beta}f^2=0. \qquad (51)$$

The solution to this equation is:

$$f(t)=\frac{-2\delta}{\beta\left[1-\sqrt{1-\frac{8\delta\nu}{3\beta}}\cosh\left(2\sqrt{-\frac{\delta}{\beta}}t\right)\right]}. \qquad (52)$$

Clearly, this solution exists when δ/β is negative and ν positive. The value $d=0$, and the solution to the CGLE does not have any phase chirp. This solution arises because the coefficients of the equation are chosen in such a way that a com-

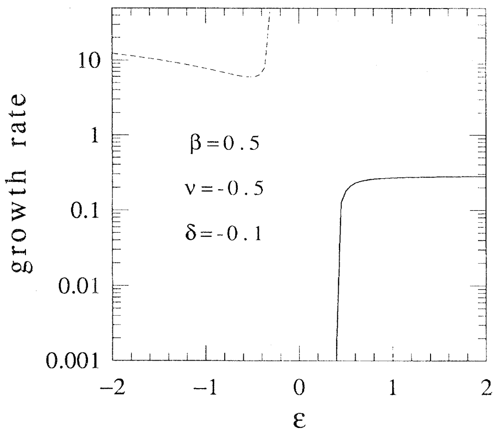

FIG. 8. Perturbation growth rate of the fixed-amplitude solutions for $\beta=0.5$, $\nu=-0.5$, $\delta=-0.1$. The solid line is for $d=d_-$, the dashed line is for $d=d_+$.

plex constant can be factored out from Eq. (1) when it is reduced to an ODE in terms of $f=a^2$. The ODE then becomes purely real.

Numerical simulations show that the chirp-free pulses given by our exact solutions are unstable at any values of parameters. Nevertheless, we may expect the existence of chirp-free pulses beyond the limitations given by Eq. (29). Chirp-free solutions are important in different applications. One of them is the problem of getting the chirp-free pulse at the output of an optical laser. Such a pulse could be used in transmission lines without additional transformations. While the pulse amplitude can be relatively easily adjusted by changing the values of amplification or damping δ, controlling the chirp is a much more difficult task and needs the knowledge of other system parameters. Our analysis shows, at least partly, the range of these parameters where we can obtain chirp-free pulses.

H. Traveling pulses

If $\beta=0$ then solitons with nonzero velocity (traveling solitons) become possible. These solutions can be obtained using simple transformation because for $\beta=0$ Eq. (1) has an additional symmetry, namely, it is invariant relative to the Galilean transformation. As a result, traveling pulselike solutions can be obtained from zero-velocity ones using this transformation

$$\psi'(z,t)=\psi(z,t-vz)\exp\left(ivt-i\frac{v^2}{2}z\right). \qquad (53)$$

Hence we can use the fixed amplitude solution of Sec. IV B, set $\beta=0$, and use the transformation (53) to get the whole family of travelling pulses. Note, that all the analysis of the Sec. B is valid in this case. The critical points in ϵ are intersections of special lines in Figs. 1 and 4 with the vertical axis $\beta=0$. This last example completes the classification of possible pulselike analytic solutions for Eq. (1).

I. Stability of solutions with fixed amplitude

We studied the stability of the fixed-amplitude solution (35) numerically, using the propagation approach described

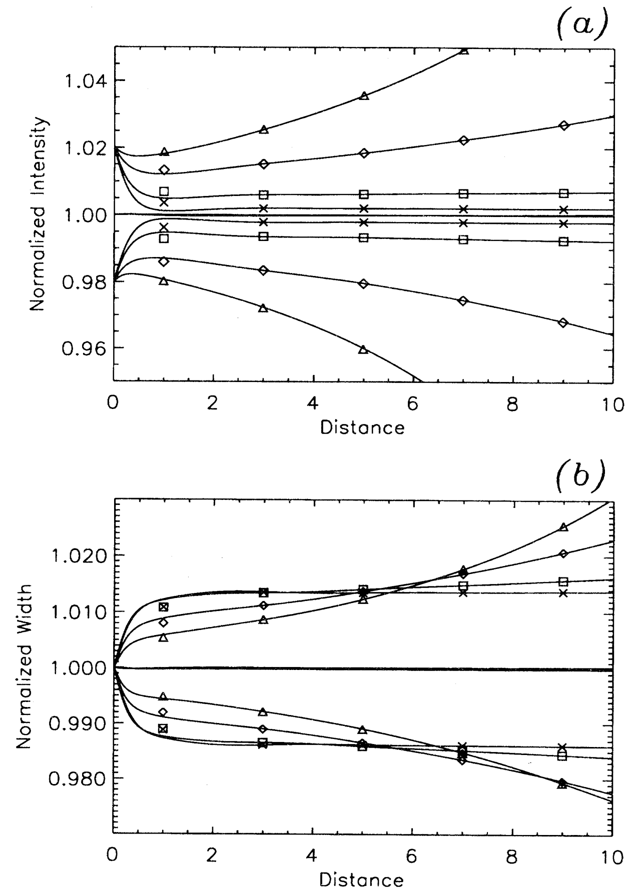

FIG. 9. (a) Evolution of the amplitude and (b) the width of the perturbed soliton with fixed amplitude for different values of f_1 and f_2: $\delta=-0.2365$, $\beta=0.2$, $\mu=-0.036\,13$ ($f_1=4.996$, $f_2=9.618$) (triangles); $f_1=5.673$, $f_2=8.941$, $\delta=-0.2496$ (diamonds); $f_1=6.576$, $f_2=8.038$, $\delta=-0.2602$ (squares); $f_1=7.076$, $f_2=7.538$, $\delta=-0.2625$ (crosses).

above [Eq. (15)] and the linearization technique (see, e.g., Ref. [28]). We found that the solution (35) is generally unstable, as the real part of the eigenvalue of the corresponding linear problem (i.e., the growth rate) is positive. However, the growth rate drastically decreases in the vicinity of $f_1=f_2$. Figure 8 shows the growth rate calculated for some particular values of δ, β, and ν, as ϵ and μ were varied. A solution corresponding to $d=d_+$ exists only for negative ϵ and it is always unstable. A solution for $d=d_-$ exists for positive ϵ and negative μ, i.e., in the region where stable propagation is possible. The growth rate for this a solution is essentially smaller and vanishes for $\epsilon\to\epsilon_*=0.4006$. At this point $f_1=f_2$ and solution does not exist for $\epsilon<\epsilon_*$.

To study the development of the instability in the vicinity of $f_1=f_2$, we applied the propagation approach. Our calculations show that if f_1 and f_2 differ more than twice, the solution exhibits typical unstable behavior, i.e., perturbation of the pulse amplitude grows exponentially (see Fig. 9, the curves with triangles and diamonds). Initial perturbation of the pulse amplitude affects the pulse width, which experiences the same changes, i.e., if the amplitude grows, the

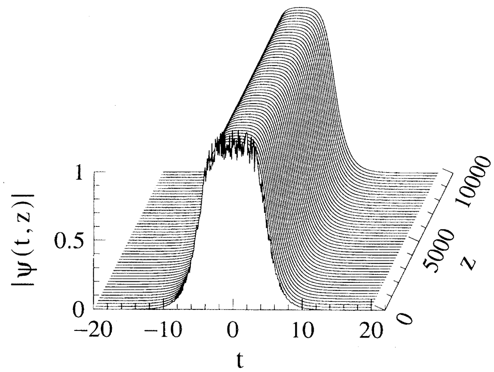

FIG. 10. Stable propagation of a flat-top soliton. The parameters chosen for this simulation are $\beta=0.5$, $\epsilon=0.4006$, $\nu=-0.5$, $\delta=-0.1$, and $d=d_-$, which gives $\mu=-0.227$. The perturbation added initially to the stationary solution is a uniform random function.

width also grows. However, as we choose f_1 and f_2 closer to each other, the dynamics of the perturbed solution changes drastically. First, the pulse amplitude initially evolves toward the stationary value and only then does the perturbation begin to grow, because the perturbation is not an exact eigenmode of the linearized problem. Secondly, the growth rate is significantly reduced, altough the pulse is still unstable (Fig. 9, curves with squares and crosses). At the same time, we notice that the initial perturbation of the pulse amplitude affects the pulse width in such a way that after the perturbation the pulse width acquires new value, and this change of the pulse width depends on the value of the amplitude perturbation. Such pulses can propagate a significant distance without visible changes (Fig. 10). This shows, that the region of existence of zero-velocity fronts is really wider than is given by our analytic expression (47).

V. DISCUSSION

In the above analysis we developed a technique which allowed us to find pulselike solutions of the cubic and the quintic CGLE following the same procedure. A particular ansatz for the solution is assumed, which reduces the initial partial differential equaiton (PDE) into a single elliptic ODE. The technique is simple and allows us to find general as well as special classes of soliton solutions of both the cubic and the quintic CGLE. All solutions are written in explicit form in terms of the parameters of the CGLE. This permits us to make clear classifications and to find the regions in the parameter space where the particular classes of solutions exist. We revealed two different classes of solutions: solutions with fixed amplitude and solutions with arbitrary amplitude. We also found, in analytic form, special lines in the parameter space, where the solutions with fixed amplitude become singular.

For the cubic case, our solutions cover all possible pulses including solitons with fixed amplitude and a class of arbitrary-amplitude solitons. In the quintic case, the solution can be explicitly written in a subspace of the full space of the coefficients of a lower dimensionality. This is a consequence of using the ansatz (5). This subspace is described by Eq. (28).

Most importantly, we discovered stable classes of solitons for both cubic and quintic cases. These are the above-mentioned arbitrary-amplitude solitons which exist on special lines in the parameter space where solutions with fixed amplitude become singular. They can propagate long distances without any changes. In the case of the cubic CGLE this class is the only stable class among all the stationary pulses. In the case of the quintic equation the class of arbitrary-amplitude solitons is also stable. Moreover, we found one more example of stable propagation of flat-top solitons at certain values of parameters. Other examples of stable soliton propagation also exist as found in numerical simulations [16] but they are beyond the set of solutions described by our ansatz.

Our approach allows several possible generalizations. For instance, the coefficients on the left-hand side of Eq. (1) also can be varied. In particular, the same type of analysis can be done when the factor 1/2 in front of the second time derivative becomes negative (which for the optical fiber case would mean that we are moving to the normal dispersion region). Possibly, the most important target for the generalization is the ansatz (5). Potentially, the inclusion of one more parameter in this ansatz may allow one to find a more general solution of the quintic CGLE. In this sense, our solutions can serve as the basis for this expansion. Besides, the perturbation theory can be developed, such that our set of solutions can be used as unperturbed objects, rather than the solutions of the NLSE. This is one of the ways to step out beyond the limitations of Eq. (29).

It may be interesting to compare our solutions with the results obtained from the perturbation theory. The classic adiabatic soliton perturbation theory, taking the solutions of the NLSE, as the unperturbed set, has been commonly used for consideration of the complex Ginzburg-Landau equation [10,16]. To understand the relation between our results and results of the perturbation theory (PT), we note that the standard PT have some major limitations. It can be applied only if the coefficients in the right-hand side of Eq. (1) are small. It does not give the value of the pulse chirp. Moreover, Eq. (1) has pulse solutions for positive dispersion as well (minus sign in front of the second derivative). Our method allows us to obtain these solutions, although we did not consider this case in this paper. On the other hand, the standard PT cannot be applied for the positive dispersion case because the NLSE itself then does not have bright-soliton solutions. Despite this, however, the PT gives some valuable results for both cubic and quintic cases.

For the cubic case, PT gives the limit for the special line S at small β. The PT correctly predicts that for $\delta>0$ the solution with fixed amplitude exists below this line and it is stable, while for $\delta<0$ the solution exists above this line but it is unstable. At the line itself the solution does not exist as its amplitude becomes infinite. Moreover, the PT would allow us to predict the arbitrary-amplitude pulses because their existence follows from the scaling transformation. However, the chirp-free pulses cannot be predicted in this way.

In the quintic case the PT in the form in which it has been developed before [10,15] has an even more limited range of applications, as it allows us to take into account only dissipative effects. The special solutions, including the class of arbitrary-amplitude pulses, flat-top pulses, and algebraic pulses, have not been predicted at all. On the other hand, our solutions could not be compared with the results of the standard PT because the parameters in the right-hand-side of the the CGLE cannot be reduced to zero simultaneously. If we take $\nu=0$, for example, the parameter μ must be chosen higher than $3\sqrt{3}$.

Let us turn now to possible applications of our analysis. The results of our work can be applied to different physical problems. The CGLE appeared first in the theory of phase transitions. Later it has been studied in plasma theory [12] and traditionally has been used to describe binary fluid convection [4,24]. These days, however, the most promising area where the solutions of the CGLE can give a new view of the problem is optical telecommunications and laser physics [23]. Consequently, we discuss here only the use of new solutions in this important area, leaving aside other possible applications.

It is known that the cubic CGLE is a good model for describing the optical transmission systems with guiding filters. The use of the nonlinear gain ($\epsilon>0$) in these systems allows the reduction or suppression of the growth of linear radiation. Our results show, in particular, that stronger spectral filtering $\beta\sim 1$ can be used in these systems than has been considered before, $\beta\ll 1$. In this case Eq. (19), derived here, gives the instability threshold. Our results show also that by removing the linear gain from the system we can achieve stable propagation of both the soliton and the background. This possibility has not been discussed before.

The existence of singularities shows a simple and effective way to control the pulse parameters (say, in lasers) by small variations in the "material parameters." The existence of solutions with arbitrary amplitudes can be used to switch the system from a "rigid regime" with fixed-amplitude solitons to a "soft" one with solitons having variable parameters. This can be done just by changing the parameters of the system (the relation between the linear and nonlinear gain, for example).

Note that the very range of parameters where the special solutions exist is of great importance and many experimental and numerical observations were performed in this range, even if this fact was not explicitly realized. We illustrate this using the example of arbitrary-amplitude pulses. Indeed, for the systems described by the cubic CGLE (e.g., soliton transmission systems), the main limitation is due to the growth of linear radiation (instability of the background state). Emission of the spontaneous noise by amplifiers also contributes to this effect. So, it is desirable to reduce this instability, and, in order to do this, keep the excess linear gain δ as low as possible. At the same time, the soliton collapse (which occurs if $\epsilon>\epsilon_S$) also should be avoided. So the optimal regime lies near the curve S, where the nonlinear gain and the spectral filtering balance each other so significant contribution from the linear gain is not necessary.

Another example of the soliton fiber system which has the working regime near the curve S is the soliton fiber laser with saturable absorption [25,26]. Such a laser is described by the cubic CGLE, but the difference from Eq. (1) is that the linear amplification coefficient δ depends on the total pulse energy E. In the mode-locked regime, δ has small negative value to stabilize the background state. If the pulse energy increases, the absorption also increases and vice versa. Clearly, stability of such a system depends on the slope of $\delta(E)$ dependence. However, to provide the self-start of the laser, δ should have small positive value for E much smaller than the energy of the stationary pulse. So, again the optimal regime lies near the S curve, where the nonlinear gain and spectral filtering compensate for each other and the absolute value of δ can be kept small.

The system which is described by the quintic CGLE is, for example, the soliton fiber laser with fast saturable absorption [21]. In this case the soliton is supported by nonlinear gain and loses energy due to three effects: spectral filtering, linear losses, and the quintic stabilizing term. However, even small linear loss is enough to keep the background state stable. So, the stationary state exists basically as the result of balance between nonlinear gain, spectral filtering, and the quintic stabilizing term; this proves the importance of the study of the arbitrary-amplitude pulses in the quintic model.

VI. CONCLUSION

In conclusion, we developed a simple technique which allows us to find pulselike solutions of both the cubic and quintic CGLE, using the same procedure. For the cubic CGLE, we have revealed the singularities of the fixed-amplitude solutions, and found arbitrary-amplitude and the chirp-free solutions. For the quintic CGLE, we have obtained a class of fixed-amplitude solutions, studied its singularities, and found several special cases. Among them are the class of arbitrary-amplitude pulses, chirp-free pulses, the flat-top solution, and others. We have investigated the stability of these solutions by direct numerical simulations and found that arbitrary-amplitude solitons are stable in the whole range of parameters where they exist.

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