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Nonextensive thermostatistics can yield apparent magnetism

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Bacry [Phys. Lett. B **317**, 523 (1993)] showed that, on the basis of a deformed Poincaré group, special relativity yields a nonadditive energy for large systems, i.e., a total energy (of the Universe) which would not be proportional to the number of particles. He consistently argued that this effect could explain (part of) the so-called dark matter. By considering noninteracting spins at thermal equilibrium in the presence of an external magnetic field, we show here that the recently introduced nonextensive (nonadditive) thermostatistics could account for a theoretically possible "dark magnetism" (the apparent number of spins being *smaller* or *larger* than the actual one).

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For a variety of (possibly interconnected) physical reasons, a large amount of work is nowadays dedicated to fundamentally nonlinear formalisms for physics. Two quite active streams along this line are quantum groups (see [1,2] and references therein) and nonextensive thermostatistics [3], to which the present effort is dedicated. In a series of papers, Bacry [1,2] recently pointed out a number of advantages of introducing a deformation of the Poincaré group (leading to "quantum special relativity"). This new quantum group preserves the main successes of the ordinary Poincaré group: conservation laws for all momenta as well as additivity of angular momentum remain valid. He also showed that quantum special relativity can yield an interesting kinematic effect, namely a nonadditive (nonextensive) energy, in the sense that the total energy need not be proportional to the total number of particles, if the number of particles is large. He then suggested [1] that, as a side benefit, this effect could account for all (or part) of the so-called "dark matter" of the Universe. We show here that similar effects are obtained on a quite different theoretical background, namely in the specific kind of equilibrium thermostatistics just mentioned. The effect is so general that, with pedagogical advantage, it can be illustrated in a very simple system, namely an assembly of noninteracting spin 1/2 atoms in the presence of an external magnetic field. This system has already been focused [4] in order to discuss the existence of the thermodynamic limit within nonextensive thermostatistics. We now focus on a different aspect, namely the theoretically possible existence of what we shall be referring to as "dark magnetism" (where the apparent number of spins is smaller or larger than the actual one).

Before we approach the magnetic system, let us briefly review the nonextensive statistical mechanics. It is based upon a generalized entropic form [3] for a physical system, namely

$$S_q(\{p_n\}) \equiv k\,\frac{1-\sum_{n=1}^{W} p_n^q}{q-1}\,, \tag{1}$$

where k is a dimensional positive constant, q any real number, and p_n the probability associated with the nth microstate ($\Sigma_{n=1}^{W} p_n = 1$), with the proviso that the sum must be carried out over states with nonzero probabilities. It can be immediately proved that the well-known additive Shannon's entropy is recovered as a special case of (1): $\lim_{q\to 1} S_q = -k_B\,\Sigma_{n=1}^{W} p_n \ln p_n$ (k_B is Boltzmann's constant).

The physics is an extensive one only for $q=1$. Otherwise, we are led into the realm of nonextensivity [3, 5, 6]. Indeed, let $\{p_n\}$ and $\{p'_m\}$ be two distributions associated with two *independent* systems (so that the joint probability is given by $p_{nm} = p_n\, p'_m$). Then

$$\frac{S_q(\{p_{nm}\})}{k} = \frac{S_q(\{p_n\})}{k} + \frac{S_q(\{p'_m\})}{k} + (1-q)\,\frac{S_q(\{p_n\})}{k}\,\frac{S_q(\{p'_m\})}{k}. \tag{2}$$

Many properties and diverse applications of this proposal have been given by a number of authors (see, for example, Refs. [4–27]). Through the usual variational procedures, a generalized (power-law instead of exponential) form for the distribution functions p_n has been found. Consistency with a generalized thermodynamics has also been established [5]. More specifically, this formalism has found applications in self-gravitating systems [7,8], Lévy-like anomalous superdiffusion [9], optimization techniques (simulated annealing) [10], hydrogen atom specific heat [11], correlated anomalous diffusion [12], ferrofluidlike systems [13], cosmic background radiation [14], two-dimensional turbulence [15], among others. The generalized entropy for a quantum system characterized by the density operator $\hat{\rho}$ reads

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$$S_q(\hat{\rho})=k\,\frac{1-\mathrm{tr}(\hat{\rho}^{\,q})}{q-1} \tag{3}$$

while the q-expectation value (to be associated with physical observables) of a quantum-mechanical operator $\hat{O}$ is defined as

$$\langle\hat{O}\rangle_q=\mathrm{tr}(\hat{\rho}^{\,q}\,\hat{O}). \tag{4}$$

Let us consider a system of identical spins each of magnetic moment $\hat{\vec{\mu}}^{(i)}=g\,e/(2mc)\,\hat{\vec{S}}^{(i)}$, where $\hat{\vec{S}}^{(i)}=(\hbar/2)\,\hat{\vec{\sigma}}^{(i)}$ ($\hat{\vec{\sigma}}$ denotes the Pauli spin matrices). The potential energy arising from the interaction of N localized spins with an external uniform magnetic field $\vec{H}$ along the z axis is written as

$$\hat{\mathscr{H}}=-\sum_{i=1}^{N}\hat{\vec{\mu}}^{(i)}\cdot\vec{H}=-\frac{g\mu_0}{\hbar}H\hat{S}_z, \tag{5}$$

where we have introduced the elementary magneton, $\mu_0=e\hbar/2mc$, and the collective operator $\hat{\vec{S}}=\Sigma_{i=1}^{N}\hat{\vec{S}}^{(i)}$ for the total spin. The eigenvectors of $\hat{\vec{S}}^2$ and $\hat{S}_z$, which constitute a basis of the concomitant 2^N-dimensional space, are labeled as $|S,M\rangle$ with $S=\delta,\ldots,N/2$, $M=-S,\ldots,S$, and $\delta\equiv N/2-[N/2]=0\,(1/2)$ if N is even (odd). The corresponding multiplicities are $Y(S,M)=Y(S)=N!\,(2S+1)\,/\,[(N/2-S)!\,(N/2+S+1)!]$.

Within the framework of a generalized statistics of index q, the mean magnetic moment of the system at temperature T is given by

$$\mathscr{M}_q\equiv\frac{g\mu_0}{\hbar}\,\langle\hat{S}_z\rangle_q=\frac{g\mu_0}{\hbar}\,\mathrm{tr}(\hat{\rho}^{\,q}\,\hat{S}_z). \tag{6}$$

The statistical operator $\hat{\rho}$ is obtained by extremalization of $S_q(\hat{\rho})$ subject to the normalization condition $\mathrm{tr}(\hat{\rho})=1$, and to the assumed knowledge of $\langle\hat{\mathscr{H}}\rangle_q$. From Eqs. (3)–(5) we arrive at

$$\hat{\rho}=\frac{1}{Z_q}\left[1+\beta(1-q)\frac{g\mu_0}{\hbar}H\hat{S}_z\right]^{1/(1-q)} \tag{7}$$

with $\beta=1/kT$ and the partition function defined as

$$Z_q=\mathrm{tr}\left(\left[1+\beta(1-q)\frac{g\mu_0}{\hbar}H\hat{S}_z\right]^{1/(1-q)}\right). \tag{8}$$

We easily verify that the $q\to1$ limit yields the standard, Boltzmann-Gibbs, exponential form for the density operator. The magnetization $\mathscr{M}_q$ as derived from the function Z_q reads [5]

$$\mathscr{M}_q=\frac{1}{\beta}\frac{\partial}{\partial H}\left(\frac{1-Z_q^{\,1-q}}{q-1}\right). \tag{9}$$

It is to be remarked that, on computing expectation values, those states that *do not* satisfy the condition

$$[1+\beta(1-q)g\mu_0HM]>0 \tag{10}$$

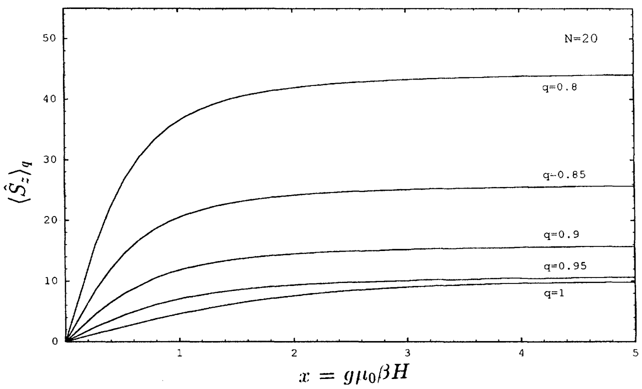

FIG. 1. $\langle\hat{S}_z\rangle_q$ as a function of x.

must be excluded from the summation implied in the trace. In other words, these states are assigned a probability amplitude $p_{S,M}=\langle S,M|\hat{\rho}|S,M\rangle\equiv0$, so that $\hat{\rho}$ is positive definite. The physical origin of this cutoff condition has been given in Ref. [8]. Let us mention here that an analogous situation is present in quantum special relativity: the κ parameter characterizing the deformation of the Poincaré group could be thought of as the upper limit of the energy of a particle [2].

We consider now the magnetic behavior of the system as a function of (i) the number of particles, (ii) the index q (for $q>0$), and (iii) the dimensionless parameter $x\equiv g\mu_0\beta H$. In Figs. 1 and 2 we show the shape of the q magnetization (in units of $g\mu_0/\hbar$), $\langle\hat{S}_z\rangle_q$, as a function of x, when N and q assume different values. The study of the asymptotic cases $H\ll kT$ and $H\gg kT$ allows for a simple treatment of the effects we wish to describe here. A weak external magnetic field or high temperatures correspond to the limit $x\to0$. In this case, expanding $(p_{S,M})^q$ up to first order in x and taking an appropriate trace over the whole state space [if x is sufficiently small, condition (10) is verified for all M], one finds

$$\langle\hat{S}_z\rangle_q\simeq\frac{1}{2^{Nq}}\,q\,x\sum_{M=-N/2}^{N/2}C^{N}_{N/2-M}\,M^2, \tag{11}$$

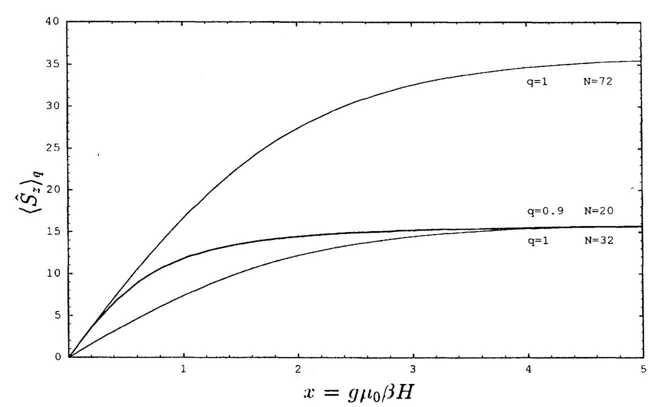

FIG. 2. $\langle\hat{S}_z\rangle_q$ vs x for $q=0.9$ and $N=20$, compared with its "standard analogs." The slope of $\langle\hat{S}_z\rangle_1$ at $x=0$ is proportional to N, and its saturation value at $x=\infty$ is $N/2$. When $q\neq1$, we define the slope and saturation value to be proportional to N^0_{eff} and N^∞_{eff}, respectively.

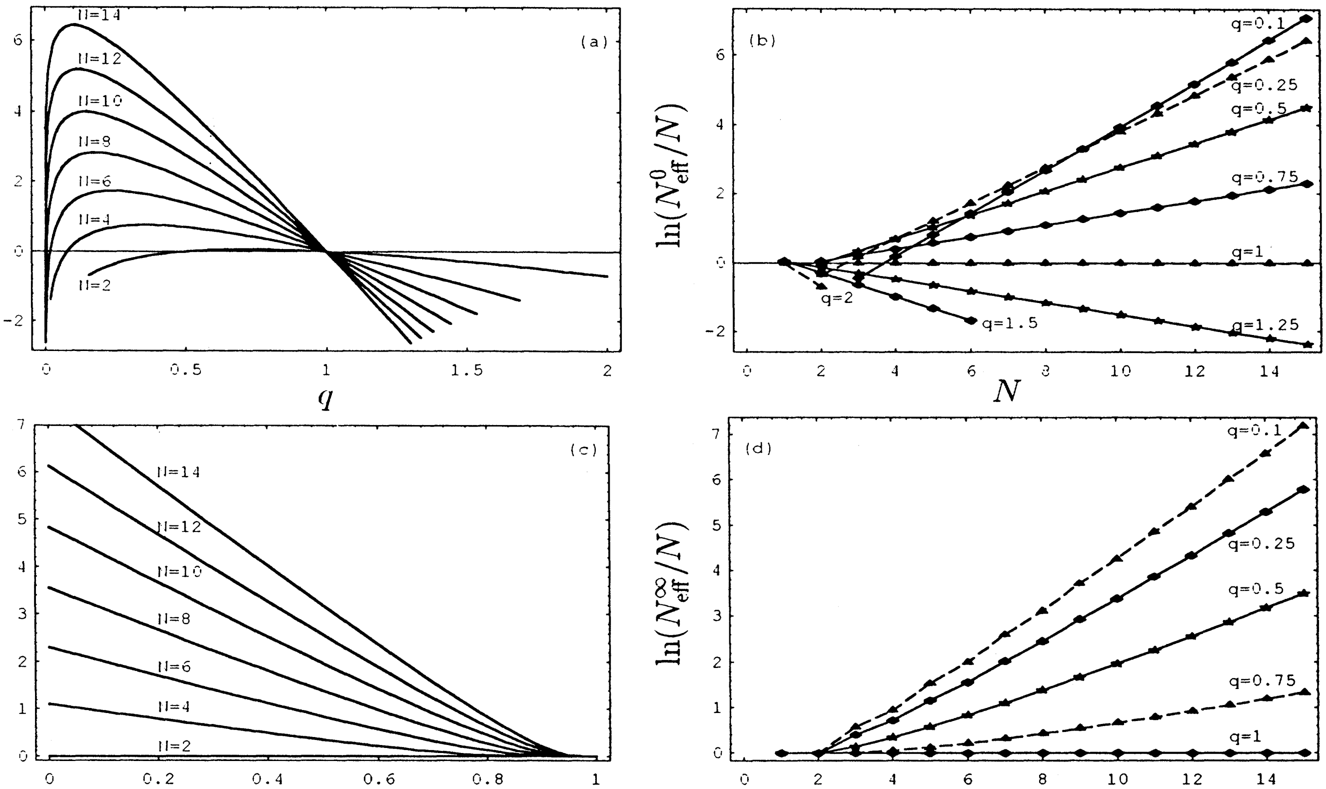

FIG. 3. $\ln(N^0_{\rm eff}/N)$ as a function of q (a) and N (b) (the condition $N^0_{\rm eff} \geqslant 1$ has been imposed); $\ln(N^\infty_{\rm eff}/N)$ versus q (c) and N (d).

where C^N_n denotes the usual binomial coefficient. From this approximation we calculate the generalized isothermic magnetic susceptibility

$$\chi_q(N;T) \equiv \lim_{H\to 0}\left(\frac{\partial \mathscr{M}_q}{\partial H}\right)_T = \frac{(g\mu_0)^2}{4\hbar kT}\, N q 2^{N(1-q)}, \tag{12}$$

which for $q=1$ is simply proportional to the number of particles, N. For $q \neq 1$, we define the (low H/kT) effective particle number $N^0_{\rm eff}$ by means of the identification

$$\chi_q(N;T) \equiv \chi_1(N^0_{\rm eff};T)$$

(see Fig. 2) so that

$$N^0_{\rm eff}(q,N) = Nq\ 2^{N(1-q)}. \tag{13}$$

In Figs. 3(a) and 3(b) we plot $\ln(N^0_{\rm eff}/N)$ as a function of q and N, respectively, for various particle numbers and q parameters (the condition $N^0_{\rm eff} \geqslant 1$ has been imposed). If $N>1$ and $q \in (0,1)$ are such that $N\ln 2 \gtrless -\ln(q)/(1-q)$, then the effective size of the system will be $N^0_{\rm eff} \gtrless N$. Therefore, for N and q, $q<1$, not very small, the apparent number of spins is *larger* than the actual one. The function $\ln(N^0_{\rm eff}/N)$ exhibits a maximum at $q = 1/(N\ln 2)$ where $N^0_{\rm eff} \approx 2^N$ if $N \gg 1$ [Fig. 3(a)]. Meanwhile, the nonextensive, subadditive theory obtained when q exceeds 1 gives the illusion of a number of spins *smaller* than N.

The other extreme situation, that of a high magnetic field or low temperatures, corresponds to $x \to \infty$. In this case and assuming $q \in (0,1)$, in the computation of the traces the only states which contribute are those with $M>0$, for each $S = \delta, \ldots, N/2$. We compare the saturation value of the magnetization (in units of $g\mu_0/\hbar$),

$$\lim_{x\to\infty} \langle \hat{S}_z \rangle_q = \left(\sum_{M=\delta}^{N/2} C^N_{N/2-M}\, M^{1/(1-q)} \right)^{1-q} \tag{14}$$

with the corresponding one for $q=1$, which is simply

$$\lim_{x\to\infty} \langle \hat{S}_z \rangle_1 = \frac{N}{2}. \tag{15}$$

We therefore introduce the (high H/kT) effective particle number $N^\infty_{\rm eff}$ in the following way (see Fig. 2):

$$N^\infty_{\rm eff}(q,N) \equiv 2\left(\sum_{M=\delta}^{N/2} C^N_{N/2-M}\, M^{1/(1-q)} \right)^{1-q}. \tag{16}$$

For every fixed q, $0<q<1$, it can be seen that $N^\infty_{\rm eff}(q,N) = N$, if and only if $N = 1$ or 2; moreover, the system seems to be *larger* if $N \geqslant 3$. We depict in Figs. 3(c) and 3(d) the function $\ln(N^\infty_{\rm eff}/N)$ vs q and N, respectively. [The case $q>1$ in this limit is of no interest. The sum in Eq. (14), now to run between $M = -N/2$ and $M = -\delta$, converges when $q \to 1^+$ to -1 $(-1/2)$ for N even (odd). Then, no effective particle number is defined in this situation.]

We mention that there exist pairs of values (q,N) for which the system, as described by a generalized statistics, has the same "standard analog" in both limits, with an effective number of particles $N^0_{\rm eff}(q,N)=N^\infty_{\rm eff}(q,N)$. In the intermediate region $H/kT \sim 1$, however, the q magnetization can differ from $\mathscr{M}_1$.

Summing up, we have shown that the effective size of the system *is not independent of the statistical averaging procedure*. Here, different q statistics yield different magnetizations. An observer that measures the magnetization would make an estimation of the number of particles involved that could be quite wrong, if she or he assumes a given value for q that is not the one appropriate to the environmental circumstances that govern the associated physical process. Of course, if $q=1$, no problems arise. But *q might be different from 1*. It has been shown that in systems where long-range interactions (like the gravitational forces) are present, q could be significantly lower than unity [8]. Is it perhaps conceivable that statistical factors are involved in the dark matter paradoxes? The present contribution shows that a sort of "dark magnetism" is indeed conceivable. As a final remark, the whole picture that has emerged here is so similar to that exhibited by Bacry [1], that the possible connection that has been recently advanced [6] between quantum groups and nonextensive statistical mechanics comes out reinforced. In particular, it is worth stressing that, in both formalisms, the internal energy is generically nonextensive. In the same spirit, we recall two other important facts, namely that the laws of additivity of spins considered here and for particles at rest in the deformed Poincaré group agree, and also that cutoffs appear naturally in both treatments.

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