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Soliton solutions in Rowland ghost gaps

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We present here a class of solitary wave solutions for nonlinear periodic Bragg gratings. The effect of periodic coefficients is to cause additional band gaps to open up. It is in these Rowland ghost gaps that solitary waves have been found.

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I. INTRODUCTION

In a diffraction grating a periodic error in the ruling gives rise to high diffraction intensities at angles close to the actual line. These additional lines, "Rowland ghosts," were examined by Rowland in 1902 and Pierce in 1879 [1]. These ghosts are caused by the double periodicity of the grating: a rapid variation with a period close to the wavelength of light, and a slow variation with a much longer period. We present here an analysis for nonlinear fiber Bragg gratings which have this same dual periodicity. A schematic of the refractive index profile can be seen in Fig. 1, where the light propagates perpendicular to the planes of constant refractive index. Such "grating superstructures" have been written in optical fibers where the two periods are typically $d \approx 0.5$ μm and $\Lambda \approx 1$ mm [2]. The rapid refractive index variation on its own gives rise to photonic band gaps where the light cannot propagate due to Bragg reflection. These band gaps correspond to regions of high reflectivity. The presence of an additional periodicity $\Lambda \gg d$ in the refractive index creates additional reflection peaks [3]. These new peaks are clustered around frequencies where a uniform grating would be strongly reflecting.

To understand this property of grating superstructures it is informative to consider why a uniform grating does not reflect strongly at frequencies far from the Bragg resonance: light diffracted off different interfaces is not in phase, as shown schematically in Fig. 2. However, if the parts with negative phase, say, are removed, as in a superstructure (Fig. 1), then strong reflection results leading to the formation of Rowland ghost gaps.

Previously much of the work on grating superstructures has been concerned with their linear reflection spectrum, i.e., predicting the position and strength of the Rowland ghosts. This linear analysis suggests that for frequencies close to a single ghost gap the grating superstructure behaves almost identically to a uniform grating [3]. Here we report on the nonlinear properties of grating superstructures and how they compare with nonlinear uniform gratings. Nonlinear uniform gratings posses solitary wave solutions, "gap solitons" [4], whose frequency content lies near the band gap of the uniform grating. This paper describes a class of solitary waves whose frequency content can be close to any ghost gap. Such solutions are generalizations of gap solitons to superstructure gratings.

II. COUPLED MODE EQUATIONS

We consider a one dimensional superstructure Bragg grating whose refractive index varies with a base period d, and with a superperiod Λ. The grating couples the forward and backward traveling modes whose propagation constants differ by $2\pi/d \equiv 2k_0$. The refractive index profile $n(x)$ of a superstructure Bragg grating is then given by

$$n(x)=\bar{n}(x)+\Delta n(x)\cos 2k_0 x, \tag{1}$$

where $\bar{n}(x)$ and $\Delta n(x)$ are periodic with period Λ. Note that for a uniform grating $\bar{n}$ and Δn are constant. For frequencies ω close to the Bragg resonant frequency ω_0 we can write the electric field as

$$\mathscr{E}(t,x)=[\mathscr{F}_+(t,x)e^{+ikx}+\mathscr{F}_-(t,x)e^{-ikx}]e^{-i\omega t}+\text{c.c.}, \tag{2}$$

where $\mathscr{F}_+$ and $\mathscr{F}_-$ are the slowly varying envelopes of the modes. With this ansatz for the electric field, Maxwell's equations with a Kerr nonlinearity $n^{(2)}$ can be approximated by the nonlinear coupled mode equations (NLCME's) [5,6]:

$$i\frac{\partial \mathscr{F}_+}{\partial x}+\frac{i}{v_g}\frac{\partial \mathscr{F}_+}{\partial t}+\kappa(x)\mathscr{F}_-+[\Delta+\delta(x)]\mathscr{F}_+ +2\Gamma|\mathscr{F}_-|^2\mathscr{F}_+ +\Gamma|\mathscr{F}_+|^2\mathscr{F}_+=0,$$

$$-i\frac{\partial \mathscr{F}_-}{\partial x}+\frac{i}{v_g}\frac{\partial \mathscr{F}_-}{\partial t}+\kappa(x)\mathscr{F}_+ +[\Delta+\delta(x)]\mathscr{F}_- +2\Gamma|\mathscr{F}_+|^2\mathscr{F}_- +\Gamma|\mathscr{F}_-|^2\mathscr{F}_-=0, \tag{3}$$

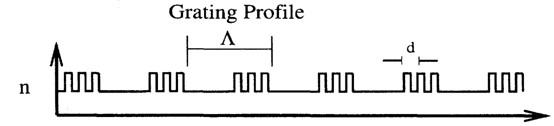

FIG. 1. Schematic of the refractive index profile for a grating superstructure. The light propagates parallel to the horizontal axis.

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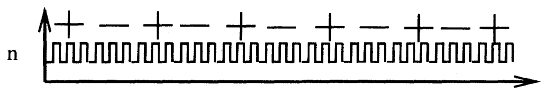

FIG. 2. Schematic of the refractive index profile for a uniform grating. The phase of the light reflected from each part is indicated by + or −.

where [5,6]

$$\kappa(x)=\frac{\pi\Delta n(x)}{\lambda},\quad \Delta=\frac{\omega-\omega_0}{v_g},\quad \Gamma=\frac{4\pi\bar{n}}{\lambda Z}n^{(2)},\tag{4}$$

and v_g is the group velocity in the absence of a grating, λ is the free space wavelength, and Z the vacuum impedance. After a trivial rescaling we can take $v_g=1$ without loss of generality. The function $\kappa(x)$ measures the strength of the coupling while the detuning Δ is a normalized frequency. Finally $\delta(x)$ describes the shift in the Bragg wavelength; for gratings written in optical fibers $\delta(x)=2\kappa(x)$ [5], which we assume is true here, although it does not significantly affect the analysis. Nonlinear effects are described by Γ which we take to be constant in space although this is not essential. For a uniform linear grating $\Gamma=0$ and κ and δ are constant. Light incident upon a uniform grating at detunings Δ within the band gap $-3\kappa<\Delta<-\kappa$ cannot propagate in the linear regime and is thus strongly reflected. While at frequencies far from the Bragg resonance ($|\Delta|\gg\kappa$) light propagates freely.

In a nonlinear uniform grating, a two parameter family of solitary waves exists [7]. These gap solitons are all singly peaked, with a frequency and width determined by the parameter Ξ where $0<\Xi<\pi$. The center frequency of a gap soliton is close to the band gap of the grating. The second parameter v, where $-1<v<1$, determines the velocity of the soliton which travels at a speed of vc/n [4].

Having briefly discussed the solutions to Eqs. (3) when the coefficients are constant, we now treat grating superstructures where κ and δ vary with a period of Λ. In any periodic medium the natural basis set to use in analysis are the Bloch functions: the solutions to the linear problem for an infinite medium and which form a complete orthogonal set. Each Bloch function $\varphi_{\ell,k}$ is labeled by two "quantum numbers:" a reduced wave number k and an integer ℓ that labels the photonic bands. At a given k therefore, the associated frequencies Δ_ℓ may be labeled uniquely by ℓ. By plotting the values of Δ_ℓ against k we obtain the photonic band diagram corresponding to the dispersion relation for a grating. Part of a typical photonic band diagram is shown in Fig. 3 for a superstructure with parameters

$$\kappa(x)=[1+0.8\cos(2\pi x/\Lambda)]\ \text{cm}^{-1}\tag{5}$$

with $\Lambda=0.9$ cm. Gaps occur when for a range of Δ's no running wave solutions can be found. It is at these frequencies that the grating is highly reflective. These "ghost gaps"

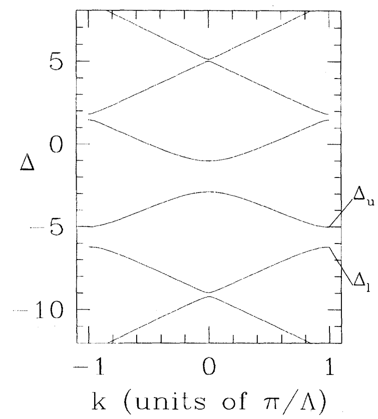

FIG. 3. Photonic band structure of the grating superstructure used in the numerical simulations. Here $\kappa(x)=[1+0.8\cos[(2\pi/\Lambda x)]]$ cm^{-1} with $\Lambda=0.9$ cm. The detuning Δ is measured in inverse cm.

are analogous to the ghost lines studied by Rowland in the context of diffraction gratings [1].

To describe the positions of the ghost gaps we need the detunings

$$\Delta_m=-\delta_0+\text{sgn}(m)\sqrt{(m\pi/\Lambda)^2+\kappa_0^2},\tag{6}$$

where $m=0,\pm1,\pm2,\ldots,$ κ_0 is the zeroth order Fourier component of κ, and similarly for δ_0. The function sgn(m) is defined by sgn(m)=1 if $m>0$, sgn(0)=0, and sgn(m)=−1 if $m<0$. If the superstructure is *weak* then there is a "ghost gap" centered around each detuning Δ_m, with a width, in normalized units ($v_g=1$), of $2\kappa_m$ where κ_m is the strength of mth Fourier coefficient of κ. If a particular Fourier component vanishes then the associated gap does not appear. Note that the Δ_m are the only frequencies where ghost gaps can appear. However, when the modulation becomes *deep* additional ghost gaps appear at frequencies Δ_m where the corresponding Fourier coefficient is zero, and the strength and positions of the other gaps change. This is illustrated in Fig. 3 where the Fourier expansion of κ contains three terms [see Eq. (5)]; zeroth order which corresponds to the large gap centered at $\Delta=-2$ cm^{-1}, and the ±1 terms which correspond to the gaps above and below this gap. Although the Fourier coefficient for the ±1 terms are equal, the sizes of the ghost gaps are different implying that the grating superstructure is deep. Also gaps now appear at frequencies where the associated Fourier coefficients vanish.

Below we consider systems where the range of frequencies of interest lies near a ghost gap. A typical ghost can be seen in Fig. 3 lying between $\Delta_u=-5.0$ cm^{-1} and $\Delta_l=-6.2$ cm^{-1}. Since for each k the associated Bloch functions are complete we can represent any field distribution as a linear combination of them with constant coefficients. For field distributions with frequencies near a particular gap, we expect that the largest terms in our expansion would correspond to the Bloch functions φ_u and φ_l at the top (Δ_u) and bottom (Δ_l) of the gap of interest, respectively (see Fig. 3). We therefore use the following ansatz for the field envelopes [cf. Eq. (2)]:

$$\mathscr{F}(x,t)=[f_u(x,t)\varphi_u(x)+f_l(x,t)\varphi_l(x)]e^{-i\Delta_0 t}, \quad (7)$$

where $\mathscr{F}=(\mathscr{F}_+,\mathscr{F}_-)$ is a two component column vector, while $\Delta_0=(\Delta_u+\Delta_l)/2$ is the frequency at the center of the band gap we are considering. A consequence of truncating the expansion is that the coefficients f_u and f_l are now slowly varying "superenvelope functions" which can also describe the effects of the nonlinearity.

Substituting Eq. (7) into the coupled mode equations leads to two equations for $f_{u,l}(x,t)$. These are simplified by projecting onto both $\varphi_u(x)$ and $\varphi_l(x)$ and using the orthogonality of the Bloch functions. The integrals are performed by assuming that the superenvelopes $f_{u,l}(x,t)$ are constant over a superlattice period Λ. This procedure is similar to that described by Salinas *et al.* [8] and the references therein. After performing the overlap integrals it can be shown that the superpositions

$$f_\pm=f_l\mp f_u \quad (8)$$

satisfy the *supercoupled mode* equations

$$i\frac{\partial f_+}{\partial x}+\frac{i}{V}\frac{\partial f_+}{\partial t}+\tilde{\kappa}f_-+2\tilde{\Gamma}|f_-|^2f_++\tilde{\Gamma}|f_+|^2f_++\Gamma_1(|f_-|^2+|f_+|^2)f_-+\Gamma_1(f_+f_-^*+f_-f_+^*)f_++\Gamma_2f_-^2f_+^*=0, \quad (9)$$

$$-i\frac{\partial f_-}{\partial x}+\frac{i}{V}\frac{\partial f_-}{\partial t}+\tilde{\kappa}f_++2\tilde{\Gamma}|f_+|^2f_-+\tilde{\Gamma}|f_-|^2f_-+\Gamma_1(|f_-|^2+|f_+|^2)f_++\Gamma_1(f_+f_-^*+f_-f_+^*)f_-+\Gamma_2f_+^2f_-^*=0,$$

which are very similar to the original coupled mode equations (3). Note however that, in contrast to Eqs. (3), for a periodic superstructure the grating strength $\tilde{\kappa}$ is *constant*, and that the terms in δ are totally absent, as they reside now in the details of the Bloch functions. The fact that Eqs. (9) can be derived implies that, around each of its ghost gaps, a superstructure grating behaves approximately like a uniform grating. However, the nonlinear coefficients in the supercoupled mode equations (9) are more complicated than in Eqs. (3), though, as we discuss below, this poses no significant problem. These nonlinear equations were derived previously in a different context [8] and solitary wave solutions have been found. We can thus adapt these previously known solutions to our problem. These new solitary waves form approximate solutions to the original NLCME's [Eqs. (3)] with periodic parameters but as we will show in Sec. III the approximation is a good one.

In the new nonlinear equations $\tilde{\kappa}$, the new group velocity V, and nonlinear coefficients all depend on the Bloch functions. The nonlinear coefficients $\tilde{\Gamma}$, Γ_1, and Γ_2 in Eqs. (9) are linear combinations of various overlap integrals of combinations of the Bloch functions and the nonlinearity. For shallow superstructures $\tilde{\kappa}$ equals the appropriate Fourier coefficient in the expansion of $\kappa(x)$ which agrees with the earlier stated result [3]. In order to determine the coefficients it is necessary to know the Bloch functions. These we obtain using a matrix method [9]. In the cases we have examined the nonlinear terms Γ_1 and Γ_2 are much smaller than $\tilde{\Gamma}$ and thus can be neglected. In this limit the new equations are formally identically to the old equations but with constant coefficients.

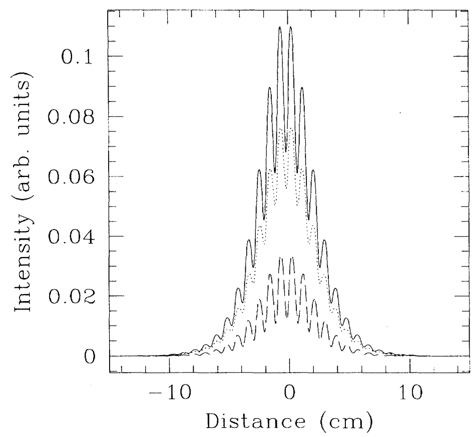

FIG. 4. Typical intensity profile of a ghost gap soliton. The solid line shows the total intensity $|\mathscr{F}_+|^2+|\mathscr{F}_-|^2$, the dotted line $|\mathscr{F}_+|^2$, and the dashed line $|\mathscr{F}_-|^2$.

III. SOLITARY WAVES IN GHOST GAPS

In Sec. II we stated that for frequencies close to a ghost gap the original coupled mode equations (3) with periodic coefficients could be approximated by the new supercoupled mode equations (9) with constant coefficients. These new equations have solitary wave solutions and hence our original coupled mode equations have solitary wave solutions with frequencies around any ghost gap. We note in passing that our approach is not limited to describing pulses and can be used to analyze the continuous wave properties of nonlinear grating superstructures.

Figure 4 shows a typical ghost gap soliton at the envelope level for the grating given in Fig. 3. This solution was obtained by solving the supercoupled mode equations [Eqs. (9)] for $f_\pm$ and then obtaining $\mathscr{F}_\pm$ from Eqs. (8) and (7). The frequency of the soliton lies within the ghost gap between $\Delta_u=-5.0$ cm^{-1} and $\Delta_l=-6.2$ cm^{-1} in Fig. 3. The intensity has been normalized such that $\Gamma=1$. In this case the parameters in the supercoupled mode equations [Eq. (9)] are $\tilde{\kappa}=0.623$ cm^{-1}, $V=0.96$, $\tilde{\Gamma}=0.519$ while Γ_1 and Γ_2 have been neglected. The superenvelopes of the fields are the standard gap solitons [4] with parameters $\Xi=0.5$ and $v=0.5$. The rapid variations on the intensity profile are due to the underlying Bloch functions while the superenvelopes of the fields described by $f_\pm$ are smooth. These variations have a period on the order of Λ and as the soliton propagates the wriggles remain stationary while the soliton moves under them. This then implies that the soliton's intensity profile oscillates with a period $\tau=\Lambda/v$ where v is the velocity of the soliton. These large scale periodic oscillations in the intensity profile of the envelopes do not exist for the ordinary gap solitons. Our numerical simulations of Eqs. (3) suggest that these solutions appear to be stable and that they can propagate over distances much longer than their width, and indeed longer than any current grating design suggesting that they can be observed.

IV. CONCLUSION

We have shown that in a nonlinear grating superstructure solitary wave solutions exist within each of the ghost gaps. The solutions appear as superenvelopes which modulate the Bloch functions of the grating. These solutions appear to be stable and can propagate over large distances. In addition, these solutions can be created at an interface by an incident pulse. The power necessary to see these solutions is comparable to the power needed to see a conventional gap soliton.

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