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Desynchronization by periodic orbits

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Synchronous chaotic behavior is often interrupted by bursts of desynchronized behavior. We investigate the role of unstable periodic orbits in bursting events and show that they serve as sources of local transverse instability within a synchronous chaotic attractor. Analysis of bursts in both model and experimental studies of two coupled Rössler-like oscillators reveals the importance of unstable periodic orbits in bursting events.

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Synchronization of chaotic systems [1,2] has attracted considerable attention lately. Recent applications of chaotic synchronization ideas to communications [3,4] and model verification [5] have further enhanced the appeal of this phenomenon. Despite this widespread attention, basic issues in chaotic synchronization have yet to be settled. One of these, and perhaps the most basic issue, is a sharp, sufficient condition stating when two identical systems will synchronize. There is a growing body of evidence [5–8] pointing to the fact that standard stability statements centered around the negativity of so-called transverse Lyapunov exponents are inadequate to guarantee high-quality synchronized behavior. While such conditions are *necessary* for synchronization, they are by no means *sufficient*. Intervals of desynchronized bursting behavior can appear even when the largest transverse Lyapunov exponent is negative, especially, as in practical settings, when there is noise in the system. Such bursting events are undesirable and potentially detrimental in application settings. Methods have been proposed for enhancing the quality of synchronous behavior by suitably altering the form of the dynamical systems involved [9]. This approach may not always be practical in real systems, however. It is therefore important to understand the origin of the bursting behavior, so that alternative methods for enhancing the synchronization quality can be devised. While in this paper we do not provide the definitive stability statement for synchronization of chaotic oscillators, we do point to one reason for the inadequacy of standard stability statements. In particular, we show that unstable periodic orbits within a synchronous chaotic attractor can play a crucial role in bursting events. We illustrate these effects with numerical and experimental studies of two coupled Rössler-like oscillators.

A common system for chaotic synchronization studies is the system of two identical coupled oscillators given by [10,11]

$$\dot{u}_j = f(u_j) + c\Gamma(u_{j+1} - u_j), \quad j = 0,1. \tag{1}$$

Here the $u_j \in \mathbb{R}^n$ represent the individual oscillator coordinates and the vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ defines the dynamics of a single oscillator. The coupling is diffusive type, with an $n \times n$ coupling matrix Γ and scalar coupling constant c. If $s(t)$ is a chaotic solution of $\dot{u} = f(u)$ then a synchronous chaotic state is defined by $u_0 = u_1 = s(t)$. This state resides on an n-dimensional *synchronization manifold* within the $2n$-dimensional phase space. The stability of the synchronous state can be determined by performing a linear stability analysis of (1) about $s(t)$. This analysis is facilitated by introducing coordinates along and transverse to the synchronization manifold, $\eta = u_0 + u_1$ and $\xi = u_1 - u_0$, respectively. Linearizing (1) about $s(t)$ then gives

$$\dot{\eta} = Df(s)\,\eta, \tag{2a}$$

$$\dot{\xi} = [Df(s) - 2c\Gamma]\xi, \tag{2b}$$

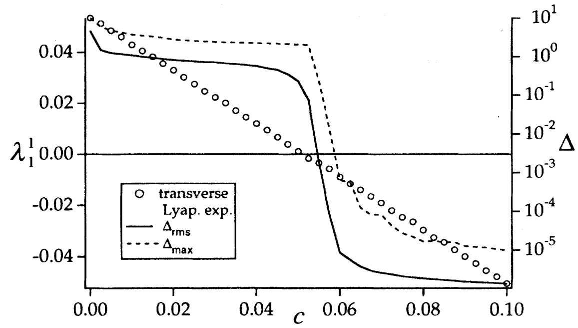

FIG. 1. Largest transverse Lyapunov exponent versus coupling for the coupled modified Rössler oscillators (left scale). Also plotted (right scale) are the rms and max values of the Euclidean distance between the two oscillators.

where $Df(s)$ is the Jacobian of f evaluated on $s(t)$. The first variational equation determines the nature of the synchronous state $s(t)$ itself; the Lyapunov exponents of $s(t)$, denoted by $\lambda_1^0 \geq \lambda_2^0 \geq \cdots \geq \lambda_n^0$, are computed from this equation. The second equation governs the transverse coordinates; the *transverse Lyapunov exponents*, denoted by $\lambda_1^1 \geq \lambda_2^1 \geq \cdots \geq \lambda_n^1$, are found from this equation. A necessary condition for stability of the synchronous chaotic state is $\lambda_j^1 < 0$ for all j. One of the main points of this paper is that *this condition is not sufficient for the stability of the synchronous state if $s(t)$ is chaotic*. The reason is that there can be invariant sets within a chaotic attractor (e.g., fixed points, periodic orbits, invariant manifolds, invariant Cantor sets) whose largest transverse Lyapunov exponent is positive, even when the largest transverse exponent of the attractor as a whole is negative. As trajectories come near these invariant sets they can be repelled from the synchronization manifold, giving rise to bursts. If there is noise in the system the bursting events can continue indefinitely. In this paper we focus on the role of periodic orbits in the bursting process.

To illustrate these effects we focus on a particular example, where each oscillator is described by the vector field

$$\dot{x} = -\alpha(rx + \beta y + z),$$
$$\dot{y} = \alpha(x + ay), \tag{3}$$
$$\dot{z} = \alpha[\mu g(x) - bz].$$

This system, which was investigated numerically and experimentally in [11], is a variant of the well-known Rössler system [12]. The time factor is set to $\alpha = 1.0$ in all numerical work, and is $\alpha = 10^4\ \mathrm{s}^{-1}$ in the circuit. The function $g(x)$ is a ramp; $g(x) = (x - x_c)$ if $x > x_c$ and 0 otherwise. This piecewise linear "nonlinearity" is easy to realize in the circuit construction. For parameters $a = 0.12$, $b = 1.0$, $\beta = 0.5$, $\mu = 15.0$, $r = 0.05$, and $x_c = 3.0$ the modified Rössler system (3) has a well-developed chaotic attractor. The coupling matrix is chosen to be diagonal, given by $\Gamma = \mathrm{diag}\,(0,1,0)$ (y coupling).

Figure 1 shows the largest transverse Lyapunov exponent λ_1^1 of the coupled system as a function of the coupling constant c. The exponents were obtained from the variational equation (2b) by computing the growth rate of a randomly

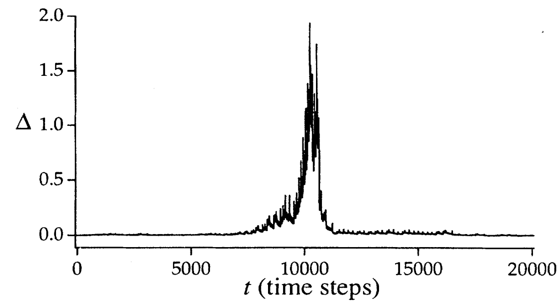

FIG. 2. Large burst in the numerical model at coupling $c = 0.055$.

chosen initial variation $\xi(0)$. Integrations were carried out with a fourth order Runge-Kutta method [13] and the growth rate was computed over 20 000 typical attractor cycles (1 cycle $= \tau = 2\pi/\sqrt{f}$) after a 1000 cycle transient. The step size was fixed at $\Delta t = \tau/100$. A zero line is drawn to indicate the zero crossing of λ_1^1 ("naive" synchronization threshold), which occurs at $c = c_0 \approx 0.051$. The degree of synchronization is measured via the Euclidean distance between the two oscillators, $\Delta = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$ [5]. Plotted on the right scale of Fig. 1 are values of Δ_{rms} and Δ_{max} computed from the coupled system for runs of 20 000 attractor cycles. In practice a small amount of white noise of order 10^{-7} is added to the right hand side of (1) during the numerical integrations. This is done to prevent spurious synchronization stemming from finite precision arithmetic. For a range of coupling constants $c > c_0$ there are small values of Δ_{rms} accompanied by large values of Δ_{max}; this situation is indicative of bursting behavior. The condition $\lambda_1^1 = 0$ is therefore inadequate to guarantee sustained high-quality synchronization. Similar bursting events have been reported by others [5,8] and have been observed by us in other coupled chaotic systems. Figure 2 shows a bursting episode for the numerical model at coupling constant $c = 0.055$. This burst is comparable to the size of the attractor, which for these parameters is $\sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \approx 3.4$. Similar bursts are observed in the coupled electronic circuit experiment. As shown below, an examination of these bursts reveals their association with unstable periodic orbits within the synchronous chaotic attractor.

Periodic orbit thresholds—theory. The destabilizing ef-

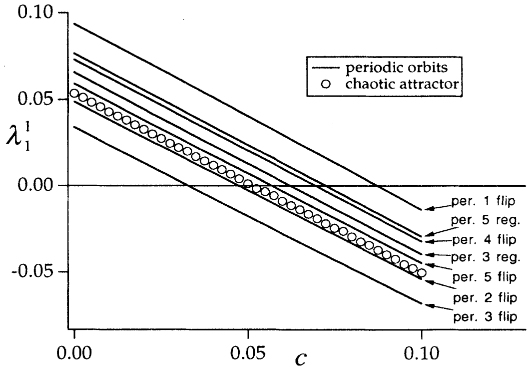

FIG. 3. Largest transverse Lyapunov exponent versus coupling for unstable periodic orbits through period 5. Also shown is the largest transverse Lyapunov exponent for the chaotic attractor.

TABLE I. Threshold coupling constants for the coupled modified Rössler system on various orbits. Third column gives experimental values adjusted for the factor of 2 stemming from the one-way driving setup.

Orbit	Threshold coupling (theory)	Threshold coupling (experiment)
period-1 flip	0.096	0.094
period-5 regular	0.077	orbit not found
period-4 flip	0.073	0.061
period-3 regular	0.065	0.074
period-5 flip	0.059	0.078
chaos	0.051	0.062
period-2 flip	0.047	0.065
period-3 flip	0.033	0.058

fects of periodic orbits can be understood by computing the largest transverse Lyapunov exponent *on the periodic orbits themselves*. This is carried out as follows. First, a periodic orbit of (3) is located. This is accomplished with a Newton search algorithm [14]. Next, the Floquet matrix [14] is constructed for the transverse variational equation (2b) on the periodic orbit. The Floquet matrix, denoted by Q, takes an arbitrary variation about the periodic orbit, $\delta(0)$, to its value one period later, $\delta(T)$,

$$\delta(T) = Q\,\delta(0). \tag{4}$$

The eigenvalues of Q are the "transverse Floquet multipliers," $\{\alpha_1, \alpha_2, \alpha_3\}$, for the periodic orbit, ordered by decreasing modulus. The largest transverse Lyapunov exponent for the periodic orbit is then given by

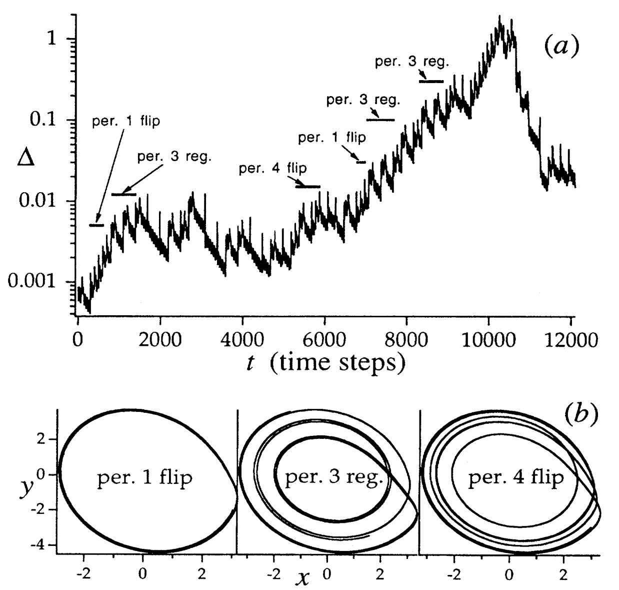

FIG. 4. (a) Burst of Fig. 2 shown on a logarithmic scale. Labeled regions are associated with the indicated unstable periodic orbits. (b) Projections of trajectories extracted from one oscillator corresponding to the first three labeled intervals in (a). From left to right the extracted segments are close to the period-1 flip saddle, period-3 regular saddle, and period-4 flip saddle.

$$\lambda_1^1 = \frac{\ln|\alpha_1|}{T}, \tag{5}$$

where T is the period of the orbit.

Figure 3 shows the largest transverse Lyapunov exponent as a function of coupling for all periodic orbits through period 5 for the modified Rössler system (3). Also shown for comparison is the largest transverse Lyapunov exponent for the chaotic attractor. For of the six computed orbits have exponents larger than the exponent of the chaotic attractor. Over the range of coupling constants shown, all of the exponents maintain the same ordering as in the uncoupled ($c=0$) limit. From this we expect dominant unstable orbits within the attractor to be dominant sources of transverse instability. This is borne out in the burst analysis below.

Periodic orbit thresholds—experiment. Experimentally, the threshold coupling at which a periodic orbit became transversely stable ($\lambda_1^1=0$) was determined as follows. First, the x, y, and z signals from a single chaotic circuit were digitized at a rate of 100 kHz (about 100 points per cycle). The method of close returns [15] was used to extract unstable periodic orbits from a three-dimensional time series of 50 000 points. From this data all periodic orbits indicated in Fig. 3 were found, with the exception of the period-5 regular saddle. Each orbit was then used to drive a second circuit by playing the orbit back with a computer through a digital-to-analog converter. The coupling configuration, was a one-way variant of the mutual coupling arrangement (1);

$$\dot{u} = f(u) + c\Gamma(u_{\mathrm{PO}} - u), \tag{6}$$

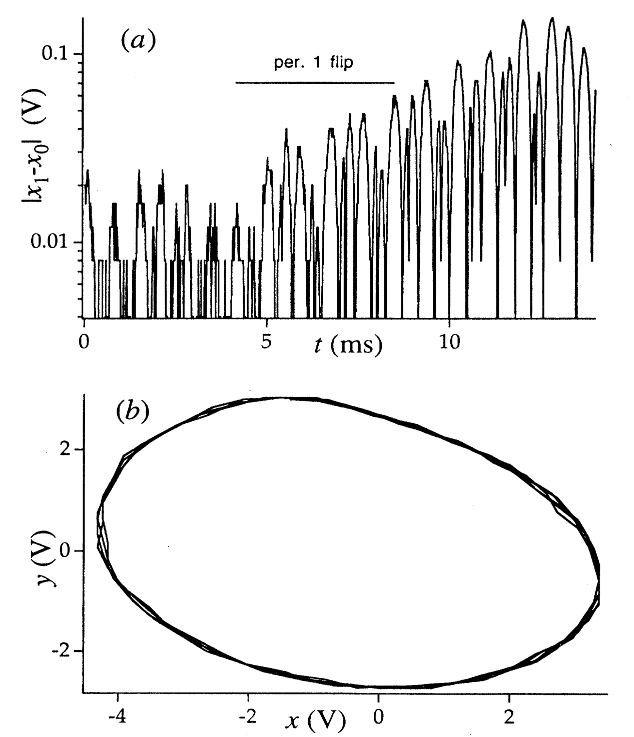

FIG. 5. (a) Portion of burst (measured via $|x_1-x_0|$) in the coupled circuit experiment on a logarithmic scale; (b) projection of trajectory extracted from one circuit over interval indicated in (a), showing underlying period-1 dynamics.

where u_{PO} is the periodic orbit. This arrangement is the same as one proposed by Pyragas [16] for synchronizing a chaotic system to a stored chaotic signal. Because of the one-way coupling configuration (6), the coupling constant c required to synchronize the response u with the drive u_{PO} is twice the value required in the mutually coupled case (1). The experimental thresholds are summarized in Table I, adjusted for this factor of 2; theoretical values are given for comparison. The chaotic threshold $c=0.062$ was determined by averaging the difference between x signals to smooth out the bursts. The extreme orbits (period-1 flip saddle and period-3 regular saddle) are in qualitative agreement with the theory. While the ordering for the other orbits is not exactly as in the theory, there are several possible sources of error: the response circuit is not identical to the circuit from which the orbits were obtained, the periodic orbits from the data are close to but not exactly on the true periodic orbits, and there are timing and digitization errors induced by the digital-to-analog converter when the orbits are played back.

Burst analysis—theory. Figure 4(a) shows the numerical burst of Fig. 2 displayed with a vertical logarithmic scale. Exponential growth regions are clearly present. Six growth regions are indicated by the unstable orbits associated with them [17]. The burst initiates with a strong growth phase mediated by the period-1 flip saddle. Subsequent visits to other periodic orbits, particularly the period-3 regular saddle and the period-4 flip saddle, amplify the burst even more. Large bursting events, such as this one, are generally found to be connected with more than one periodic orbit. Figure 4(b) shows x-y projections of the trajectory of one of the oscillators corresponding to the first three growth intervals in Fig. 4(a). The extracted segments match well with the numerically computed orbits, establishing the connection between the labeled growth intervals and the periodic orbits.

For additional confirmation we have estimated the growth rates of Δ over the intervals labeled in Fig. 4(a) and compared these rates to the transverse Lyapunov exponents of the associated periodic orbits. The growth rate of Δ can be estimated by assuming that over the interval of interest Δ can be written as

$$\Delta(t)=g(t)e^{\lambda T}, \tag{7}$$

where λ is the growth rate, T is the known period of the associated periodic orbit, and $g(t)$ is an unknown periodic function with period T, $g(t+T)=g(t)$. A plot of $\Delta(t+T)$ versus $\Delta(t)$ then gives a straight line with intercept zero and slope $e^{\lambda T}$. Least squares fits to this model for the first three growth regions shown in Fig. 4(a) give growth rates of 0.027, 0.011, and 0.0077, respectively. The corresponding theoretical rates are as follows: period-1 flip saddle, 0.035; period-3 regular saddle, 0.0076; period-4 flip saddle, 0.015. Other intervals give similar results. The estimated growth rates are in reasonable agreement with the transverse Lyapunov exponents, particularly for the period-1 flip saddle and the period-3 regular saddle.

Burst analysis—experiment. Figure 5(a) shows a burst for the mutually coupled experimental system at coupling constant $c=0.0735$ (beyond the coupling constant for chaotic synchronization, $c=0.062$). During the indicated time interval the circuits are near the period-1 orbit. Figure 5(b) shows a plot of y vs x from one of the circuits over this time interval. Almost three full cycles of the unstable period-1 orbit are present. Over this interval the two signals diverge exponentially at a rate of about 100 s^{-1}. Of the bursts studied in detail, almost all were initiated by the unstable period-1 orbit.

In summary, we have shown through numerical models and electronic circuits that unstable periodic orbits within a synchronous chaotic attractor can serve as local sources of transverse instability. Trajectories that come near periodic orbits with large transverse Lyapunov exponents can initiate desynchronized bursting behavior. Sustained bursting is found in both the model (with small additive noise) and the experiment. This phenomenon must be taken into account to yield high-quality chaotic synchronization. Future work will focus on formulating a more stringent stability criteria for high-quality synchronization of chaotic oscillators.

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