RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 52, NUMBER 2 AUGUST 1995

Unusual universality of branching interfaces in random media

Mehran Kardar,[1] Attilio L. Stella,[2] Giovanni Sartoni,[3] and Bernard Derrida[4,5]

[1]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

[2]*Dipartimento di Fisica and Sezione Istituto Nazionale di Fisica Nucleare, Università di Padova, 35131 Padova, Italy*

[3]*Dipartimento di Fisica and Sezione Istituto Nazionale di Fisica Nucleare, Università di Bologna, 40126 Bologna, Italy*

[4]*Laboratoire de Physique Statistique, Ecole Normale Supérieure, 24 rue Lhomond, F-75005 Paris, France*

[5]*Service de Physique Théorique, Centre d' Etudes de Saclay, F-91191 Gif-sur-Yvette, France*

(Received 14 November 1994)

We study the criticality of a Potts interface by introducing a *froth* model which, unlike its solid-on-solid Ising counterpart, incorporates bubbles of different phases. The interface is fractal at the phase transition of a pure system. However, a position space approximation suggests that the probability of loop formation vanishes marginally at a transition dominated by *strong random bond disorder*. This implies a linear critical interface, and provides a mechanism for the conjectured equivalence of critical random Potts and Ising models.

PACS number(s): 05.70.Jk, 68.35.Rh, 75.10.Hk, 82.65.Dp

The effect of quenched impurities on phase transitions is important and quite fascinating. A simple "Harris criterion" [1] indicates that critical behavior is modified by (bond) randomness in systems with a positive heat capacity exponent. Impurities can also change first order transitions to second order [2], to the extent that in two dimensions there are no discontinuous phase transitions [3]. There is a growing body of numerical [4] and experimental [5] evidence that, at least in some situations, the asymptotic criticality is similar to the random bond Ising model, *irrespective of the underlying symmetry*. Here we provide some justification for this observation based on an unexpected universality of the critical interface in the presence of strong bond randomness.

Interfaces are key to second order phase transitions; the interfacial free energy vanishes at criticality with the Widom exponent of μ. The behavior of Ising interfaces has been extensively studied by a solid-on-solid (SOS) model which simplifies numerical and theoretical analysis [6]. Potts models also have a potential continuous transition with broken discrete symmetry [7]. Potts interfaces are harder to study as they are complicated branching objects which become fractal at criticality. We introduce several approximations to such interfaces in $1+1$ dimensions, eventually arriving at a model that is amenable to a position space renormalization group (RG) treatment; exact on a hierarchical lattice. The simplified model also allows us to examine interfaces of random bond Potts models. This leads to an intriguing generalization of directed polymers in random media [8], with potential applications beyond those discussed here.

Consider the interface between two distinct phases of an ordered q state Potts model. At very low temperatures the interface is a weakly fluctuating surface, much as in the Ising case. However, on approaching the critical point, bubbles of any of the other $q-2$ phases may appear at the interface, in addition to islands in the bulk phases. As in SOS models of Ising interfaces, we shall ignore isolated islands and overhangs. The resulting *froth* is a collection of bubbles, each bounded by two SOS surfaces. Even this simplified model is too hard to analyze: The allowed configurations in $1+1$ dimensions are a subset of those encountered in directed percolation (no dangling branches), with weights depending on q. To make the problem tractable, we confine the interface to the bonds of a diamond hierarchical lattice (DHL) (inset of Fig. 1). In one iteration on this lattice the interface can cross either of two branches, or create a bubble by going through both. The bubble has a "fugacity" of $q-2$, the number of possible intermediate phases. This procedure is repeated iteratively creating the possibility of loops within loops ad infinitum. Not all configurations of the original froth model (those with multiplicities that are not powers of $q-2$) are included within this scheme. It is possible to construct more complicated models and RG schemes without these deficiencies; for example by considering the diamond plus diagonal hierarchical lattice (DDHL) in Fig. 1. We have checked that the qualitative results are unchanged by the choice of RG scheme, and will thus focus on the simpler DHL where the recursion relation for the interface partition function is

$$Z_{n+1}=2Z_n^2+(q-2)Z_n^4. \tag{1}$$

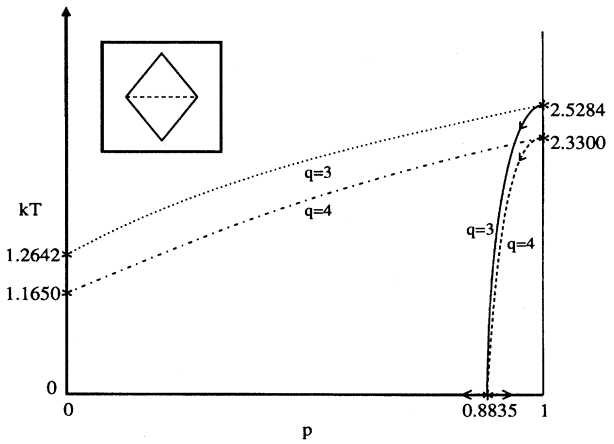

FIG. 1. Phase boundaries for $q=3$ and 4, with mixtures of positive and negative bonds (solid, dashed), and positive bonds (dotted, dashed dotted). Arrows indicate RG flow towards the $T=0$ fixed point. The solid lines in the inset figure indicate the cell replacing each bond at every stage of the DHL construction. The central (dashed) diagonal is also present in the DDHL.

TABLE I. The exponent $\mu(q)$ for the vanishing of the interfacial free energy, obtained from the DHL and the DDHL, compared to the exact values in $d=2$.

q	DHL	DDHL	Exact
1	1.634	1.429	4/3
2	1	0.847	1
3	0.886	0.749	5/6
4	0.830	0.700	2/3

Here $Z_1=\exp(-2\beta J)$ is the Boltzmann weight of one broken bond and $\beta=1/(k_BT)$.

The above recursion relation has a stable fixed point at $Z^*=0$. For initial values flowing to this point $Z_n \propto \exp[-Lf_s]$, where $L=2^n$ is the length of the lattice after n iterations and f_s is the *interfacial free energy*. There is another stable sink with $Z^*\to\infty$ where $Z_n\propto\exp[-L^2f_b]$. This phase is a dense foam of bubbles where the interface analogy breaks down. These two phases are separated by an unstable fixed point at a *finite* $Z^*(q)$. Clearly any finite fixed point for Z corresponds to $f_s=f_b=0$. A similar fixed point mechanism for the vanishing of the surface energy is present in wetting phenomena [9]. Denoting $t=Z_1-Z^*(q)$, we find $t'=2^{y(q)}t$ with $y(q)=2+\ln[1-Z^*(q)]/\ln 2$. Since the interface free energy satisfies the homogeneity condition $f_s(t)=b^{-1}f_s(b^{y(q)}t)$, it vanishes as t^μ with $\mu(q)=1/y(q)$. The exponents obtained from the DHL and DDHL RG schemes are compared with the exactly known values for the Potts model in $d=2$ in Table I. The exponent is (accidentally) exact for $q=2$ on the DHL, and shows the correct trend on increasing q. However, this approach does not show the expected change to first order transitions at $q=4$ [7]. The critical interface is *fractal*, its mass (number of occupied bonds) growing as $M=d\ln Z/d\ln Z_1\propto L^{d_f}$ with $d_f=y(q)$. The probability of forming a loop is *independent* of scale and given by $P=2^{d_f-1}-1$.

As shown by Hui and Berker [2], randomness in the couplings of a Potts model may change first order transitions to second order. In particular, there can be no discontinuous symmetry breaking in *two dimensions* [2,3], and all disordered Potts models have continuous phase transitions. This opens up the possibility of an infinite set of new universality classes. A numerical study [4] of the eight-state Potts model has confirmed the second order nature of the transition. Intriguingly, based on their simulations, the authors of Ref. [4] conjecture that all these new universality classes are in fact similar to the random bond Ising ($q=2$) model. There is also some experimental support for this conjecture from the effects of oxygen impurities on the phase transition of $(2\times2)-2H$ on Ni(111). The pure system is a realization of the $q=4$ Potts model, while in the presence of impurities the asymptotic behavior appears to be Ising-like [5]. This contradicts other evidence based on RG schemes [10,11] that, at least for weak disorder, the critical exponents do depend on q. In fact, the latter RG scheme [11] was partly motivated by an earlier work [12] suggesting that the (nonuniversal) critical line of the pure Baxter model shows Ising behavior in the presence of randomness.

To gain further insight into random bond criticality, we examine the singular behavior of the interfacial energy in the framework developed earlier. The initial weights $\{Z_1(i)\}$ for the bonds of the DHL are chosen randomly from a probability distribution $\mathscr{P}_1(Z)$. The recursion relation for a specific realization of bonds,

$$Z_{n+1}=Z_n(1)Z_n(2)+Z_n(3)Z_n(4)+(q-2)Z_n(1)Z_n(2)Z_n(3)Z_n(4), \quad (2)$$

can be used to construct a functional recursion relation for $\mathscr{P}_n(Z)$. We use a binary initial distribution of energies with a fraction p of positive bonds J_1, and $1-p$ of positive *or negative* bonds J_2, i.e.,

$$\mathscr{P}_1(Z)=p\,\delta(Z-e^{-2\beta J_1})+(1-p)\,\delta(Z-e^{-2\beta J_2}). \quad (3)$$

Under iteration, the simple fixed points of the pure system are replaced by stable distributions: The analog of $Z^*=0$ describes directed paths on the hierarchical lattice and is governed by the distribution discussed in Ref. [13]. This distribution for $\ln Z$ is characterized by a mean that shifts as $-f_sL\ll 0$ and a width that grows as L^ω with $\omega\approx 0.30$ (compared to the exact value of $\omega=1/3$ on a two dimensional lattice [8]). In the dense foam phase with $\overline{\ln Z}=-f_bL^2\gg 0$, the central limit theorem should apply and we expect fluctuations of $\ln Z$ to grow as L. An argument similar to the Harris criterion [1] shows that the relevance of randomness at the critical fixed point is determined by the sign of $y(q)-1$. Thus any randomness is expected to modify the interfacial criticality of Potts models with $q\geq 2$. By analogy with the other two limiting distributions, we expect the critical point to flow towards a third zero temperature stable distribution with $\overline{\ln Z}=0$, and $\delta\ln Z\propto L^{\omega_c}$.

These expectations are consistent with numerical iterations of the recursion relation starting from a large initial ensemble of $\{Z_1(i)\}$ [14]. The RG flows are towards zero temperature, leading to the phase diagrams indicated in Fig. 1 for $J_2=-J_1$ (and $J_2=+J_1/2$) for $q=3$ and 4. The choice of an initial distribution with a fraction of *negative bonds* ensures that the phase boundary extends to zero temperature [15]. Since the eventual fixed distributions are at zero temperature, criticality can be examined by directly looking at the recursion relations at $T=0$. With this enormous simplification, we have iterated energies,

$$E_{n+1}=\min\{E_n(1)+E_n(2),E_n(3)+E_n(4),E_n(1)+E_n(2)+E_n(3)+E_n(4)\}. \quad (4)$$

The last term is of course absent for $q=2$. We could iterate exactly the probability distribution for bond energies up to $n=7$. The Monte Carlo iteration was typically extended up to $n=20$ for our determinations of the $T=0$ critical properties. It is not immediately apparent from Fig. 1 that the phase boundary for an initial mixture of ferromagnetic bonds is also governed by a $T=0$ fixed point that has a mixture of positive and negative bonds. Although in a regular RG scheme positive bonds only generate positive bonds, it can be checked easily that for all temperatures intermediate between the extreme critical points, the recursion of Eq. (2) leads to a mixture of positive and negative bonds. Thus the critical behavior is likely to be the same for both cases.

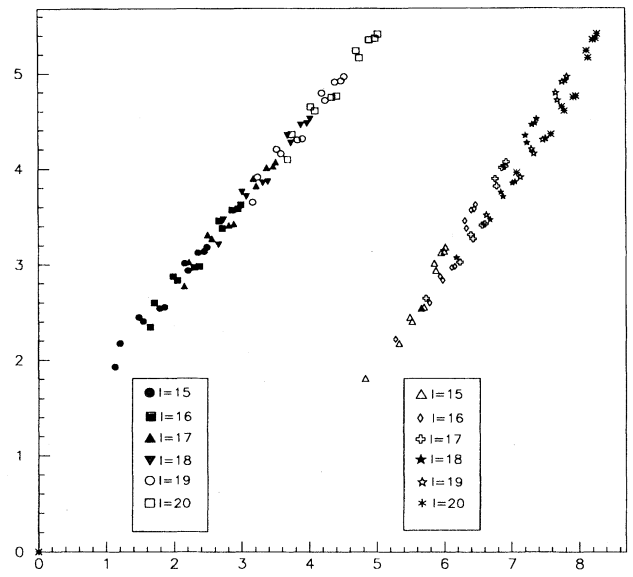

FIG. 2. Data collapse of $E(L,p\geq p_c)/L^{\omega_c}$, against $(p-p_c)^{\mu}L^{(1-\omega_c)}$ (left group), or $|p-p_c||\ln(|p-p_c|)|^{c}L^{(1-\omega_c)}$ (right group) in the abscissa. The scales are logarithmic.

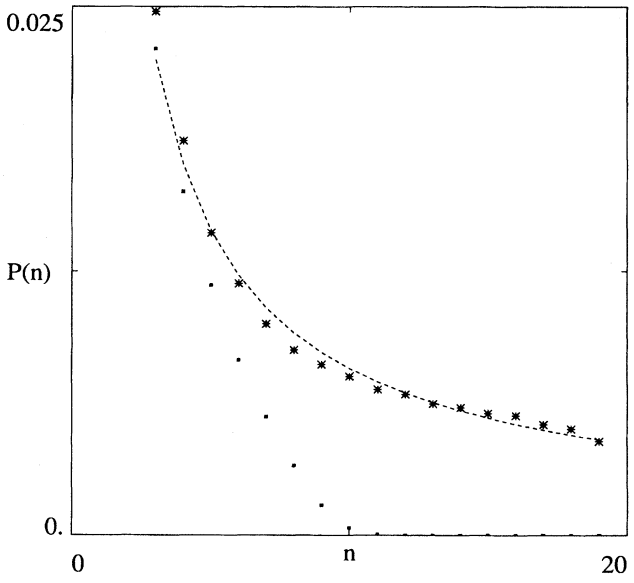

FIG. 3. The iterated probability for loop formation P_n. The decay is fitted to c/n (asterisks and dashed line) at p_c. Data for $p>p_c$ (squares) can be fitted to an exponential.

A striking feature of the $T=0$ recursion relation is that it is *independent of* q for $q\neq 2$. Thus the critical behavior for the vanishing of f_s should not depend on q. This expectation goes beyond the approximations of the model and the hierarchical lattice. Consider the configurations contributing to a low temperature expansion of the interface free energy of the Potts model on any lattice. The index q appears only in entropic factors giving the multiplicity of possible colorings. If, as is usually the case in random systems, the scaling properties of the interface are controlled by a zero temperature distribution, these properties will be independent of q. Of course, the SOS configurations allowed for $q=2$ are very different from the froth that appears for $q>2$, and thus in principle, we expect two different universality classes. Strictly speaking, since four colors are needed to cover an arbitrary "map" in two dimensions, there are restrictions on configurations allowed for $q=3$. If relevant, this leads to a third potential random interface behavior in $d=2$. Indeed the recursion relations of the DDHL at $T=0$ discriminate between $q=2$, $q=3$, and $q\geq 4$.

We focused our studies of critical behavior at $T=0$ on the mixture of bonds of strengths $+J$ and $-J$. The critical concentration in the presence of bubbles was identified by trial and error as $p_c=0.883\pm 0.001$. At the critical point, the mean and variance of energy should scale as $\overline{E(L)}=L^{\theta_c}$ and $\mathrm{var}[E(L)]=L^{2\omega_c}$, with $\theta_c=\omega_c$ if there is only one energy scale. The fits at p_c are consistent with this expectation, giving $\theta_c=0.34\pm 0.05$ and $\omega_c=0.31\pm 0.08$. Rather surprisingly, ω_c is very close to the value of ω for directed paths with no loops. Figure 2 shows that the data for the energy on approaching p_c from above can be collapsed by a finite-size scaling form,

$$E(L,p\geq p_c)=A|p_c-p|^{\mu}L+BL^{\omega_c}, \qquad (5)$$

with $\mu=0.91\pm 0.04$. In the absence of loops for $q=2$, $\mu=1$ exactly, since the mean value of the final energy is simply linear in the mean value for individual bonds. There is also a subleading correction to the energy from fluctuations that scales as L^{ω_c} [16].

To better understand the closeness of exponents in the presence and absence of loops, we looked directly at the fractal structure of the critical interface. This is achieved by examining the probability P_n that a loop forms at the nth iteration. As indicated in Fig. 3, P_n decays exponentially to zero for $p>p_c$, while the decay slows down on approaching p_c. The data at p_c can be fitted to a decay as c/n, with $c=0.060\pm 0.003$. This fit suggests the *phenomenological* differential recursion relations

$$dP/dn=-f_sP-P^2/c, \quad df_s/dn=f_s+f_sP. \qquad (6)$$

The first equation reproduces $P(n)=c/n$ at criticality; the second is just the behavior of the mean (free) energy (or mass of the cluster). The fixed point at $f_s=P=0$ describes a critical cluster that is asymptotically linear, with loops appearing predominantly at short length scales. Linearizing the second equation gives $y=1$. However, the marginality of P at criticality leads to logarithmic corrections to various scaling quantities. For example, the mass of the critical cluster grows as $L(\ln L)^c$. Similarly, the interfacial free energy vanishes as $|p-p_c||\ln(|p-p_c|)|^c$. Figure 2 shows that a data collapse is also possible using such logarithmic corrections. The best fit is achieved for $c=0.5\pm 0.6$, compared to $c=0.060\pm 0.003$ obtained directly from Fig. 3.

The asymptotic linearity of the critical cluster extends beyond the simple example of the DHL, and was also checked for the DDHL. In the latter, an interface configuration that covers both branches, as well as the central diagonal, is allowed only for $q>3$. Thus this lattice supports three types of

recursion relations at $T=0$, corresponding to $q=2$, 3, and ≥ 4. In spite of these differences, within our numerical accuracy, we could not detect any significant changes in the exponents from the simpler DHL. As indicated in Table I, even in the absence of loops ($q=2$), the critical cluster in the pure system is a fractal. This is because it takes advantage of the diagonal bond and is no longer simply directed. The critical cluster of the random system does not take advantage of the diagonal bond, or the possibility of loops, staying asymptotically linear. This suggests that the marginal irrelevance of operators that may complexify the structure [such as P in Eq. (6)] is in fact quite generic. Although we have presented the results in the context of Potts models, they are probably more generally applicable to systems of discrete symmetry. For example, it has been suggested that regions of the Baxter model with diverging heat capacity also exhibit Ising-like behavior in the presence of random bonds [17,12]. It is also tempting to generalize the conclusions to higher dimensions: Strong enough disorder may result in a continuous transition [2] with the interfacial criticality governed by a zero temperature fixed point. If the apparent irrelevance of bubbles can be generalized from hierarchical lattices, the exponent μ will be the same as in the Ising model. How can we reconcile this apparent super-universality of random bond criticality, with earlier results [10,11] which do indicate exponents that depend on q? The latter calculations were performed for weak disorder, and lead to a finite temperature fixed distribution. Is it possible that stronger disorder leads to different behavior, dominated by zero temperature fixed points? This scenario is precisely what is observed in a recent position space RG of a three-state random bond model [18]. Of course, another possibility is that our approximate interface model does not fully capture the physics of the system. Clearly, further investigations are desirable.

In summary, we have introduced a simple position space approximation for studying the interfacial properties of Potts models in $d=2$. In the pure case, the critical interface is a fractal froth, and interfacial tension vanishes with an exponent $\mu(q)$. In a random medium, within our numerical accuracy, we find that the critical interface is asymptotically linear, and the interfacial free energy vanishes linearly with logarithmic corrections, i.e., $\mu=1$ independent of q. We argue that, *if governed by a zero temperature fixed point*, $\mu(q)$ should be independent of q for all sufficiently large q. The above model also provides the simplest generalization of directed paths in random media [8] to ramified objects, with potential applications to fracture cracks, lightning patterns, etc.

We acknowledge discussions with N.C. Bartelt, A.N. Berker, and T.L. Einstein. This research was initiated during visits to Oxford University (M.K. and A.S., who also acknowledges an Accademia dei Lincei Royal Society grant and INFM support), and the Isaac Newton Institute (B.D. and M.K.). The work at MIT was supported by the NSF through Grant Nos. DMR-93-03667 and PYI/DMR-89-58061.

[1] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[2] K. Hui and A. N. Berker, Phys. Rev. Lett. **62**, 2507 (1989).
[3] A. Aizenman and J. Wehr, Phys. Rev. Lett. **62**, 2503 (1989).
[4] S. Chen, A. M. Ferrenberg, and D. P. Landau, Phys. Rev. Lett. **69**, 1213 (1992).
[5] L. Schwenger, K. Budde, C. Voges, and H. Pfnür, Phys. Rev. Lett. **73**, 296 (1994).
[6] G. Forgacs, R. Lipowsky, and Th. Nieuwenhuizen, in *Phase Trensitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1991), Vol. 14.
[7] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
[8] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[9] B. Derrida, V. Hakim, and J. Vannimenus, J. Stat. Phys. **66**, 1189 (1992).
[10] B. Derrida and E. Gardner, J. Phys. A Math Gen. **17**, 3223 (1984).
[11] A. W. W. Ludwig, Nucl. Phys. B **285**, 97 (1987).
[12] V. S. Dotsenko and Vl. S. Dotsenko, Adv. Phys. **32**, 129 (1983); V.S. Dotsenko, J. Phys. A **18**, L241 (1985).
[13] B. Derrida and R. B. Griffiths, Europhys. Lett. **8**, 111 (1989); J. Cook and B. Derrida, J. Stat. Phys. **57**, 89 (1989).
[14] The initial ensemble is consistent with $\mathscr{P}_1$. At each iteration, groups of four bonds are sampled and from each group a new weight is obtained through Eq. (2).
[15] In view of the assumed compactness of bulk phases, our model cannot properly describe the effects of dilution ($J_2=0$). A dilute Potts model at percolation threshold has $T_c=0$. Remarkably enough, this different $T=0$ fixed point also displays universality with respect to q [A. Coniglio, Phys. Rev. Lett. **46**, 250 (1981)].
[16] J. P. Bouchaud and H. Orland, J. Stat. Phys. **61**, 877 (1990).
[17] D. Matthews-Morgan and D. P. Landau, Phys. Rev. Lett. **53**, 670 (1984).
[18] A. Falicov and A. N. Berker (unpublished).