Vortex filament dynamics in plasmas and superconductors

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The dynamical description of a thin vortex filament in a perfect fluid is generalized to a charged fluid on a neutralizing background. The filament curvature and torsion evolve integrably according to a higher nonlinear Schrödinger equation, which is found using the Hasimoto transformation. Because of screening, local induction is accurate, and the logarithmic long-range divergency is removed. Linear waves, such as the Kelvin mode, are supported by the filament, as well as nonlinear waves and solitons. The effects of vortex stretching are briefly discussed. Applications of the results are found in electron magnetohydrodynamics and in type II superconductor theory.

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I. INTRODUCTION

The study of infinitely thin vortices in two-dimensional (2D) and three-dimensional (3D) fluid dynamics is motivated by several facts. Some qualitative understanding of vortex dynamics may be obtained from simple arguments disregarding any internal structure of vortices. There is considerable mathematical tractability (and beauty) in models of point vortices or infinitely thin vortex filaments (VF's) [1,2]. Observations in nature of strongly localized vorticity distributions can be quite impressive (e.g., the tornado) and intriguing (as in superfluid helium). Computational fluid dynamics may benefit from a separate introduction of strong vortices in roll-up or shear layers [3]. The spontaneous appearance of vortex filaments in strong turbulence is observed in numerical (e.g., [4]) and laboratory (e.g., [5]) experiments.

The equations of motion of a single VF as expressed in its curvature and torsion, the Betchov–Da Rios equations [6,7], were elegantly transformed into the cubic nonlinear Schrödinger equation (NSE) by Hasimoto [2], and may thereby in principle be regarded as solved. Apart from the linear Kelvin waves [8], the VF supports solitons and waves of various kinds, e.g., as given in Refs. [2,9–11].

Despite this beautiful success story of VF theory, some features remain to be dealt with. It is not the purpose of this work to accomplish this by providing an ultimate discussion of nonlocal effects, vorticity structure and axial flow effects, reconnection, or vortex stretching. (For example, the extension to a VF with axial flow [12] seems to fit experimental observations better [13].) But a main result is that our generalization to a charged fluid on a neutralizing background eliminates the logarithmic divergence that stains the above-mentioned success; viz, the need to introduce an *ad hoc* long-range cutoff, the localized induction approximation (LIA), in the expression for the VF self-velocity derived from a Biot-Savart law, is removed. The London penetration depth is a natural cutoff scale. There have been analytical efforts to excuse for this imperfection in neutral fluids, reviewed in the monograph by Saffman [14]. Some numerical work has shown that qualitatively, but not quantitatively acceptable results are obtained when relying on the LIA [3,15]. In the charged fluid context, this discussion becomes obsolete.

Despite the fact that the equation we derive is a perturbed NSE, it turns out to be as exactly integrable as the unperturbed equation. This is a consequence of the nonstretching property of the filament curve [16]. By including some structure dynamics of the vortex core, nonintegrability is introduced via a second coupled equation. This effect enters as an even smaller perturbation than the previous one. We conclude that a certain linear wave on the filament will grow in amplitude, but that nonlinearly the influence will be small, and we do not consider it further.

We focus on two areas of application of the results derived in this paper. First, in type II superconductor theory, where a few old results will reappear [17–19], moral support for a rather recent development is given [20,21]. The well-known result of Kelvin waves on a flux line is reproduced, along with some other waves and solitons that we believe to be previously unknown in this context. This is done using a macroscopic treatment, which could be identified as a time-dependent London model. The "moral support" concerns the computation of flux line dynamics, especially in 3D geometry. Flux line cutting (reconnection) appears to be strongly discriminated by quasi-2D models.

Second, in plasmas there are events on time and space scales where ion dynamics is too slow to be relevant. The electron magnetohydrodynamical (EMH) model [22] developed to describe these phenomena in terms of electron and magnetic field motion is actually the charged fluid model we consider in this paper.

In Sec. II we derive the equations of motion for a charged fluid, and in particular the expression for the ve-

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locity of a VF. The Hasimoto transformation is carried out in Sec. III, leading to a discussion of a Hamiltonian and other invariants of the motion in Sec. IV, where we also discuss integrability. Linear theory follows in Sec. V. The discussion of physical applications is given in Sec. VI A for superconductors and in Sec. VI B for plasmas. A summary follows in Sec. VII.

II. EQUATIONS OF MOTION

The present model is known in plasma physics as electron magnetohydrodynamics (EMH) and is reviewed in Ref. [22]. The stability of vortices with respect to short wave perturbations was shown by Ivonin [23] and some special solutions were given by Isichenko and Marnachev [24]. We briefly show how to obtain the standard EMH equation of motion for the magnetic field. The true applicability to plasmas is dealt with in Sec. VI B.

The equation of motion for an electron fluid is (neglecting collisions with the neutralizing background)

$$m\left[\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right]=\frac{-\nabla P}{n}-e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) , \tag{1}$$

where m and $-e$ denote the electron mass and charge, and $\mathbf{v}$, n, and P denote the electron fluid velocity, number density, and pressure. In the regime where the electron displacement current can be neglected, Ampère's law for the magnetic field $\mathbf{B}$ takes on the appearance

$$\nabla\times\mathbf{B}=-\mu_0 en\mathbf{v} , \tag{2}$$

and the electric field $\mathbf{E}$ is sufficiently determined by Faraday's law.

Assuming $P=P(n)$ and taking the curl of Eq. (1), it can be given the form

$$\frac{\partial\mathbf{\Omega}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{\Omega}) , \tag{3}$$

where

$$\mathbf{\Omega}=\nabla\times\mathbf{p} , \quad \mathbf{p}=m\mathbf{v}-e\mathbf{A} .$$

The statement of Eq. (3) is that the curl of the generalized momentum is frozen in the fluid. This fact has been pointed out by several authors in the past [25]. Using (2) in the definition of $\mathbf{\Omega}$ we find $\mathbf{\Omega}=-e[\mathbf{B}+\nabla\times(\lambda^2\nabla\times\mathbf{B})]$, where $\lambda=c/\omega_{\rm pe}$ is the penetration depth of the magnetic field, sometimes referred to as the London length. The electron plasma frequency is given by $\omega_{\rm pe}^2=ne^2/m\varepsilon_0$.

In this paper we will consider the homogeneous case, and hence the expression for the generalized vorticity $\mathbf{\Omega}$ is written

$$\mathbf{\Omega}=\mathbf{B}-\lambda^2\nabla^2\mathbf{B} . \tag{4}$$

In this expression we have rescaled the magnetic field and generalized vorticity, $\mathbf{B}'=-e\mathbf{B}/m$, $\mathbf{\Omega}'=\mathbf{\Omega}/m$, and dropped the primes. Equation (3) with (4) and the rescaled version of (2),

$$\mathbf{v}=\lambda^2\nabla\times\mathbf{B} , \tag{2$'$}$$

inserted constitutes the basic equation of motion.

Two remarks are in order here. First, Eq. (3) can be obtained in an elegant way from the one-particle Hamiltonian

$$H=P(\mathbf{q},\mathbf{p})+e\phi+\frac{(\mathbf{p}+e\mathbf{A})^2}{2m} .$$

P is some generalized potential. As this Hamiltonian is the same for all electrons, or electron fluid particles, the relative Poincaré invariant [26] $I=\oint_C \mathbf{p}\cdot d\mathbf{q}$ taken at one instant in time is equally invariant when interpreted as a circulation integral $\int_{(\partial S=C)}\mathbf{\Omega}(\mathbf{q})\cdot d\mathbf{S}$ for an arbitrary closed curve C in the fluid. This is again a statement of freezing of $\mathbf{\Omega}$ in the fluid. This topic has been pursued by Yankov and Petviashvili [27], who also considered a more general Hamiltonian. The demand that all fluid particles have the same Hamiltonian is equivalent to the particle relabeling symmetry. Ripa and Salmon emphasize this view and interpret the invariant as a Noether invariant [28]. Either way, by relaxing this symmetry to subsets of all particles, an Ertel theorem has been obtained in an elegant manner [27,28]. We remind the reader that there is another simplest Poincaré invariant expression, $\oint_C \mathbf{q}\cdot d\mathbf{p}$ as well, which might be more convenient to use in some situations.

Second, we note that $\mathbf{\Omega}$ consists of a magnetic and an inertial term. In the limit where the magnetic term completely dominates, Alfvén's theorem of the freezing-in of the magnetic field is obtained, albeit for an electron fluid. In the converse limit we find Kelvin's circulation theorem.

We now turn to the question of vortex filaments (VF's) in a charged fluid [29]. Because of freezing-in, they will move with the fluid and stay localized. Our idealized VF will be described as follows. Consider a filament along the curve $\mathbf{r}=\mathbf{c}(s)$, where s is the arclength. The curve may be closed or it may extend to infinity. Filaments that end on a surface have image filaments that close their curves, possibly at infinity. The generalized vorticity distribution is

$$\mathbf{\Omega}(\mathbf{r})=\int\hat{\mathbf{t}}\Phi\delta(\mathbf{r}-\mathbf{c}(s))ds .$$

$\hat{\mathbf{t}}$ is the unit tangent vector and Φ is the conserved circulation around the VF, in units of action in dimensional variables. Two sets of boundary conditions will be important to us: homogeneous and periodic. By homogeneous boundary conditions we mean that the curve approaches a straight line at infinity. Periodic boundary conditions means that the curve is finite and closed.

Because there is a finite energy density associated with the filament, there will be an "unbending" force directed toward the center of curvature. The "inertial response" of a vortex is the Magnus force. Hence, it acquires a velocity in the binormal direction. We return to this picture in Sec. V, and give a more geometric derivation of the velocity here.

The self-induced velocity at any point in the fluid is given by a Biot-Savart law

$$\mathbf{v}(\mathbf{r})=\int\nabla G(|\mathbf{r}-\mathbf{r}'|)\times\mathbf{\Omega}(\mathbf{r}')dV' , \tag{5}$$

where, for an $\mathbf{\Omega}$ of the form of Eq. (4),

$$G(R)=\frac{e^{-R/\lambda}}{4\pi R},$$

is the Green's function. It reduces to that of an Euler fluid in the limit $\lambda\to\infty$. The integral in Eq. (5) is effectively a line integral along the VF, i.e., $\Omega dV'\to\Phi ds$. To find the self-induced velocity of a point $\mathbf{r}=\mathbf{c}(0)$ on the filament, we expand the expression for $\mathbf{c}$ around that point, using the local Frenet frame. Parameter subscripts will denote differentiation. Then, as $\hat{\mathbf{t}}=\mathbf{c}_s$ the right-handed orthonormal frame $(\hat{\mathbf{t}},\hat{\mathbf{n}},\hat{\mathbf{b}})$ satisfies the Frenet formulas

$$\begin{bmatrix}\hat{\mathbf{t}}_s\\ \hat{\mathbf{n}}_s\\ \hat{\mathbf{b}}_s\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\ -\kappa&0&\tau\\ 0&-\tau&0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{t}}\\ \hat{\mathbf{n}}\\ \hat{\mathbf{b}}\end{bmatrix},$$

where κ is the (first) curvature and τ the torsion (or second curvature). They depend on s and on time t, but we will suppress as much of these dependencies as possible in writing. Now we can expand $\mathbf{c}(s)$ and $\hat{\mathbf{t}}(s)$ as

$$\begin{aligned}\mathbf{c}(s)=\mathbf{r}+\hat{\mathbf{t}}&\left[s-\frac{s^3}{6}\kappa^2-\frac{s^4}{8}\kappa\kappa_s+\cdots\right]\\ &+\hat{\mathbf{n}}\left[\frac{s^2}{2}\kappa+\frac{s^3}{6}\kappa_s+\frac{s^4}{24}[\kappa_{ss}-\kappa(\kappa^2+\tau^2)]+\cdots\right]\\ &+\hat{\mathbf{b}}\left[\frac{s^3}{6}\kappa\tau+\frac{s^4}{24}\frac{(\kappa^2\tau)_s}{\kappa}+\cdots\right],\\ \hat{\mathbf{t}}(s)=\hat{\mathbf{t}}&\left[1-\frac{s^2}{2}\kappa^2-\frac{s^3}{2}\kappa_{ss}+\cdots\right]\\ &+\hat{\mathbf{n}}\left[s\kappa+\frac{s^2}{2}\kappa_s+\frac{s^3}{6}[\kappa_{ss}-\kappa(\kappa^2+\tau^2)]+\cdots\right]\\ &+\hat{b}\left[\frac{s^2}{2}\kappa\tau+\frac{s^3}{6}\frac{(\kappa^2\tau)_s}{\kappa}+\cdots\right],\end{aligned}$$

where $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$, κ, τ, and g on the right-hand side are evaluated at $s=0$. The expansion procedure is valid if $\varepsilon_\kappa=\kappa\lambda\ll 1$, and all terms of order λ^2 will be equally important if $\kappa\sim\tau\sim\partial/\partial s$. Assuming sufficient differentiability, the expression for the velocity will be valid for any and all points on the filament. Expanding $R=|\mathbf{r}-\mathbf{c}(s)|$ in s and inverting to get $s=s(R)$ the velocity is found from Eq. (5) to be

$$\mathbf{v}(s)=\frac{\Phi}{4\pi}\left\{\hat{\mathbf{b}}\kappa\Lambda+\frac{3\lambda^2}{4}\left[\hat{\mathbf{t}}\frac{2}{3}\kappa^2\tau-\hat{\mathbf{n}}\frac{(\kappa^2\tau)_s}{\kappa}+\hat{\mathbf{b}}\left[\kappa_{ss}-\kappa\tau^2+\frac{\kappa^3}{2}\right]\right]\right\}\tag{6}$$

for an element at $\mathbf{c}(s)$, where s can be chosen as arbitrarily as the $s=0$ point can. An approximation to the leading order term has been given in the context of type II superconductors [18]. We do encounter a singularity here. Λ is defined as the logarithmically divergent (at zero)

$$\Lambda=\int_\xi^\infty\frac{e^{-R/\lambda}}{R}dR+1=Ei(\varepsilon_\xi)+1,$$

where $\varepsilon_\xi=\xi/\lambda\ll 1$. ξ is a suitable short-range cutoff, taken to be an effective radius of the filament. For an extensive discussion of this kind of regularization, see Saffman [14]. It is reasonable to choose ξ such that $|\mathbf{v}(s)|$ is smaller than some model limiting velocity, e.g., the electron Alfvén speed $v_{Ae}=B/\sqrt{\mu_0 mn}$ [22]. If a sound wave is present in the system [by modification of Eq. (1)], the sound speed should not be challenged, in order to ensure that internal compressible dynamics of the filament core does not come into play. Strictly speaking, for consistency this cutoff should be imposed on the regular terms as well, but this would only result in terms of the order of $\varepsilon_\kappa^2\varepsilon_\xi$ and a factor of $\exp(-\varepsilon_\xi)\approx 1$.

The most notable difference from the dynamics of an Euler fluid filament is that *the velocity only depends on nearby parts of the filament,* artificial long-range cutoffs hence being superfluous. The leading order term in (6) is expected to be strongly dominating, but we will pay some attention to the effects of the smaller terms because they introduce new features. In the next two sections, where we discuss the Hasimoto transformation and invariants, it will be seen that these terms introduce vortex stretching effects, implying nonintegrability. One should be aware of these, although the main line of our exposition is to neglect them. The order of smallness of these terms is the same as that of the first nonconstant term of the energy, c.f., Eq. (10), but nonintegrability only enters to an even higher order.

To simplify the description of the motion of the filament, we separate the local stretching of the filament from the motion of the curve along which the VF lies. The velocity $\mathbf{u}(s)$ of the *curve* is obtained by subtracting from $\mathbf{v}(s)$ the integrated stretching $\hat{\mathbf{t}}\int_0^s\gamma\, ds'$ [30], where the stretching rate

$$\gamma=\hat{\mathbf{t}}\cdot\mathbf{v}_s=\frac{\Phi}{4\pi}\frac{3\lambda^2}{4}\frac{5}{3}(\kappa^2\tau)_s.$$

The result is

$$\mathbf{u}(s)=\frac{\Phi}{4\pi}\left\{\hat{\mathbf{b}}\kappa\Lambda+\frac{3\lambda^2}{4}\left[-\hat{\mathbf{t}}\kappa^2\tau-\hat{\mathbf{n}}\frac{(\kappa^2\tau)_s}{\kappa}+\hat{\mathbf{b}}\left[\kappa_{ss}-\kappa\tau^2+\frac{\kappa^3}{2}\right]\right]\right\},\tag{7}$$

the velocity of the curve. [To keep track of a point of attachment of the curve to the filament, the lower limit term of the integral should be included.] We see, for example, that the velocity of a plane, circular vortex ring is

$$\mathbf{u}_R=\hat{\mathbf{b}}\frac{\Phi\kappa}{4\pi}\left[\Lambda-\frac{3\varepsilon_\kappa^2}{8}\right].$$

In the model, there seems to be no mechanism of shrinking of a vortex ring, although this phenomenon seems to be accepted ever since an argument of energy decrease was put forward long ago [19]. Friction, as expressed by the Hall and Vinen parameters, has been shown to account for ring shrinkage in superfluid helium [30], and is a natural candidate in a charged fluid as well.

Because of the freezing-in of $\mathbf{\Omega}$ and incompressibility, the volume of the filament is locally conserved. But then the stretching must be compensated by a change in the cross-sectional area (assumed proportional to ξ^2): $\xi_t = -\gamma\xi/2$, and hence

$$\Lambda_t = \frac{\gamma}{2} e^{-\xi/\lambda} \approx \frac{\gamma}{2} . \tag{8}$$

By this construction, sections of the filament may slide along the curve whenever $\gamma \neq 0$, but the curve is in each moment nonstretching. Note that the total length L of a closed filament does not change: $L_t = \oint \gamma \, ds = 0$.

It will be useful to know the energy of the filament. In the rescaled fields, the energy density is given by $(n\lambda^2/2)\mathbf{B}\cdot\mathbf{\Omega}$. Thus,

$$\begin{aligned} E &= \tfrac{1}{2}\int n\lambda^2 \mathbf{B}\cdot\mathbf{\Omega}\, dV \\ &= \tfrac{1}{2}\int\int{}' nG(|\mathbf{r}-\mathbf{r}'|)\mathbf{\Omega}(\mathbf{r})\cdot\mathbf{\Omega}(\mathbf{r}')dV\,dV' \end{aligned} \tag{9}$$

and to lowest geometry-dependent order the energy line density (tension) is found to be

$$\varepsilon = \frac{n\Phi^2}{4\pi}\left[\Lambda - \frac{3}{8}\varepsilon_\kappa^2\right] . \tag{10}$$

In Ref. [19] this expression was given (in other units) with a less accurate value of Λ.

III. HASIMOTO TRANSFORMATION

The intrinsic evolution equations for κ and τ, given $\mathbf{u}(s)$, probably first derived by Germano [31] and recently derived in a general manner by Ricca [32], are given in Sec. IV. For the moment we will pass to another way of expressing the motion.

In a celebrated paper [2] Hasimoto introduced the following transformation and applied it to the dominating term in the Euler fluid counterpart of Eq. (6) (which has the same appearance). With a phase carrier

$$\varphi = \exp\left[i\int_0^s \tau\, ds'\right] , \tag{11}$$

the vector and scalar

$$\mathbf{N} = (\hat{\mathbf{n}} + i\hat{\mathbf{b}})\varphi , \quad \psi = \kappa\varphi$$

are defined. After some manipulations it is found that ψ satisfies [30]

$$i\psi_t + \psi\int_0^s \kappa\hat{\mathbf{b}}\cdot\mathbf{u}_{s'}ds' - i(\mathbf{N}\cdot\mathbf{u}_s)_s = 0 . \tag{12}$$

Rescaling time as $t' = t\Phi/4\pi$ for the remainder of this section and Sec. IV and dropping the prime, the first term in Eq. (7) inserted into (12) is found to give [2]

$$i\psi_t + \Lambda(\psi_{ss} + \tfrac{1}{2}|\psi|^2\psi) = \frac{\Lambda}{2}|\psi(s=0,t)|^2\psi , \tag{13}$$

i.e., the NSE. The rhs can be removed by shifting the phase of φ. The NSE is known to be a fully integrable equation, so that in principle the dynamics of a VF is known. The transformation from ψ to $\mathbf{c}$ has been carried out for several particular solutions, e.g., in Refs. [2,9–11]. Among these are linear and nonlinear waves and single and multiple soliton solutions. An introduction to these matters is given by Lamb [33]. The interpretation of various invariants of the NSE in this context has been given by Ricca [34], and will be briefly considered in Sec. IV. Langer and Perline [35] give several *geometric* interpretations that illuminate a general class of curve motion, to which belong the different cases treated by Lamb.

Applying now Eq. (12) to all of (7), we obtain

$$i\psi_t + \Lambda(\psi_{ss} + \tfrac{1}{2}|\psi|^2\psi) + \frac{3\lambda^2}{4}(\psi_{ssss} + \tfrac{3}{8}|\psi|^4\psi) + \frac{3\lambda^2}{8}[|\psi|^2_{ss}\psi + 3(\psi\psi_s)_s\psi^*] = A(t)\psi , \tag{14a}$$

$$A(t) = \frac{\Lambda}{2}|\psi|^2 + \frac{3\lambda^2}{4}[\tfrac{3}{8}|\psi|^4 + \tfrac{1}{2}(\psi\psi^*_{ss} + \psi^*\psi_{ss} - |\psi_s|^2)]\Big|_{s=0} . \tag{14b}$$

In analogy to what was said about (13), we can eliminate the rhs of (14a) by shifting the phase of φ an amount $-i\int_0^t A(t')dt'$. The nonlocal velocity term that we omitted in (7) would take the form $B(t)\psi_s$, and this is also removable by a simple transformation [30]. These complications are consequences of using the arclength coordinate s, and of the definition of φ, Eq. (11). In the following, we will consider

$$\begin{aligned} i\psi_t + \Lambda(\psi_{ss} + \tfrac{1}{2}|\psi|^2\psi) &+ \frac{3\lambda^2}{4}(\psi_{ssss} + \tfrac{3}{8}|\psi|^4\psi) \\ &+ \frac{3\lambda^2}{8}[|\psi|^2_{ss}\psi + 3(\psi\psi_s)_s\psi^*] = 0 \end{aligned} \tag{14c}$$

to be the basic equation. One should keep in mind that simple solutions ψ (see Sec. V) become nontrivial if $A(t)$ is eliminated in the above manner.

Consider now how stretching modifies Eq. (14). The straightforward way is to add Eq. (8) to the system and no longer assume Λ to be constant when applying (12) to $\mathbf{u}(s)$. The order Λ term in (14c) is then replaced by

$$(\Lambda\psi)_{ss} + \psi\int^s \frac{\Lambda}{2}(\psi\psi^*_s + \psi^*\psi_s)ds .$$

[The lower integration limit will appear in $A(t)$.] Equation (8) can be written

$$i\Lambda_t - \frac{5\lambda^2}{8}(\psi\psi^*_s - \psi^*\psi_s)_s = 0 .$$

In the next section we will discuss what the effect of stretching might be and how large it is.

IV. HAMILTONIAN AND OTHER INVARIANTS. INTEGRABILITY

Disposing of the rhs the Hamiltonian of Eq. (13) has the simple form

$$H_0=\Lambda\int(\tfrac{1}{4}|\psi|^4-|\psi_s|^2)ds\ .$$

The second and third brackets in (14c) add the higher-order nonlinearity and dispersion terms

$$H_1=\frac{3\lambda^2}{4}\int(\tfrac{1}{8}|\psi|^6-|\psi_{ss}|^2)ds$$

and the nonlinear dispersion term

$$H_2=\frac{3\lambda^2}{8}\int[\tfrac{1}{2}|\psi|^2(\psi\psi_{ss}^*+\psi^*\psi_{ss})-2|\psi|^2|\psi_s|^2]ds\ ,$$

respectively, to the Hamiltonian. Apart from the full Hamiltonian $H=H_0+H_1+H_2$, the quasiparticle number N and momentum Π are conserved in a hierarchical way. We have

$$N=\int\nu\,ds\equiv\int(|\psi|^2)ds\ ,$$

$$\Pi=\int(\pi_0+\pi_1+\pi_2)ds\ ,$$

$$\pi_0=i\Lambda(\psi\psi_s^*-\psi^*\psi_s)\ ,$$

$$\pi_1=i\frac{3\lambda^2}{4}[(\psi\psi_s^*-\psi^*\psi_s)_{ss}-2(\psi_s\psi_{ss}^*-\psi_s^*\psi_{ss})]\ ,$$

$$\pi_2=i\frac{3\lambda^2}{8}3|\psi|^2(\psi\psi_s^*-\psi^*\psi_s)\ ,$$

where the partitioning was made to correspond to that of the Hamiltonian. The integral Π_0 of only π_0 is conserved as well. We turn to a discussion of the conservation law

$$\nu_t+(\pi_0+\pi_1+\pi_2)_s=0\ . \tag{15}$$

Equation (14c) and its complex conjugate can be written

$$-i\psi_t=D(\psi,\psi^*)\ ,\quad i\psi_t^*=D(\psi^*,\psi)\ .$$

First we note that $D(\psi,\psi^*)$ is homogeneous in φ to an order of 1. Let us call this property φ homogeneity. Evidently, Eq. (15) can be written

$$(\kappa^2)_t=i[\psi^*D(\psi,\psi^*)-\psi D(\psi^*,\psi)]\ ,$$

which is 2κ times the first of the two intrinsic equations given by Germano and Ricca [31,32],

$$\kappa_t=(\kappa\alpha+\beta_s-\tau\gamma)_s-\tau(\tau\beta+\gamma_s)\ , \tag{16a}$$

$$\tau_t=\left[\frac{\tau(\kappa\alpha+\beta_s-\tau\gamma)+(\tau\beta+\gamma_s)_s}{\kappa}\right]_s+\kappa(\tau\beta+\gamma_s)\ , \tag{16b}$$

where we let α, β, and γ denote the tangent, normal, and binormal velocity, respectively. It is a *general property* of a φ-homogeneous NSE obtained by the Hasimoto transformation that conservation of the integral of the square of the curvature is equivalent to quasiparticle number conservation.

Ricca has virtually shown this for the special case of the leading order term in $\mathbf{u}(s)$ [34]. There, κ^2 is proportional to $|\mathbf{u}|^2$ and ν may be interpreted as a kinetic energy density of the filament. In an Euler fluid a single straight VF has no nonzero proper velocity. The same holds true in a charged fluid, but there is still a finite rest energy, cf. Eq. (10). The second term is proportional to κ^2, and hence we see that the energy conservation interpretation is possible here as well. There is an analogy with an elastic, nonstretching string, which has the bending energy proportional to κ^2. Variation of the string bending energy under the condition of preserved length is formally equivalent to variation of the filament energy [36]. This, in turn, is equivalent to the variation of H with respect to ψ^* [16].

The presence of invariants is more complete than indicated above. The term H_1+H_2 is one of the higher invariants of the unperturbed NSE. This means that Eq. (14c) is integrable. Soliton solutions are given in Ref. [36]. It can be noted that the functional form of the solitons is the same as for the unperturbed NSE, although the relation between amplitude, carrier velocity, and envelope velocity is slightly more complicated. This is a general feature of equations for nonstretching filaments [16,36]. The single soliton is given by

$$\psi(s,t)=\pm\operatorname{Im}\zeta\frac{\exp\left[-i\left[\Lambda\operatorname{Re}\zeta^2t-\dfrac{3\lambda^2}{4}\operatorname{Re}\zeta^4t+\operatorname{Re}\zeta s-\phi\right]\right]}{\cosh\left[\operatorname{Im}\zeta(s-s_0)+\Lambda\operatorname{Im}\zeta^2t-\dfrac{3\lambda^2}{4}\operatorname{Im}\zeta^4t\right]}\ , \tag{17}$$

where ϕ is the phase and s_0 the initial central position. The complex parameter ζ assures independent tuning of envelope velocity and amplitude. In particular, a nontraveling soliton (standing breather) is possible.

It should be noted that the perturbation due to axial flow, derived by Fukumoto and Miyazaki [12], led to another integrable equation, Hirota's equation. This is an equation closely related to ours. It combines the NSE and the modified Korteweg–de Vries equation, and it has envelope solitons. Equation (14c) combines two equations that both have envelope solitons, which combine in the simple way indicated by Eq. (17).

A priori we expect stretching to bring nonintegrability. To get an idea of the nature of the influence of stretching on the Hamiltonian structure, we use a perturbative approach to determine Λ. The leading order terms in Eq.

(16a) give

$$(\kappa^2)_t = -2\Lambda(\kappa^2\tau)_s - 4\kappa^2\tau\Lambda_s \ .$$

This expression carries a lowest order relation between κ^2 and Λ, which we insert in Eq. (8). Dropping for a moment the Λ_s term, integration with respect to time yields

$$\Lambda = \left[\Lambda_0^2(s) - \frac{5\lambda^2\kappa^2}{8} \right]^{1/2} .$$

The leading order correction to $\mathbf{u}(s)$ is seen to be of the order

$$\mathbf{u}_\Lambda \sim -\hat{\mathbf{b}}\frac{5\lambda^2\kappa^3}{16\Lambda_0} , \tag{18}$$

about an order of magnitude smaller than the smallest terms in Eq. (7). The term we left out will not change this. Equation (18) corresponds to two new terms on the left-hand side of Eq. (14c),

$$-\frac{5\lambda^2}{16\Lambda_0}[(|\psi|^2\psi)_{ss} + \tfrac{3}{4}|\psi|^4\psi] ,$$

and a contribution to the Hamiltonian

$$H_3 = -\frac{5\lambda^2}{32}\int \frac{1}{\Lambda_0}[\tfrac{1}{2}|\psi|^6 + |\psi|^2(|\psi|^2_{ss} + \psi^*\psi_{ss})]ds \ . \tag{19}$$

In terms of the NSE one can say that nonlinearity and nonlinear dispersion are added. The term (19) is not real, and it must be supplemented with the neglected term. Since only the binormal velocity component is affected by the Λ variation, and the neglected term apparently is complicated, we are unlikely to obtain yet another integrable set of terms, cf. Ref. [16]. It appears that stretching implies nonintegrability, which might affect the detailed behavior of solitons. Most important, the elasticity of soliton collisions may be lost. We will not delve into the details of this here. When it comes to linear theory, we will see that a qualitative difference can be distinguished.

We finally note that there is a special sense to the usual structure

$$-i\psi_t = \frac{\delta H}{\delta\psi^*} , \tag{20a}$$

$$i\psi_t^* = \frac{\delta H}{\delta\psi} , \tag{20b}$$

where ψ^* is usually a dynamically equivalent time reverse of ψ. Instead, (20b) gives the dynamics of the chiral reverse ($\tau \to -\tau$) of the original filament curve. This is because of the influence of the pseudovector field $\mathbf{B}$, or $\mathbf{\Omega}$. Indeed, time *and* magnetic field reversal leaves dynamics unchanged.

V. LINEAR PROPERTIES

The first question that arises in the context of linear theory is, "linear with respect to what?" We have several choices. In particular, we will consider linearized equations for ψ and for a small displacement $\delta\mathbf{r}$.

A linear perturbation $\delta\mathbf{r}$ in the location of an element of a filament whose length is initially L results in a stretching energy $\delta E_1 = \varepsilon\delta L = -\int \varepsilon\delta\mathbf{r}\cdot\delta\mathbf{r}_{ss}ds$, if we disregard a tangential perturbation. On the other hand, the variation of E with L kept constant is

$$\delta E_2 = \int \delta\mathbf{r}\cdot \left[\frac{n\Phi}{2}\hat{\mathbf{t}}\times\delta\mathbf{r}_t - \frac{3n\Phi^2\lambda^2}{16\pi}\delta\mathbf{r}_{ssss} \right] ds \ . \tag{21}$$

The cross product of $\mathbf{\Omega}$ with Eq. (5) was substituted into the variation of (9). Totaling δE to zero, the linear frequency is obtained:

$$\omega = \frac{\Phi k^2}{4\pi}\left[\Lambda - \frac{3\varepsilon_k^2}{8} - \frac{3\lambda^2 k^2}{4} \right] . \tag{22}$$

This is the frequency of the so-called Kelvin waves [8]. If the ground state is a straight filament, the perturbation is a plane or helical sinusoidal rotating ripple. The perpendicular force in the expression (21) for δE_2 consists of the Lorentz force (the magnetic part of $\mathbf{\Omega}$) and of the Magnus force (the inertial part of $\mathbf{\Omega}$). We will return briefly to this in Sec. VI A. Corresponding results in the hydrodynamical case are more complicated, even in the leading order term, because of lack of screening.

A linear perturbation of the $\psi=0$ state corresponds to a helical ripple on the filament. This analysis presupposes that the perturbation satisfies $\tau \ll \kappa$. The linear dispersion relation of Eq. (14) is (for an initially straight filament)

$$\omega = \frac{\Phi k^2}{4\pi}\left[\Lambda - \frac{3\lambda^2 k^2}{4} \right] , \tag{23}$$

which honors the conjecture of Yankov and Grechikha [29] and fixes their ε to $\frac{3}{4}$. This and Eq. (22) are close to the helicon dispersion relation, which comes as no surprise, as helicons have been shown to be exact solutions to the equation of motion [24]. Nozières and Vinen touched upon this as well [18]. Including the variation of Λ, a term $-\Phi(2ik\Lambda_s + \Lambda_{ss})/4\pi$ is added to the rhs of (23). This means that if Λ is made nonuniform by some mechanism, waves will travel in the direction of decreasing Λ, or increasing filament thickness. Thus, the effect of stretching, manifested in the variation of Λ, is to inflate a linear perturbation to a nonlinear wave.

Assuming a ground state of the form of a plane circle, $\psi_0 = \kappa$, we find to leading order

$$\omega = \frac{\Phi\Lambda k^2}{4\pi}\left[1 - \frac{2\kappa^2}{k^2} + \frac{3\kappa^4}{4k^4} \right]^{1/2} . \tag{24}$$

Imposing periodicity by the integer p, $k = p\kappa$ and (24) becomes

$$\omega = \frac{\Phi\Lambda\kappa^2}{4\pi}\left[(p^2-1)^2 - \frac{p^2}{4} \right]^{1/2} .$$

Hence, for $p = 2, 3, \ldots$ the circular filament supports linear waves. The $p=1$ mode only corresponds to a displacement in the plane of the circle, and a tilt.

VI. APPLICATIONS

We turn to a discussion of the possible applications of the preceding sections. In Sec. VI A we treat type II superconductivity, and in Sec. VI B we discuss plasmas in the EMH model.

A. Type II superconductors

Flux lines in a type II superconductor ($\varepsilon_\xi \ll 1$ being the inverse Ginzburg-Landau parameter) exhibit behavior that can be described in a macroscopic model. Given the quantized magnetic flux $\phi_0 = h/2e$ and a thermodynamically determined value of ξ (referred to as the coherence length), the motion of a flux line should be well described by results obtained in this paper. However, unless the applied field H is only slightly larger than the lower critical field H_{c1}, the flux lines will ultimately approach each other to within distances smaller than the penetration depth λ, and nonlocal and collective effects will be important [17]. Still, a few general conclusions can be drawn.

The law of motion of a flux line as expressed by its velocity (7), in particular to leading order, takes the same form as that of a neutral fluid VF. Applied flow and drag forces will modify dynamics. In superfluid helium this has been studied numerically by Schwarz [15], who finds the local self-induction term to be dominant. Schwarz has examined pinning, reconnection (flux line cutting), vortex tangle formation, etc. These phenomena are currently being investigated in superconductors. A flux line simulation will give important information even in this case, *mutatis mutandis.* For instance, pinning is a bulk effect, sometimes modeled by friction. In recent years reconnection has been found in anisotropic superconductor simulations [20]. An essential step was to abandon the 2D approach in favor of the fully 3D interaction [21]. A universal route to vortex reconnection has been suggested, well in line with these simulations [37]. Furthermore, in the sparse flux limit not only Kelvin waves [17], but also the helical distortions and the solitons should be present, or possible to excite. The presence of pinning makes these modes hard to detect in reality, as it effectively disperses the spectrum.

A qualitative dynamical 3D model of pinning has been proposed [29], where electron-density depletion centers might catch a VF. Energy is released [cf. Eq. (10), $\varepsilon \sim n$] in the form of waves propagating away from the pinning center. Ricca discusses how to generalize the equations of motion by treating density variations as deformations on a Riemannian manifold [32]. These two references point at two different ways of treating a fluid of inhomogeneous density. We should mention here that in superconductors the number of pinning centers per flux line is usually large, whereas in superfluid helium this is not so.

The appearance of the Magnus and Lorentz forces in parallel in Sec. V deserves some comment. A vortex in an external flow experiences the Magnus force. In a naïve sense magnetic flux may be considered as "electromagnetic vorticity," and the Lorentz force is just an "electromagnetic Magnus force." (That charge times flux actually gives the angular momentum contained in the fields is easily confirmed; see, e.g., Peshkin [38].) This terminology of ours should not be taken as a standpoint in the revived discussion about the existence of a Magnus force in superconductors, where *microscopic* arguments are needed [39]. In the London model there is no doubt.

Our treatment of a charged fluid focused on an isotropic medium, though there has long been an interest in anisotropic media. We wish to refer to Kogan's paper [40] as an early report in this direction. Later developments, with an eye to high T_c superconductors, were reviewed by Brandt [21], as were some experiments in his review with Essmann [41]. Recent progress in the study of the dynamics of the flux line lattice might be traced through Ref. [42]. We remark that in this work a quasi-2D approach is made, as the position vector of each filament is given as a single-valued function of a Cartesian coordinate. This causes the omission of an important class of reconnection possibilities.

B. EMH plasma model

Starting from a two-fluid model of a plasma, the magnetohydrodynamical (MHD) model [43] is obtained by neglecting electron mass and taking ion velocity to be the fluid velocity. However, events occurring on time scales shorter than the ion plasma time, or on space scales shorter than the ion fluid penetration depth, where ions are dominated by inertia, will not be described by the MHD model. To evade the complexity of a two-fluid model, the opposite extreme, the EMH model, is invoked [22]. On space scales larger than λ (which we have defined to be the *electron* fluid penetration depth), the inertial part of the generalized momentum is usually neglected, and so, for instance, a mechanism of magnetic field convection by electrons is found. Small scales are generated, and therefore the inertia is kept. (Still there is no displacement current in Ampère's law.) A locus for energy transfer from the magnetic field to the electrons or vice versa is provided by the generalized momentum.

The derivation of the fluid equation in Sec. II is a standard EMH derivation. In consequence, a kind of hydrodynamics emerges with a screening property. Another difference from the Euler fluid is the high energy cost of creating voluminous twist and bend structures of the generalized vorticity lines. Either a smooth distribution is apt, or smooth highly localized structures will appear as argued above. We feel that the study of thin vortex filaments is relevant in this context.

As is seen in Sec. IV, energy is transported (in the guise of curvature) by the filament. Typically, a filament could carry energy from a 3D turbulence region to a region where dynamics is essentially 2D (cf. Hopfinger and Brownand, Ref. [5]). The filament could then be an organizer of coherent vortices. The converse is also imaginable.

VII. SUMMARY

We have examined a set of properties of a vortex filament in a charged fluid. By "charged fluid" we can understand, e.g., the object of a time-dependent London model for type II superconductors, or of the EMH plasma model.

Because of screening, the LIA is not needed and the expression for the VF velocity is more accurate than in the neutral fluid case. Applying the Hasimoto transformation to the velocity, an integrable extension of the NSE is found to govern the evolution. The form of the solutions is similar to that of unperturbed NSE solutions. Taking vortex stretching into account, a small, nonintegrable contribution is added.

The VF supports linear waves, nonlinear waves, and solitons. The soliton velocities depend on the electrostatic screening in a simple way. Similar to observations in water tank experiments, we except the filament to provide a mechanism for energy transfer in space, as well as between different realizations (2D or 3D, kinetic or magnetic).

Flux line simulations for superconductors, analogous to simulations for superfluid helium, are believed to be an important development where an extended London model could be used. By allowing energy input, e.g., from an external magnetic field, the flux lines will grow, and more complex, entangling dynamics arises. Knowledge about the formation of vortex glass might become available. Future work will focus on some ideas pertaining to inhomogeneous charged fluids [29] and other generalizations of the simplistic model treated in this paper.

As we pointed out in the Introduction, there are several areas of investigation in VF theory still lying open. The extension to charged fluids given in this paper increases the set of applications, and makes further work in the field even more important.

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