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Modified method for synchronizing and cascading chaotic systems

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In this contribution a modification of the Pecora-Carroll [Phys. Rev. Lett. **64**, 821 (1990)] one-way (or drive-response) synchronization method is suggested, such that both drive and response have the same dimensionality. As a result, it is possible reproduce the driving signal with a single connection, increasing, thus, the number of potential connections of a given system. The main features of the method presented in this work are discussed with an application to the Rössler and Lorenz models [O. E. Rössler, Phys. Lett. A **57**, 397 (1976); E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963)], including the possibility of designing different chaotic receivers to be used in the field of secure communications and the setup of an array of chaotic units in which several possible connections are allowed for.

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Several recent studies have shown the possibility of synchronizing chaotic systems [1,2], although, due to their sensitive dependence on the initial conditions, at first sight this may appear to be against common sense. In particular, Pecora and Carroll (PC) [1] have considered the situation of unidirectional coupling, in which a chaotic signal from a drive system is used to force a second response system. In the PC method a subsystem of the drive, that will be common between the two systems, will be used to make the response synchronize with the drive. The stability of the synchronized state can be written in terms of the corresponding transverse Lyapunov exponents, which measure the growth of small perturbations of the differences between the two systems.

A very interesting situation [3] is obtained when the response system, with respect to a given drive, acts as the drive of a second response system, this connection being called cascading. Cascading two PC subsystems in this way allows one to regenerate the driving input signal. It is in this sense that it is possible to show [3] that this system can act as a chaotic filter, potentially useful in the field of secure communications [4] (see, however, Ref. [5]). In its simplest version the idea is to use a chaotic signal to mask the information to be transmitted, the latter bearing a small fraction of the power spectrum, and this is an alternative to the classical noise masking procedure. The receiver should have the appropriate chaotic filter, which in this setting can be obtained by using two cascaded chaotic subsystems.

Cuomo *et al.* [4] have implemented a Lorenz chaotic filter that could be useful in the field of secure communications. These authors were able to design a receiver circuit that is a single three-dimensional chaotic circuit comprising the cascade $\xrightarrow{x}(y',z')\xrightarrow{y'}(x'',z'')$, which has, in principle, an overall dimension of four. The device is implemented by noticing that variable z'' in the second subsystem does not influence x'', but rather performs its own dynamics, or in other words, variable z does not enter into $\dot{x}$. The result is that one can write the usual expression for a Lorenz [6] chaotic circuit acting as the drive (transmitter),

$$\dot{x}_1=\sigma(y_1-x_1),\ \ \dot{y}_1=Rx_1-y_1-x_1z_1,\ \ \dot{z}_1=x_1y_1-bz_1, \quad (1)$$

while the response (receiver) circuit can be written in the form [4],

$$\dot{x}_2=\sigma(y_2-x_2),\ \ \dot{y}_2=Rx_1-y_2-x_1z_2,\ \ \dot{z}_2=x_1y_2-bz_2. \quad (2)$$

The parameters have the usual meaning, and it is to be noticed that the driving signal x_1 appears in $\dot{y}_2$ and $\dot{z}_2$, while x_2 appears in $\dot{x}_2$, and, thus, one has some kind of *hybrid* dynamical system.

One could also think of designing a more general cascade acting as a nonlinear information processing unit, but this is difficult to achieve if one strictly sticks to the PC method. The reason is that in a cascade of low-dimensional chaotic systems the connectivity is limited because not all the possible subsystems of a given chaotic system are stable from the viewpoint of synchronization, and in order to build such a cascade any two contiguous connections must be different [3].

Thus, and within the examples considered in Ref. [1], in the case of the Rössler [7] model there is a single stable subsystem, namely (x,z), while in the case of the Lorenz [6] model both (x,z) and (y,z) subsystems are stable. Thus, the Rössler system cannot be a candidate to build a chaotic filter, for which one needs a cascade such that the driving signal is regenerated (although one could obtain the same effect through the use of a suitable modification of the method [8]). Moreover, if one wishes that the chaotic filter is compact, i.e., that it has the same dimensionality as the drive, such as is the case of the Lorenz circuit of (2), it is necessary that one of the variables (z in the case of the Lorenz system) does not appear in the evolution of the input variable to the cascade [x in (2)].

The aim of this work is to introduce a strategy consisting in a generalization of the PC method, that should allow one to design in a systematic way a response system with the same dimension as the drive and that yields the same result obtained within the original PC method with a cascade of two subsystems. One of the most interesting potential appli-

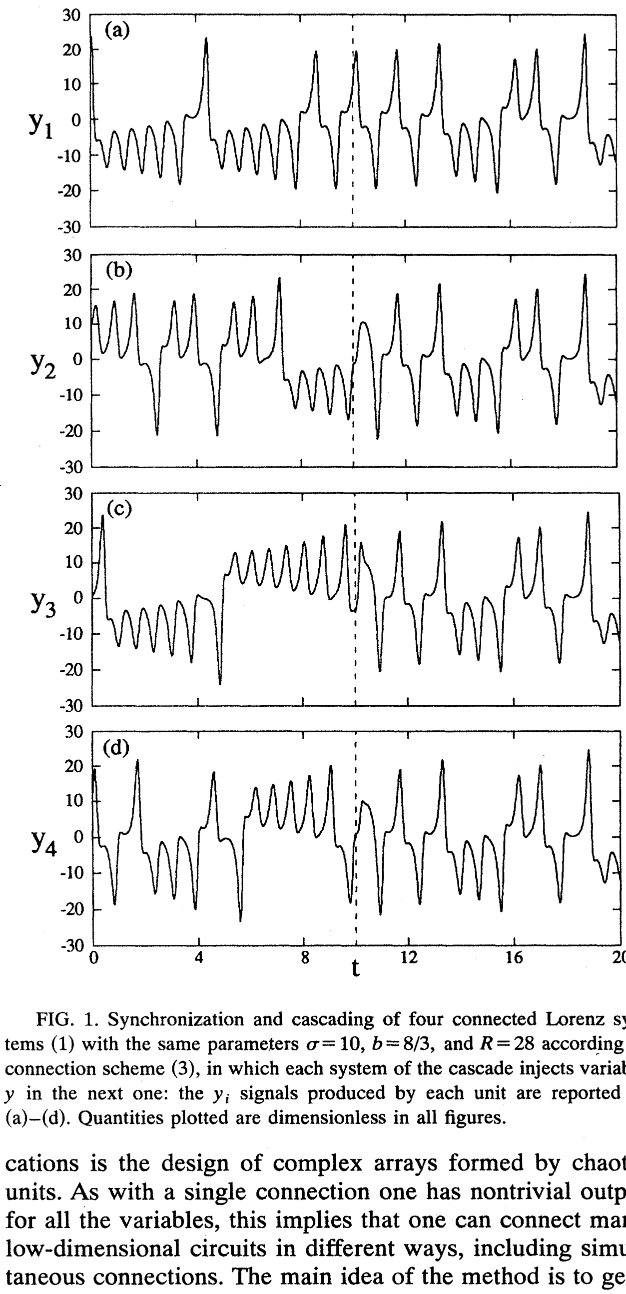

FIG. 1. Synchronization and cascading of four connected Lorenz systems (1) with the same parameters $\sigma=10$, $b=8/3$, and $R=28$ according to connection scheme (3), in which each system of the cascade injects variable y in the next one: the y_i signals produced by each unit are reported in (a)–(d). Quantities plotted are dimensionless in all figures.

cations is the design of complex arrays formed by chaotic units. As with a single connection one has nontrivial output for all the variables, this implies that one can connect many low-dimensional circuits in different ways, including simultaneous connections. The main idea of the method is to generalize the coexistence of different signals in the same chaotic circuit, allowing the driving signal to enter at one or more terms in the evolution equations of the response, as happens for x_1 and x_2 in the circuit described by (2). Thus, the suggestion is to introduce the driving signal in a single term of the evolution equations of the response (although more terms can be also considered), e.g., $\sigma\, y_1(t)$ in the connection reported in (3). It is possible to find some analogy between this kind of construction and the suggestion in Ref. [9] of splitting the response system into two parts, one linear and the other nonlinear, entering the drive signal only in the latter.

In order to understand better the basic idea of the method, it will be illustrated in the case that several Lorenz systems (1) are connected through variable y. By looking at the equations it is possible to see that y enters generically at four different places in the evolution equations. If one drives a receiver circuit injecting y_1 in the term corresponding to the $\dot{x}_2$ equation, then it is possible to show that the connected systems will indeed synchronize. Figure 1 shows the case of a cascade of four Lorenz models, described by the following equations, where the signal that is driving the dynamics of each response system and that comes from the corresponding drive is underlined for clarity (also in the rest of the text):

$$\dot{x}_1=\sigma(y_1-x_1),\quad \dot{y}_1=Rx_1-y_1-x_1z_1,\quad \dot{z}_1=x_1y_1-bz_1,$$

$$\dot{x}_2=\sigma(\underline{y_1(t)}-x_2),\quad \dot{y}_2=Rx_2-y_2-x_2z_2,\quad \dot{z}_2=x_2y_2-bz_2,$$

$$\dot{x}_3=\sigma(\underline{y_2(t)}-x_3),\quad \dot{y}_3=Rx_3-y_3-x_3z_3,\quad \dot{z}_3=x_3y_3-bz_3,$$

$$\dot{x}_4=\sigma(\underline{y_3(t)}-x_4),\quad \dot{y}_4=Rx_4-y_4-x_4z_4,\quad \dot{z}_4=x_4y_4-bz_4, \tag{3}$$

the four systems exhibiting perfectly synchronized behavior.

The stability of a drive-response couple in the synchronized state can be formulated in a quite general way by studying the stability of the synchronization manifold ($x_2=x_1$, $y_2=y_1$, and $z_2=z_1$) against perturbations transverse to the manifold. The stability condition implies that all possible perturbations die off, which implies that for small perturbations the (linearized) evolution equation for $\mathbf{e}=(e_1,e_2,e_3)=(x_2-x_1,y_2-y_1,z_2-z_1)$ has eigenvalues whose real part is negative. For the connection given in (3) it can be shown that the following linearized equation for the time evolution of the relative errors between any two contiguous systems is obtained:

$$\dot{\mathbf{e}}=\begin{pmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{pmatrix}=\begin{pmatrix}-\sigma & \underline{0} & 0\\ (R-z) & -1 & -x\\ y & x & -b\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}=\mathbf{Z}\,\mathbf{e}\ . \tag{4}$$

This is analogous to the error growth equation for the Lorenz system, except for the fact that a $\underline{0}$ entry appears in the same place in which the driving signal enters.

Due to the nonlinear character of the vector flow that one tries to synchronize, the $\mathbf{Z}$ matrix in (4) does not have constant coefficients, and, thus, the discussion must be done in terms of Lyapunov exponents, which in this work have been obtained by using the method of Ref. [10]. In this context, the idea of the method presented here is to set to zero the appropriate terms in $\mathbf{Z}$ such that one only gets negative trans-

TABLE I. Transverse Lyapunov exponents for various connections of two Lorenz and Rössler systems using the method suggested in this work. When the connection is explicitly reported in the text, the corresponding equation number is given.

System	Connection	Lyapunov exponents
Lorenz: $\sigma=10$	$\sigma\underline{y(t)}$, Eq. (3)	$(-1.8003,-1.8663,-10.0000)$
$b=8/3$, $R=28$	$R\underline{x(t)}$	$(-3.9513,-4.0420,-5.6734)$
	$-x\underline{z(t)}$	$(0.0000,-2.6667,-11.0000)$
Rössler: $a=b=0.2$, $c=4.6$	$a\underline{y(t)}$, Eq. (5)	$(-0.0583,-0.1243,-4.2432)$

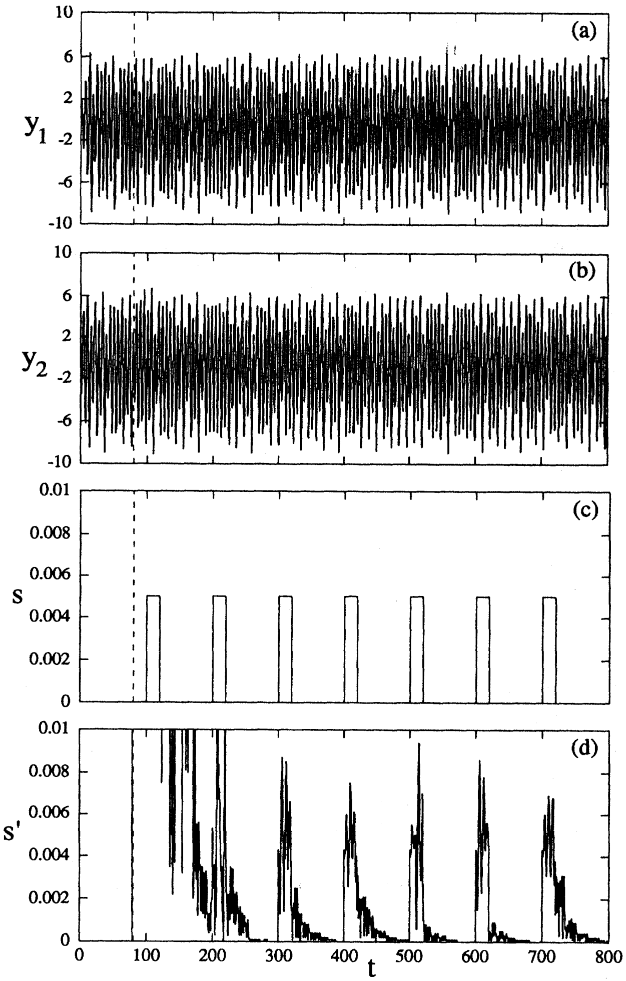

FIG. 2. Synchronization of two connected Rössler systems [7] with the same parameters $a=0.2$, $b=0.2$, and $c=4.6$, according to connection scheme (5) in which $y_1(t)$ is injected in the response system: (a) y_1 signal from the driver (transmitter) system including a square pulse $s(t)$ shown in (c); (b) induced y_2 signal in the response (receiver); (c) square pulse signal $s(t)$ added to y_1 in (a); (d) retrieved signal $s'(t)=|y_2-y_1|$ as filtered by the response.

verse Lyapunov exponents, although it is very difficult to know *a priori* which connection will synchronize. All the nonpositive transverse Lyapunov exponents for connections between two identical Lorenz and Rössler systems are gathered in Table I.

As a second application, a connection that may act as a chaotic filter based on Rössler's model of spiral chaos [7] has been set up. The original PC synchronization method yields only one stable subsystem, while within the present method two Rössler systems will synchronize if y is injected in the ay term of $\dot{y}$ in the receiver,

$$\dot{x}_1=-(y_1+z_1),\ \dot{y}_1=x_1+ay_1,\ \dot{z}_1=b+z_1(x_1-c),$$

$$\dot{x}_2=-(y_2+z_2),\ \dot{y}_2=x_2+a\underline{y_1(t)},\ \dot{z}_2=b+z_2(x_2-c). \quad (5)$$

The application to the design of a chaotic filter for y potentially useful in the field of secure communications can be seen from Fig. 2, in which a step function indicates how one could transmit information in digital form (codified in the form of 0*s* and 1*s*). Thus, Fig. 2(c) represents the signal $s(t)$ added to the drive, while in Fig. 2(d) the retrieved signal $s'(t)$ (that appears to be significatively different from the rest state) is shown. From the form in which the method is implemented it can be shown that, if the signal added to the chaotic drive does not vary too rapidly in a time compared to the highest transverse Lyapunov exponent, the response system will synchronize with the sum of both signals. Thus, Fig. 2(d) represents only a transient behavior under the presence of the perturbation. Thus, in order that one can retrieve the information in a significative way, the duration in time of the pulses to be transmitted is limited by the highest Lyapunov exponent: the lower it is in absolute value the longer the pulse can be.

One of the most interesting potential applications of the present connection scheme is to arrays of chaotic systems, because it is possible to obtain a highly complex network in which several connections among the different units coexist. This is one of the features observed in the brain, and these complex networks could act as useful nonlinear information

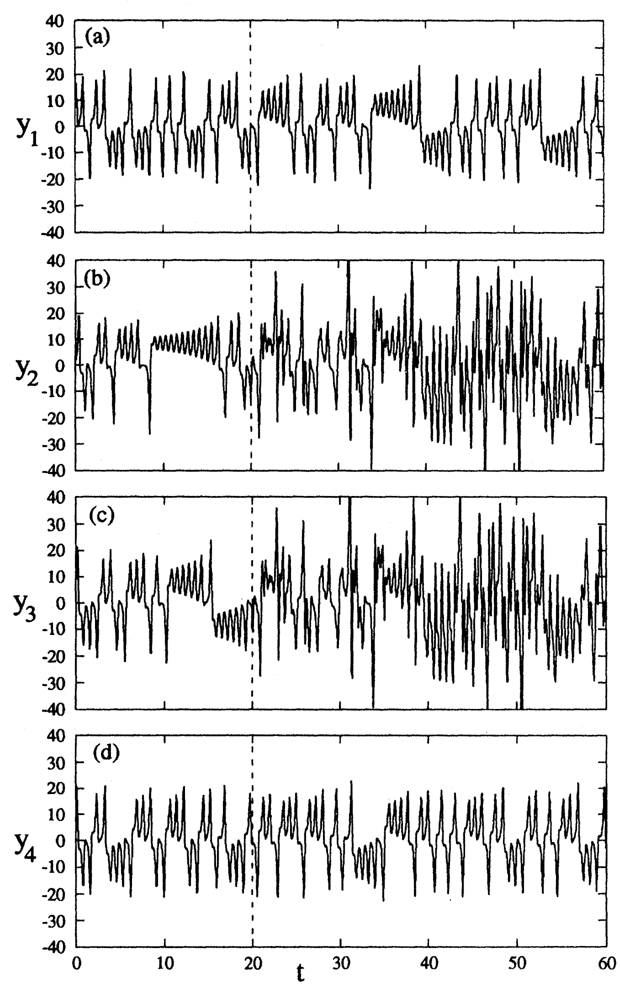

FIG. 3. Connection of four chaotic Lorenz systems according to scheme (6) shown in (a)–(d) (see Fig. 1 for the parameters). System in (a) is linked through connection $R\ x$ to systems in (b) and (c), while system in (d) is linked through connection $-x\ z$ to systems in (b) and (c). Systems in (b) and (c) exhibit mutual synchronization, although not with (a) and (d).

processing units, that could eventually outperform classical fixed point artificial neural networks. In Fig. 3 one such setting is shown, namely, one in which four Lorenz systems are connected in the form

$$\begin{aligned}
&\dot{x}_1=\sigma(y_1-x_1),\ \dot{y}_1=Rx_1-y_1-x_1z_1,\ \dot{z}_1=x_1y_1-bz_1,\\
&\dot{x}_2=\sigma(y_2-x_2),\dot{y}_2=R\underline{x_1(t)}-y_2-x_2\underline{z_4(t)},\ \dot{z}_2=x_2y_2\dot{x}-bz_2,\\
&\dot{x}_3=\sigma(y_3-x_3),\ \dot{y}_3=R\underline{x_1(t)}-y_3-x_3\underline{z_4(t)},\ \dot{z}_3=x_3y_3-bz_3,\\
&\dot{x}_4=\sigma(y_4-x_4),\ \dot{y}_4=Rx_4-y_4-x_4z_4,\ \dot{z}_4=x_4y_4-bz_4.
\end{aligned}\tag{6}$$

Thus, the first system is linked through connection $R\ \underline{x(t)}$ to the second and third systems, while the fourth system is linked through connection $-x\underline{z(t)}$ to the second and third systems. This is an example in which two systems receive, each, signals from two other systems, and then, some kind of *competition* may occur regarding the effects of the two signals. The result is that the second and third systems become synchronized one to each other, exhibiting a different behavior than the first and fourth systems, although they are not connected directly. One could analyze the stability of this kind of setting by a generalization of the stability analysis outlined before Eq. (4), although the calculations are more complex now due to the fact that two different chaotic systems need to be followed.

The conclusion of this work is that the method put forward by Pecora and Carroll (PC) [1], and that allows synchronization of chaotic systems by one-way coupling, can be easily extended to the case in which there is no subsystem in common between the two connected systems. This can be achieved by introducing a signal from the driver into a precise term of the response, allowing regeneration of the input signal with a single connection. Among the possible uses of this variant of the original method, one has the design of chaotic filters potentially useful in the field of secure communications in which a deterministic signal to be transmitted is masked with a chaotic signal produced by a chaotic system. One could design the receiver system to be formally identical to the transmitter, except for the fact that the compound signal to be filtered is introduced at some given place of the circuit. This allows a compact design of the receiver unit without the need of designing it as a cascading of two systems as in the PC scheme. In this work, a chaotic filter using two Rössler units is suggested, making it possible to use for this purpose a system with a single stable subsystem in the PC sense. The practical implementation of the present idea in a circuit such as the one devised by Cuomo *et al.* [4] would consist in the injection of the driving signal in the appropriate part of the circuit through an operational amplifier (the use of a resistance would yield mutual driving [11]).

Another use of the present suggestion is the possibility of cascading a large number of low-dimensional systems with different possible connections without reducing the dimensionality of the response systems, including discrete maps. This kind of connection can be applied to the case where the units represent model neurons. One of the features of the brain is that a large number of neurons are connected in many different ways, and its computation power appears to stem from this property. The method introduced in this work presents the advantage of allowing one to set up a network with a virtually unlimited number of circuits that are connected in many different ways. Thus, these arrays of chaotic systems might be useful as information processing systems that would work by synchronizing one to each other, mimicking the behavior observed in physiological studies.

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