# Nonlinear inverse bremsstrahlung and highly anisotropic electron distributions

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(Received 5 April 1995)

A procedure is proposed to deal with the approximate solution of the kinetic equation for the velocity distribution function of electrons in a fully ionized plasma in the presence of strong, high frequency radiation. The Legendre polynomial expansion is applied after the kinetic equation has been written in an oscillating frame, where some directions are appropriately scaled, with the aim of making approximately isotropic, on the average, distributions that are otherwise anisotropic. The equations are derived for the isotropic part of the electron distribution in the scaled frame and for the scaling factor. The procedure is meant to display its potential in cases where the electron distribution is expected to be highly distorted by the external field. As a case study, a high-$Z$ plasma, ignoring electron-electron collisions, and a linearly polarized laser field are considered. Within the proposed procedure, the existence of self-similar solutions is addressed as well. For small anisotropy, we recover and improve upon a known result. For large anisotropy, a completely different self-similar solution is found, allowing, in a simple way, the study of many plasma properties under conditions in which the laser field strongly alters the electron velocities. In the unscaled laboratory frame any part of the electron distribution is oscillating, anisotropic, and evolving in shape.

PACS number(s): 52.40.−w, 52.50.Jm

## I. INTRODUCTION

The description of physical processes and phenomena occurring in an ionized medium or in a plasma in the presence of intense laser fields requires, as a rule, knowledge of the electron velocity distribution function (EDF). When the laser and medium parameters are such that the field-free EDF is not strongly altered by the presence of the field, the Boltzmann equation is profitably solved using the well-known procedure based on the Legendre polynomial expansion (LPE) of the unknown solution. Such "small anisotropy" cases have been thoroughly investigated, and the results may be relied upon to study plasma properties and processes.

An important, qualitative criterion for successful practical application of the LPE is

$$\frac{v_e}{v_T} \ll 1 , \tag{1}$$

where $v_e = eE_0/m\omega$ is the peak velocity of the plasma electrons in the laser field, $\mathbf{E}=\mathbf{E}_0\cos\omega t$, and $v_T=\sqrt{T_e/m}$ is the electron thermal velocity. For literature on the LPE see [1–3] and references therein. In particular, several investigations have demonstrated and exploited the fact that, in the case of small anisotropy, non-Maxwellian, self-similar (SS) EDF may be formed [4–12]. Critical in all these investigations is the condition that electron-ion collisions dominate, and electron-electron ($e$-$e$) ones are negligible. A recent investigation [13], however, has extended the knowledge on this topic, showing that a SS EDF exists also when $e$-$e$ collisions cannot be neglected. The distinctive feature of the SS EDF in analytical form in Ref. [13] is that it changes smoothly, under the appropriate limits, over to the previous result, pertinent to the absence of $e$-$e$ collisions, and to the Maxwellian, established by randomizing $e$-$e$ collisions.

For large anisotropy, i.e., when one expects the field-electron interaction to distort significantly the EDF, few investigations are available and the results are still partial and sparse [7,11,14,15]. On the other hand, the increasing availability of powerful laser sources in research laboratories and plasma physics applications necessitates knowledge of the characteristics and properties of laser-embedded plasmas.

In a recent work [15] we have addressed numerically the EDF with large anisotropy for a fully ionized plasma embedded in a linearly polarized laser field. We have also studied the evolution of an initially isotropic EDF in two-dimensional velocity space. In particular, we have found that after some field periods (the actual number depending on the ratio $\nu_{ei}/\omega$, with $\nu_{ei}$ the electron-ion collision frequency and $\omega$ the laser frequency) the shape of the EDF becomes quasistationary in the reference frame oscillating with the plasma electrons. The EDF appears

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to be anisotropic and stretched out along the external field polarization direction. This result has been observed for up to 100 field periods. For larger times, whether the EDF evolves towards a larger anisotropy or, on the contrary, towards an isotropic shape remains an open question.

Exploiting information from numerical calculations [15], in the present paper we propose a procedure that allows fast convergence of the LPE also in the case of large anisotropy, thus opening the possibility of constructing analytical EDF for this case as well. The proposed procedure is presented in detail in Sec. IV. Here, the basic concept is formulated as follows. We write the kinetic equation in a scaled oscillating velocity reference frame. The scaling concerns the component of the electron velocity along the field direction (i.e., the parallel direction), and it is accomplished by introducing a time-dependent scaling factor to transform an anisotropic configuration as much as possible into an isotropic one. The frame oscillates with the external field frequency. In the scaled oscillating frame, conditions are imposed to force the EDF to take an approximately isotropic form. The hopefully small residual anisotropy is then handled by applying the usual scheme of LPE. In other words, we propose to use the LPE only after an appropriate transformation has brought the EDF into a space in which it appears largely isotropic. Eventually, after the scheme of the LPE has been worked out and an approximate EDF found, appropriate reverse transformations will restore the anisotropy embodied in the scaling factor: in the fixed laboratory frame we will finally have a time-dependent, oscillating, anisotropic EDF.

In Sec. II we provide the main equation for the interaction between a collisional plasma and an external, linearly polarized laser field. We consider the physical situation in the plasma in which inverse bremsstrahlung is the main absorption mechanism. In other words, we do not take into account processes like resonance absorption, for which the laser field inhomogeneity plays a crucial role, nor nonlinear effects like parametric instabilities. In Sec. III we briefly review the conventional procedure of LPE for small anisotropy. Section IV is devoted to the generalization of this procedure to the case of large anisotropy. In Sec. V we consider two model SS solutions of the resulting equations. Section VI contains short final remarks.

## II. THE KINETIC EQUATION

The evolution of the EDF in a uniform, collisional, fully ionized plasma can be described by the equation [1,16]

$$\frac{\partial f}{\partial t}+\frac{e\mathbf{E}_0}{m}\cdot\cos\omega t\frac{\partial f}{\partial v}=\frac{1}{2}\frac{\partial}{\partial v_n}\left[\nu_{\rm ei}(v)(v^2\delta_{nm}-v_nv_m)\frac{\partial f}{\partial v_m}\right], \quad (2)$$

where $e$ and $m$ are the electron charge and mass, $v_n$ is the $n$th component of the electron velocity, $\nu_{\rm ei}(v)=4\pi e^4 n_e Z\ln\Lambda/m^2v^3$, $n_e$ is the number of electrons per unit volume, $Z$ is the ion charge, $\ln\Lambda$ is the Coulomb logarithm, in the modified form suggested by Silin [16] for interactions between plasmas and fast oscillating fields, and $\delta_{nm}$ is the Kroneker delta symbol.

In Eq. (2) the $e$-$e$ collision term has not been included. In general, for sufficiently intense fields and/or high ionic charge $Z$, the $e$-$e$ collision term is negligible [4]. In such a case, the inequality

$$\left[\frac{v_e}{v_T}\right]^2 \gg \frac{3}{Z} \quad (3)$$

must hold. It is satisfied in the cases considered throughout this work [17]. Equation (2), based on general kinetic equations of particles ensembles, is derived, e.g., in [1, 18], where the conditions of its validity are discussed in detail. Generally, Eq. (2) is not solved directly. Provided the anisotropy is small, simplified equations are derived by expanding the unknown solution in spherical harmonics. For the solutions of such well-known simplified equations, see, e.g., [4,5,13].

To solve Eq. (2), it is useful to change to a velocity reference frame oscillating with the same frequency as the external field [5,8]. For the electron velocity one has

$$\mathbf{u}=\mathbf{v}-\mathbf{v}_e\sin\omega t\ . \quad (4)$$

In the coordinates $(u_\perp,u_z)$, with $u_z$ and $u_\perp$, respectively, the velocity component parallel and perpendicular to the direction of the external field polarization, Eq. (2) has the form

$$u_\perp\frac{\partial\varphi}{\partial t'}=\frac{\partial}{\partial u_\perp}\left[\frac{\delta u_\perp u_t}{(u_\perp^2+u_t^2)^{3/2}}\left[u_t\frac{\partial\varphi}{\partial u_\perp}-u_\perp\frac{\partial\varphi}{\partial u_z}\right]\right]+\frac{\partial}{\partial u_z}\left[\frac{\delta u_\perp^2}{(u_\perp^2+u_t^2)^{3/2}}\left[u_\perp\frac{\partial\varphi}{\partial u_z}-u_t\frac{\partial\varphi}{\partial u_\perp}\right]\right], \quad (5)$$

where $\varphi(\mathbf{u},t')=f(\mathbf{u}+\mathbf{v}\ \sin(t),t)$, $t'=t$. Equation (5) is dimensionless: the velocity components are in units of $v_e$, time is in units of $\omega^{-1}$, $u_t=u_z+\sin(t')$, and

$$\delta=\frac{\nu_{\rm ei}(v_e)}{2\omega}\ . \quad (6)$$

Explicitly Eq. (5) contains only the small parameter $\delta$, which expresses the strength of the laser-plasma interaction. Another important parameter is the ratio $v_e/v_T$, stemming from the initial conditions. In previous work [15], we solved Eq. (5) and investigated the time evolution of the resulting EDF. In the present work, taking advantage of the numerical information of Ref. [15], we aim to develop a systematic procedure able, in principle, to provide accurate analytical EDF in the domain of high anisotropy.

## III. SMALL ANISOTROPY

To clarify the connection of our approach to the usual procedure, we first briefly review the well-known method of expanding the unknown EDF into Legendre polynomials.

This method is most effective in the case of small anisotropy. We outline the basic steps, writing Eq. (2) in spherical coordinates in a fixed velocity frame:

$$v_x = v\sin\theta\cos\phi ,$$

$$v_y = v\sin\theta\sin\phi ,$$

$$v_z = v\cos\theta ,$$

$$\frac{\partial f}{\partial t} + \cos t\left[\cos\theta\frac{\partial f}{\partial v} - \frac{\sin\theta}{v}\frac{\partial f}{\partial \theta}\right] = \frac{\delta}{v^3\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial f}{\partial\theta}\right] . \quad (7)$$

In Eq. (7) dimensionless variables are also used. In a spherical frame, the nature of the collision operator (the right side of the equation) appears in its clearest form. It changes only the anisotropic part of the EDF, which is present because of the external field. When the field vanishes, any isotropic function can be taken as a solution. The Maxwellian EDF is formed only because of the *e-e* collisions.

The LPE for the EDF of Eq. (7) is written as

$$f(v,\theta,t) = f_0(v,t) + f_1(v,t)P_1(\cos\theta) + f_2(v,t)P_2(\cos\theta) + \cdots , \quad (8)$$

where the coefficients $f_0, f_1, f_2, \ldots$ are unknown. Substituting (8) into (7) and leaving only the first two terms under the assumption of small anisotropy, we have

$$\frac{\partial f_0}{\partial t} + \cos t\left[\frac{1}{3}\frac{\partial f_1}{\partial v} + \frac{2}{3v}f_1\right] = 0 , \quad (9a)$$

$$\frac{\partial f_1}{\partial f} + \cos t\frac{\partial f_0}{\partial v} + \frac{2\delta}{v^3}f_1 = 0 . \quad (9b)$$

This is the system of two equations for the isotropic and the leading anisotropic part of the EDF, $f_0$ and $f_1$. The next step is connected with some basic assumption about the properties of $f_0$ and $f_1$. If $f_0$ is a slowly oscillating function of time (within a field period), while $f_1$ is a rapidly oscillating one, it is possible to solve (9b) to get

$$f_1 = -\frac{\partial f_0}{\partial v}\left[\frac{\sin t}{1+\dfrac{4\delta^2}{v^6}} + \frac{2\delta}{v^3}\frac{\cos t}{1+\dfrac{4\delta^2}{v^6}}\right] . \quad (10)$$

Now, substituting (10) into (9a) and using the smallness of $\delta$ and removing fast oscillations by averaging over the field period, we get the equation for $f_0$, e.g., [4],

$$\frac{\partial f_0}{\partial t} = \frac{\delta}{3v^3}\frac{\partial^2 f_0}{\partial v^2} - \frac{\delta}{3v^4}\frac{\partial f_0}{\partial v} . \quad (11)$$

The fast convergence of the series (8) is crucial for the derivation of Eq. (11), which implies the inequality $f_0 \gg f_1$. The latter is equivalent to the condition $v_T \gg v_e$. Thus the domain of validity of Eq. (11) is restricted to moderately intense laser fields. The entire procedure is useful insofar as it can be truncated after the first few terms. In the present version, the procedure is of little use when $v_T \approx v_e$ or $v_T < v_e$.

## IV. LARGE ANISOTROPY

### A. Preliminary considerations

Here we show how the LPE can be extended to the large anisotropy domain by exploiting information from existing numerical calculations. The numerical solution of Eq. (2) in Ref. [15] has shown that, after a relatively short time interval of fast changes, the EDF averaged over the external field period acquires some regular shape in the oscillating frame. Specifically, the shape is stretched out along the field polarization direction and, of course, is far from that of a spherically symmetric function (see, for instance, Fig. 8 of Ref. [15]).

An appreciation that a quasistationary, anisotropic EDF has been established is given by Fig. 7 of Ref. [15], which shows the time evolution up to the first 100 field periods of the ratio $E_\perp(t)/E_\parallel(t)$, with $E_\perp$ and $E_\parallel$ being, respectively, the ensemble-averaged perpendicular and

TABLE I. Differences in the applicability (in the sense of fast convergence) of the Legendre polynomial expansion (LPE) in unscaled (A) and scaled (B) oscillating velocity frames in the cases of small and large anisotropy. Velocities are in units of $v_e$. When $a \approx 1$, $\langle u \rangle \approx \langle s \rangle \approx V_T$; when $a \gg 1$, $\langle s \rangle \approx v_e$, and $\langle u \rangle = a\langle s \rangle \gg v_e$.

| Anisotropy | Ensemble average velocity, $\langle u \rangle$ | Peak oscillatory velocity $v_e$ | Domain of expected values of $a$ | Ratio $\langle u \rangle / v_e$ | Applicability of LPE |
|---|---|---|---|---|---|
| | | (A) Unscaled oscillating frame | | | |
| small | $v_T$ | $1^1$ | | $\gg 1$ | yes |
| large | $v_T$ | 1 | | $\approx 1$ | no |
| | | (B) Scaled oscillating frame | | | |
| small | $a\langle s \rangle$ | 1 | $\approx 1$ | $\gg 1$ | yes |
| large | $a\langle s \rangle$ | 1 | $\gg 1$ | $\gg 1$ | yes |

parallel kinetic energy. This ratio, which is rigorously equal to 2 for an undistorted, initial Maxwellian EDF, undergoes significant changes during (approximately) the first 20 field cycles to become almost constant afterwards (but numerically smaller than 2).

The idea at the basis of the extension of the LPE into the large anisotropy domain is to perform a time-dependent transformation of the parallel (with respect to the field polarization) velocity scale in the oscillating frame, amounting to a contraction. An appropriate time-dependent transformation coefficient $a(t)$ is introduced. As a result, we work in a scaled oscillating frame, where the originally anisotropic, quasistationary EDF is expected to be squeezed into a distribution function with an isotropic bulk. The small residual anisotropy left after the transformation then may be treated following the standard procedure of LPE.

We point out that (i) the transformation coefficient $a(t)$ needs to be a function of time, because the anisotropy of the EDF, as a rule, changes with absorption of energy from the external field; (ii) the deviation of $a(t)$ from unity is a measure of the departure of the EDF from an isotropic shape; (iii) as in the usual LPE, to achieve fast convergence it is necessary that the terms accounting for the EDF anisotropy be much smaller than the isotropic term. However, it is not necessary that the electron oscillatory velocity $v_e$ be smaller than the thermal velocity $v_T$. The release of the constraint (1) for fast convergence of the LPE derives from the circumstance of working in a scaled oscillating frame (see also Table I).

### B. The equation for the isotropic part of the electron distribution in the scaled oscillating frame

To develop the procedure outlined above, we define a scaled, oscillating frame (see also Fig. 1):

$$s_\perp = u_\perp = s\sin\vartheta \,, \quad a(\tau)s_z = u_z = a(t)s\cos\vartheta \,, \quad \tau = t \,, \tag{12}$$

where $a(\tau)$ is an unknown time-dependent factor altering the length of $u_z$. Thus $s_z$ is the contracted parallel velocity component, while $u_z$ is the uncontracted counterpart.

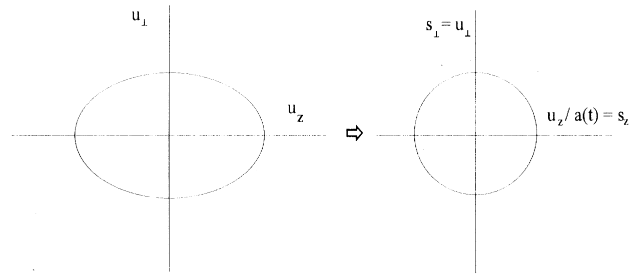


FIG. 1. Schematic of time-dependent scaling of the parallel electron velocity in the oscillating frame.

Considering the relations (12), or the similar relations between scaled moving and fixed coordinates,

$$s_\perp = \tilde{v}\sin\theta = s\sin\vartheta \,, \quad a(\tau)s_z = \tilde{v}\cos\theta - \sin\tau = a(t)s\cos\vartheta \,, \quad \tilde{v} = v/v_e \,, \tag{13}$$

Eq. (5), or Eq. (2), is transformed into

$$s\sin\vartheta\frac{\partial F}{\partial \tau} - s^2\cos^2\vartheta\sin\vartheta\left[\frac{\dot{a}}{a}\right]\frac{\partial F}{\partial s} + s\sin^2\vartheta\cos\vartheta\left[\frac{\dot{a}}{a}\right]\frac{\partial F}{\partial \vartheta} = I_{\rm coll}(F) \,, \tag{14}$$

where $F(\mathbf{s},t) = \varphi(\mathbf{u},t')$, $\dot{\mathbf{a}} = da/d\tau$, and $I_{\rm coll}(F)$ is the collision integral. In analogy with Eq. (8), we write the EDF as a series of Legendre polynomials,

$$F(s,\vartheta,\tau) = F_0(s,\tau) + F_1(s,\tau)P_1(\cos\vartheta) + F_2(s,\tau)P_2(\cos\vartheta) + \cdots \,. \tag{15}$$

Substituting (15) into (14) and integrating over the angle $\vartheta$, we get the equation for the isotropic part of the EDF in the scaled oscillating frame,

$$2s\frac{\partial F_0}{\partial \tau} - \frac{2}{3}s^2\left[\frac{\dot{a}}{a}\right]\frac{\partial F_0}{\partial s} = I^0_{\rm coll}(F_0) \,, \tag{16}$$

where the collision integral $I^0_{\rm coll}(F_0)$ is

$$I^0_{\rm coll}(F_0) = \delta\frac{\partial F_0}{\partial s}\left\{\sin^2\tau(J_0+J_2) + s\sin\tau\left[\left[2a-\frac{1}{a}\right](J_1+J_3) + \frac{1}{a}(J_1-J_3)\right] + \left[a-\frac{1}{a}\right]s^2\left[aJ_2+aJ_4-\frac{J_0-2J_2+J_4}{a}\right]\right\} + \delta\frac{\partial^2 F_0}{\partial s^2}\left\{\sin^2\tau s(J_0-J_2) + 2\sin\tau s^2\left[a-\frac{1}{a}\right](J_1-J_3) + \left[a-\frac{1}{a}\right]^2 s^3(J_2-J_4)\right\} \,. \tag{17}$$

In (17),

$$J_n=\int_0^{\pi}\frac{\sin\vartheta(\cos\vartheta)^n}{R^{3/2}}d\vartheta \quad (n=0,1,2,\ldots) \tag{18}$$

and $R=s^2\sin^2\vartheta+(as\cos\vartheta+\sin\tau)^2$. In the collision integral (17), $F_2$ and the next components of the EDF have been neglected because they are of the order of $\delta$, which is much smaller than 1. We note that we do not need to solve the equation for $F_1$ nor for $F_2$, because the collision integral $I^0_{\rm coll}(F_0)$ in the oscillating frame is not equal to zero and contains all the necessary information. On the contrary, the collision integral $I^0_{\rm coll}(f_0(v))$ in the fixed frame is identically zero as can be seen from (7) and (9a). Therefore, in this case, the term containing $f_1(v,t)$ must be included, as it results to be the leading one.

Now we demonstrate that Eq. (16) averaged over the field period turns into the well-known Eq. (11) in the limit of small anisotropy. In this case, the moving frame does not differ practically from the fixed one, as $v_e$ is much smaller than $v_T$. In the moving fame, the small anisotropy case is recovered under the limit

$$a\rightarrow 1\ , \tag{19}$$

and (16) and (17) give

$$2s\frac{\partial F_0}{\partial\tau}=\delta\frac{\partial F_0}{\partial s}[\sin^2\tau(J_0+J_2)+2s\sin\tau J_1] +\delta\frac{\partial^2F_0}{\partial s^2}\sin^2\tau s(J_0-J_2)\ . \tag{20}$$

The integrals (18) at $a=1$ give

$$J_0=\frac{2}{s^3}\ ,\quad J_1=-\frac{2\sin\tau}{s^4}\ ,\quad J_2=\frac{2}{3s^3}\ . \tag{21}$$

Therefore, from (20) we get

$$s\frac{\partial F_0}{\partial\tau}=-\delta\frac{\partial F_0}{\partial s}\frac{\sin^2\tau}{s^3}\left[\frac{2}{3}\right]+\delta\frac{\partial^2F_0}{\partial s^2}\frac{\sin^2\tau}{s^2}\left[\frac{2}{3}\right]\ . \tag{22}$$

Averaging Eq. (22) over the field period, and assuming that $F_0$ is a slowly oscillating function of time, we get Eq. (11). We note that, working in the oscillating frame, we have recovered Eq. (11), which has been derived in Sec. III in the fixed frame. In the case of small anisotropy, as observed, the moving frame does not differ substantially from the fixed one. This result therefore helps to remove the seeming contradiction between the parallel derivation of the same SS EDF given by Langdon [4] and Balescu [5] [see remark in Ref. [5] after Eq. (2.19)].

From the above derivation, we have learned that the equation for the isotropic part of the EDF, $F_0$, is obtained by averaging Eq. (20) over the field period. The integrals $J_n$ are expanded in power series of $\sin(\tau)/s$. For the integral with even $n$, terms up to the second order need to be retained, while for those with odd $n$, it is sufficient to keep the first nonzero term. Under the condition $a\times s>1$, we get

$$s^3J_0=\frac{2a}{a^2-\frac{\sin^2\tau}{s^2}}\approx\frac{2}{a}+\frac{2}{a^3}\left[\frac{\sin\tau}{s}\right]^2+\cdots\ ,$$
$$s^3J_1=\frac{2\sin\tau/s}{a^2-\frac{\sin^2\tau}{s^2}}\approx-\frac{2}{a^2}\frac{\sin\tau}{s}+\cdots\ , \tag{23}$$

and so on. Substituting (23) into (16) and (17) and averaging over the field period, we get

$$s\frac{\partial F_0}{\partial\tau}-\tfrac{1}{3}s^2\frac{\dot a}{a}\frac{\partial F_0}{\partial s}=\frac{\delta}{s^3}\frac{\partial F_0}{\partial s}[-c_1(a)+c_2(a)s^2] +\frac{\delta}{s^2}\frac{\partial^2F_0}{\partial s^2}[c_1(a)+c_2(a)s^2]\ , \tag{24}$$

where

$$c_1(a)=\frac{10a-3k-2a^2k}{8a^2(a^2-1)}$$

and

$$c_2(a)=\frac{-6a+k+2a^2k}{4a^2}\ , \tag{25}$$

with

$$k=\frac{\ln\frac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}}{\sqrt{a^2-1}}\ . \tag{25a}$$

Within the range $1<a<5$, $c_1$ and $c_2$ are approximated by the simpler expressions

$$c_1(a)\approx\frac{2}{3a^2(a^2+1)}\quad\text{and}\quad c_2(a)\approx\frac{5(a^2-1)^2}{26(a^4+1)}\ . \tag{26}$$

The comparison of the exact values of $c_1$ and $c_2$ to their approximate ones, Eq. (26), is displayed in Fig. 2.

Rearranging the terms, Eq. (24) is rewritten as

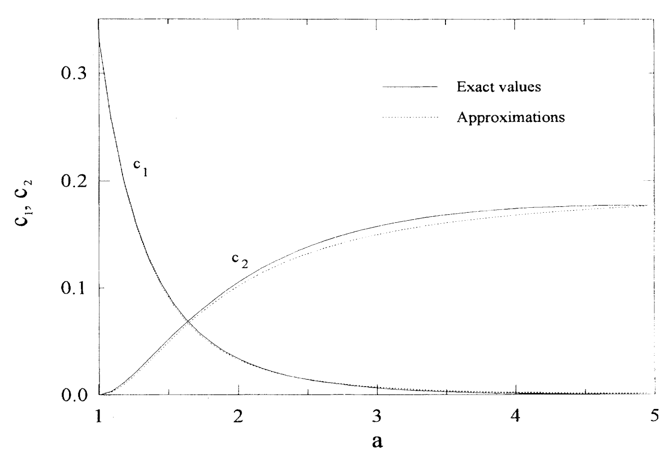


FIG. 2. The $c_n$ coefficients of Eq. (24) vs $a$ $(n=1,2)$.

$$s^2\frac{\partial F_0}{\partial \tau}=\frac{\dot a}{a}\frac{s^3}{3}\frac{\partial F_0}{\partial s}+\delta\frac{\partial}{\partial s}\left[\frac{c_1}{s}\frac{\partial F_0}{\partial s}+c_2 s\frac{\partial F_0}{\partial s}\right] . \quad (27)$$

Under the limit $a\rightarrow 1$, Eq. (27) goes over to Eq. (11). Thus, Eq. (27) is the "final" equation we need to solve for the isotropic part of the EDF in the scaled, oscillating frame for arbitrary values of $a(\tau)$. The EDF must satisfy the normalization condition

$$4\pi a(\tau)\int_0^\infty F(s)s^2 ds=1 . \quad (28)$$

It is possible to exclude $a(\tau)$ from the normalization condition by the transformation $G_0(s,\tau)=a(\tau)F_0(s,\tau)$. In this case, Eq. (27) is rewritten as

$$s^2\frac{\partial G_0}{\partial \tau}=\frac{\partial}{\partial s}\left[\frac{\dot a}{a}\frac{s^3}{3}G_0+\delta\left[\frac{c_1}{s}+c_2 s\right]\frac{\partial G_0}{\partial s}\right] \quad (29)$$

and the normalization condition is

$$4\pi\int_0^\infty G_0(s)s^2 ds=1 . \quad (30)$$

### C. The anisotropic parts of the distribution function

The high-order coefficients of the LPE are necessary to estimate the anisotropy of the EDF, and they have not entered Eq. (27). If the conditions of fast convergence are met, the sole Eq. (27) [or Eq. (29)] will be sufficient to describe the plasma characteristics.

The equation for $F_1$ is obtained from Eq. (14) as

$$\tfrac{2}{3}s\frac{\partial F_1}{\partial \tau}=\delta\left[\frac{c_3}{s^2}\frac{\partial F_0}{\partial s}+\frac{c_4}{s}\frac{\partial^2 F_0}{\partial s^2}\right]\sin\tau+\cdots , \quad (31)$$

where the terms of order $\delta F_1$ and $\delta F_2$ have been omitted as small. Assuming only fast solutions of this equation, from Eq. (31) we get

$$F_1\approx-\frac{3\delta}{2}\left[\frac{c_3}{s^3}\frac{\partial F_0}{\partial s}+\frac{c_4}{s^2}\frac{\partial^2 F_0}{\partial s^2}\right]\cos\tau , \quad (32)$$

where

$$c_3=4ac_1(a) \text{ and } c_4=\frac{8+10a^2-7ak-2a^3k}{2a^2(a^2-1)} . \quad (33)$$

From Fig. 3, we see how these coefficients decrease with increasing $a(\tau)$. $F_1$ describes the regular oscillations of the electron bulk; $F_2$ describes its shrinking or stretching. Just the latter effect was observed in the numerical calculations [15]. The equation for $F_2$ derived from Eq. (14) can be written as

$$\tfrac{2}{5}s\frac{\partial F_2}{\partial \tau}=\frac{4s^2}{15}\frac{\dot a}{a}\frac{\partial F_0}{\partial s}+\delta\left[\frac{c_5}{s}-\frac{c_6}{s^3}\right]\frac{\partial F_0}{\partial s}$$
$$-\delta\left[c_7+\frac{c_8}{s^2}\right]\frac{\partial^2 F_0}{\partial s^2}+\cdots , \quad (34)$$

where again the terms of order $\delta F_1$ and $\delta F_2$ have been omitted, and

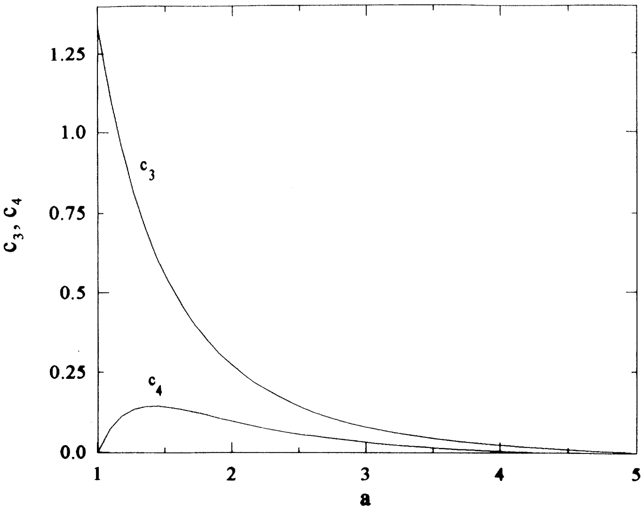


FIG. 3. The $c_n$ coefficients of Eq. (31) vs $a$ ($n=3,4$).

$$c_5(a)=\frac{66a+60a^3+k-56a^2k-8a^4k}{16a^2(a^2-1)} ,$$
$$c_6(a)=\frac{38a+52a^3+3k-40a^2k-8a^4k}{16a^2(a^2-1)^2} ,$$
$$c_7(a)=\frac{-54a-36a^3+5k+32a^2k+8a^4k}{16a^2(a^2-1)} ,$$
$$c_8(a)=\frac{214a+92a^3-21k-116a^2k-16a^4k}{16a^2(a^2-1)^2} . \quad (35)$$

Figure 4 shows how the coefficients $c_5-c_8$ change with $a(\tau)$.

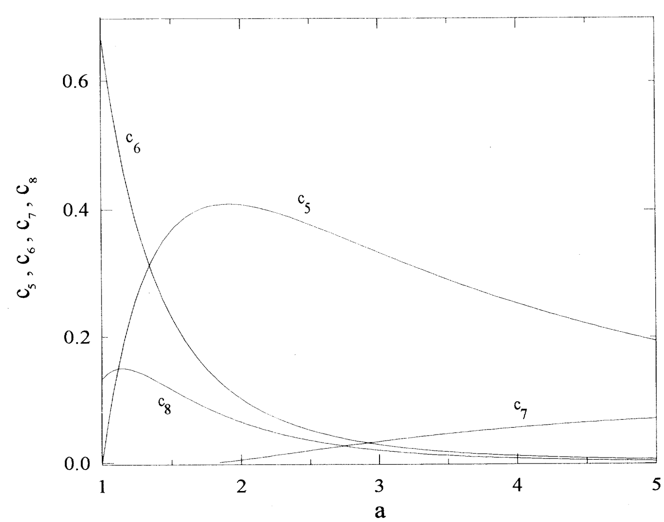


FIG. 4. Behavior vs $a$ of the coefficients $c_n$ ($n=5,6,7,8$) appearing in Eq. (34).

### D. The equation for the scaling coefficient $a(\tau)$

We need an equation giving the time evolution of the coefficient $a(\tau)$. It is derived from Eq. (34) as follows. We assume that $a(\tau)$ is such that the goal of making the EDF $F(s,\tau)$ "isotropic" on the average has been achieved. In such a case we have

$$\langle s_\perp^2 \rangle = 2\langle s_z^2 \rangle \ , \tag{36}$$

where the brackets $\langle\ \rangle$ define the average over $F(s,\tau)$ of the quantity inside them:

$$\langle s_i^n \rangle = \frac{\int s_i^n F(\mathbf{s},\tau) d^3 s}{\int F(\mathbf{s},\tau) d^3 s} \ .$$

Substituting expansion (15) into Eq. (36), which is now rewritten as

$$\int \sum_l F_l(s,\tau) P_l(\cos\vartheta) \sin^2\vartheta s^2 d^3 s = 2\int \sum_l F_l(s,\tau) P_1(\cos\vartheta) s^2 \cos^2\vartheta d^3 s \ , \tag{37a}$$

we obtain

$$\int \sum_l F_l(s,\tau) P_l(\cos\vartheta)(1-3\cos^2\vartheta) s^2 d^3 s = 0 \ . \tag{37b}$$

Recalling that $1-3\cos^2\vartheta = -2P_2(\cos\vartheta)$, from normalization and orthogonality properties of the Legendre polynomials, it follows that

$$\int_0^\infty F_2(s,\tau) s^4 ds = 0 \ . \tag{38}$$

Multiplying Eq. (34) by $s^3$, integrating over the velocity, and exploiting Eq. (38), the following equation for $a(t)$ is obtained:

$$\frac{4}{3}\frac{\dot a}{a}\int_0^\infty s^4 F_0(s,\tau) ds = \delta(c_6 - c_8) F_0(0) - 2\delta(c_5 + 3c_7)\int_0^\infty s F_0(s,\tau) ds + O(F_1, F_2, \dots) \ . \tag{39}$$

Omitting from the right-hand side (RHS) of Eq. (39) the small terms containing $F_1$, $F_2$, etc., Eq. (39) is rewritten as

$$\frac{\langle s^2 \rangle}{4\pi}\dot a = \delta c_9(a) F_0(0) - \delta c_{10}(a)\int_0^\infty s F_0(s,\tau) ds \ , \tag{40}$$

where

$$c_9 = \frac{3a}{4}(c_6 - c_8) \quad \text{and} \quad c_{10} = \frac{3a}{2}(c_5 + 3c_7) \ .$$

The behavior of $c_9$ and $c_{10}$ as functions of $a$ is shown in Fig. 5. Equation (40) has a clear physical meaning and allows an insight into the essence of the laser-plasma interaction. The first term on the RHS of Eq. (40), being always positive, describes the plasma heating by the laser field, which directly sets up the ratio of the longitudinal to the perpendicular temperature. Note the analogy with the case of small anisotropy, for which the heating rate is proportional to $F_0(0)$ [13]. The factor before $a$, in Eq. (40), namely the mean energy, is in some sense the factor of inertia, reflecting the resistance of the collisional plasma to increasing the difference between the longitudinal and the perpendicular temperatures. The remaining term on the RHS describes the transfer of the energy from the longitudinal degree of freedom to the perpendicular one. It is interesting to note that the integral appearing on the RHS of Eq. (40) may be written as

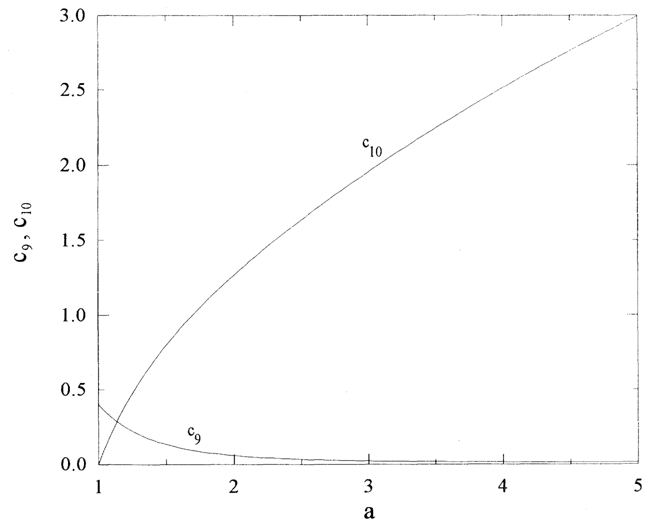


FIG. 5. Behavior vs $a$ of the coefficients $c_n$ $(n=9,10)$ appearing in Eq. (40).

$$\delta\int_0^\infty s F_0 ds = \int_0^\infty \frac{\delta}{s^3} s^4 F_0 ds \approx \langle \nu_{ei} s^2 \rangle \ , \tag{41}$$

representing the mean energy dissipated by the Coulomb collisions. By construction, when Eq. (40) holds, the anisotropic part $F_2$ is much smaller on average than $F_0$. Equation (27) for $F_0$ and Eq. (40) for $a(\tau)$ form the system of two coupled equations, which substitutes the kinetic equation, Eq. (2), with the restriction that the averaging over the field period has been performed. Of course, the proposed procedure can be considered complete only when the calculated $F_0$ and $a(\tau)$ give $F_1 \ll F_0$, Eq. (32). When this inequality is satisfied, all the LPE coefficients are small compared to the first one, $F_0$.

As we show below, this goal is made easier by exploiting the assumption of Eq. (36), i.e, values of $a(\tau)$ exist such that, on the average, the $F(s,\tau)$ is isotropic. From Eq. (12), with $\varphi(\mathbf{u},t) = F(\mathbf{s},\tau)$, we have the following relation:

$$\langle \varphi \rangle_u = a\langle F \rangle_s \ , \tag{42}$$

where $\langle \varphi \rangle_u = \int \varphi(\mathbf{u},\tau) d^3 u$, and $\langle F \rangle_s = \int F(\mathbf{s},\tau) d^3 s$. Besides,

$$\langle u_\perp^2 \rangle = \frac{\int u_\perp^2 \varphi(\mathbf{u},t) d^3 u}{\int \varphi(\mathbf{u},t) d^3 u} = \langle s_\perp^2 \rangle \tag{43a}$$

and

$$\langle u_z^2 \rangle = a^2(\tau) \langle s_z^2 \rangle \ . \tag{43b}$$

Using the relation (36), together with (43a) and (43b), we get

$$a^2(\tau) = \frac{2\langle u_z^2 \rangle}{\langle u_\perp^2 \rangle} \ . \tag{44}$$

Thus, the squared scaling function $a(\tau)$ is twice the ratio of the average parallel to the average perpendicular kinetic energies (evaluated in the unscaled, oscillating frame) [19]. To some extent, Eq. (44) makes unnecessary the self-consistent solution of Eqs. (27) and (40). As done in Ref. [15] (see, especially, Fig. 7), we first find numerically $a(\tau)$ in the unscaled, oscillating frame and then concentrate on the approximate solution of Eq. (27) in an effort to find $F_0$ analytically for the calculated values of $a(\tau)$. Examples of this procedure are worked out in the next section for such time intervals that the establishment of the SS solutions is an outcome of the EDF evolution. We conclude by observing first, that if the relation (44) is taken as an "ansatz," Eq. (40) for $a(\tau)$ is readily obtained; second, that the integrals (18) have been approximately evaluated under the assumption that $a(\tau) \geq 1$. The latter assumption expresses the expectation (corroborated by numerical calculations) that a linearly polarized laser electric field along the $u_z$ direction will yield on the average a flattening at the pole's (oblate) distribution (see Fig. 1). Thus, Eqs. (27) and (40), in their present form, are not suited to treat the case when $a(\tau) < 1$, i.e., if the field action produces an elongation towards the pole's (prolate) distribution.

## V. SELF-SIMILAR MODEL SOLUTIONS

In this section we consider solutions of our equations for the two opposite cases of small and large anisotropy. In the case of small anisotropy, the solution is known (see, e.g., [4]). We consider this case, too, because our equations, for a small but finite degree of anisotropy, allow the possibility of finding corrections to and improving upon the known result. Besides, the small anisotropy case serves as a check of our entire procedure. For the second case (large anisotropy), to the best of our knowledge, no accurate solution is known.

### A. Small but finite anisotropy

From Fig. 2 we see that the coefficients $c_2(a)$ and $c_{10}(a)$ for $a \leq 1.3$ are, respectively, much smaller than $c_1(a)$ and $c_9(a)$. Under this condition, we can simplify Eqs. (29) and (40) to the case of small anisotropy in the scaled, oscillating frame, letting $a \approx 1$ and obtaining

$$s^2 \frac{\partial G_0}{\partial \tau} = \frac{\partial}{\partial s} \left[ \frac{\dot a}{a} \frac{s^3}{3} G_0 + \delta \frac{c_1}{s} \frac{\partial G_0}{\partial s} \right] , \tag{45}$$

$$\dot a \int_0^\infty s^4 G_0 ds \approx \delta c_9(a) G_0(0) \ . \tag{46}$$

With the transformation $T = \int \delta c_1 d\tau$ and the SS transformations

$$s' = s / W(T) \ ,$$

$$G' = G_0 W^3 \ , \tag{47}$$

letting $\dot a / a \approx \dot a$, the heating equation and the SS EDF are found to be

$$W^5 = W_i^5 + \left[ 25 - \frac{5K_9}{3c_1} \right] T \tag{48a}$$

$$G'(s') = G_i \exp(-s'^5) \ , \tag{48b}$$

where $W_i$ is an initial value and $K_9 = c_9(1) G'(0) / \int_0^\infty s'^4 G' ds'$. The normalization condition gives $G_i = 0.2672$. We observe that the usual SS solution of Eq. (11) [4] for $W(\tau)$ does not contain the last term in the brackets of Eq. (48a).

In the range $1 < a < 5$, the function $c_9(a)$ can be approximated by the expression

$$c_9(a) \approx \frac{2(a+2)}{15a^3} \ . \tag{49}$$

Using Eqs. (49) and (26) in the limit $a \to 1$, we find

$$a \approx 1 + \tfrac{2}{5} \ln \left[ 1 + 5 \frac{\delta \tau}{W_i^5} \right] , \tag{50}$$

$$W \approx (W_i^5 + 5\delta\tau)^{1/5} \ , \tag{51}$$

which show that heating proceeds slower than in Ref. [4].

### B. Large anisotropy

This regime ($a \gg 1$) may be described by the equations

$$s^2 \frac{\partial G_0}{\partial \tau} = \frac{\partial}{\partial s} \left[ \frac{\dot a}{a} \frac{s^3}{3} G_0 + \delta c_2 s \frac{\partial G_0}{\partial s} \right] , \tag{52}$$

$$\dot a \int_0^\infty s^4 G_0 ds \approx -\delta c_{10}(a) \int_0^\infty s G_0 ds \ . \tag{53}$$

Making the SS transformation of Eq. (47), from Eq. (52) we get

$$\frac{\partial}{\partial s'} \left[ \dot W W s'^3 G' + W^2 \frac{\dot a}{a} \frac{s'^3}{3} G' + \frac{\delta c_2}{W} s' \frac{\partial G'}{\partial s'} \right] \approx 0 \ , \tag{54a}$$

$$\dot a W^3 \approx -\delta K_{10} \ , \tag{54b}$$

where $\dot W = dW/dT$ and

$$K_{10} = \frac{\int_0^\infty s' G'(s') ds'}{\int_0^\infty s'^4 G'(s') ds'} c_{10} \approx \frac{45}{52} a \ , \tag{55}$$

having used the approximation $c_{10} \approx (15/26) a$.

Furthermore, making the approximation $c_2 \approx \frac{5}{26}$, the SS solution to Eq. (52) is obtained as

$$G'(s') = G_i \exp(-s'^3) \ , \tag{56}$$

provided that

$$3 \dot W W^2 - \tfrac{45}{52} \delta - 9 \delta c_2 = 0 \ . \tag{57}$$

From Eqs. (54b) and (57) the heating equation and the time behavior of $a(\tau)$ are easily found to be

$$W(\tau)=(W_i^3+\tfrac{135}{52}\delta\tau)^{1/3}\ , \tag{58}$$

$$a(\tau)=a_0\left[1+\frac{135}{52}\frac{\delta\tau}{W_i^3}\right]^{-1/3}\ . \tag{59}$$

By construction, Eq. (59) is valid for values of $a(\tau)$ considerably larger than unity, and while $a(\tau)$ is predicted to be a decreasing function of time, its time behavior is correct only at times for which $a(\tau)$ is larger than at least 3. For larger times, one must use other expressions, appropriate to small values of $a(\tau)$. Equation (56) is a SS EDF in the scaled moving frame, appropriate to the case for which $a(\tau)$ is considerably larger than unity, and is one of the main results of this paper. Equation (58) makes it possible, in a relatively simple way, to calculate processes and properties of plasmas corresponding to our model, when the laser field strongly distorts the EDF.

To express the obtained SS EDF in the fixed laboratory frame, we must perform the transformations required to go back to that reference frame. First, we make the reverse transition from self-similar variables to those of the scaled moving frame:

$$F(s,\tau)=F_I\frac{1}{W^3(\tau)a(\tau)}\exp\left\{\frac{-s^3}{W^3(\tau)}\right\}\ , \tag{60}$$

with $F_I$ the normalization factor. Second, we remove the scaling in the parallel direction:

$$s^2=u_\perp^2+\frac{u_z^2}{a^2(\tau)}\ . \tag{61}$$

Finally, we make the back transformation from the moving frame to the fixed one:

$$u^2=v_\perp^2+\frac{[v_z+v_e\sin(\omega t)]^2}{a^2(\tau)}\ . \tag{62}$$

As a result, we get an oscillating distorted EDF,

$$f_0(\mathbf{v},t)=\frac{f_1}{W^3(t)a(t)}\times\exp\left\{-\frac{\{a^2v_\perp^2+[v_z+v_e\sin(\omega t)]^2\}^{3/2}}{W^3(t)a^3(t)}\right\}\ , \tag{63}$$

with $f_I$ the appropriate normalization factor, and $W(t)$ and $a(t)$ given, respectively, by Eqs. (58) and (59). In the equations, conventional velocity and time units have been restored.

## VI. CONCLUDING REMARKS

We have shown that it is possible to use the Legendre polynomial expansion within a fast converging scheme to determine approximate electron distribution functions when the plasma-laser interaction is expected to cause strong alteration of the electron velocities.

The basic concept is that of working in an oscillating frame, in which the coordinate axis is scaled according to the physical situation created by the given external field, with the aim of restoring as much as possible, on the average, an isotropic distribution.

Specifically, we have considered a linearly polarized, homogeneous, and single-mode laser field. Besides developing the basic equations for numerical calculations for arbitrary times and degree of anisotropy, we have also addressed the existence of approximate self-similar solutions. Checking our equations in the case of small anisotropy, we recover and improve upon the result first obtained by Langdon [4]. For large anisotropy, we obtain an analytical self-similar solution, which is one of the basic results of this work. This self-similar solution should prove useful in several issues of plasma behavior under the action of strong laser fields. We are confident that the ideas presented will prove useful in similar plasma situations. However, the best way to assess the potential of these ideas is to perform parallel calculations, using more familiar procedures along with the present one. We note that our procedure was prompted by the numerical calculations of Ref. [15].

Investigations aimed at extending the ideas presented here, as well as application of some of the reported results to the study of relevant plasma characteristics, will be undertaken in the future.

## ACKNOWLEDGMENTS

The authors express their thanks to the University of Palermo Computational Centre for the computer time generously provided to them. This work was supported in part by the Italian Ministry of University and Scientific and Technological Research, the National Group of Structure of Matter, and the Sicilian Regional Committee for Nuclear and Structure of Matter Research. Additional partial support from the Italian Research Council, through different application-oriented projects, is acknowledged as well. One of the authors (P. P.) thanks the INFM for the contract granted to him.

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