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Vicious walkers and directed polymer networks in general dimensions

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A number, p, of vicious random walkers on a D-dimensional lattice is considered. "Vicious walkers" describes the situation when two or more walkers arrive at the same lattice site and annihilate one another, and consequently their walks terminate. In certain cases the generating function $R(u)[S(u)]$ for the number of configurations $R_s[S_s]$ of walkers which reunite [survive] after s steps is expressed in terms of generalized hypergeometric functions. The critical exponents associated with these functions are in agreement with known results for Brownian paths. The critical dimension $D=2$ also agrees with that found for the continuum limit, and logarithmic corrections are discussed. Vicious walker configurations correspond to directed polymer networks in $d=D+1$ dimensions, and in the case $D=1$ they also correspond to directed integer flows in which the flow in any bond is in the range 0 to p.

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I. INTRODUCTION

The problem of vicious walkers, together with many physical applications, is described by Fisher [1] in a paper where a number of fundamental results are derived and useful techniques are discussed. The general model is one of p random walkers on a D-dimensional lattice who at regular time intervals simultaneously take one step with equal probability in the direction of one of the k lattice vectors. The walkers are described as vicious since if two or more of them arrive at the same site they annihilate one another. Questions which may be posed concern the probability $P_S(s)$ that they all survive for at least s steps, and the probability $P_R(s)$ that they all survive for $s-1$ steps and all make a reunion on the next step.

Fisher was mainly interested in the asymptotics for $s\to\infty$, and replaced the walkers by particles undergoing Brownian motion which is the continuum limit of the problem. He found that for $D=1$

$$P_S(s)\sim s^{-\psi} \quad \text{and} \quad P_R(s)\sim s^{-\Psi}\ , \tag{1}$$

where

$$\psi=\frac{p(p-1)}{4} \quad \text{and} \quad \Psi=\frac{(p^2-1)}{2}\ . \tag{2}$$

These continuum results have very recently been extended to higher dimensions by Mukherji and Bhattacharjee [2,3]. They find that above the critical dimension $D_c=2$ the losses due to vicious encounters have no effect on the exponents, which are

$$\psi=0 \quad \text{and} \quad \Psi=\frac{(p-1)D}{2} \quad \text{for } D\geq 2\ , \tag{3}$$

with logarithmic corrections at $D=D_c$. The decay in the case of the reunion probability is due solely to the constraint that the walkers must meet up somewhere after s steps. This paper will be concerned with exact combinatorial formulas for walks on a lattice, which we find to have the same asymptotic form as the above continuum results.

A related problem is one of enumerating the configurations of a polymer network on a d-dimensional directed lattice. The bonds of the lattice are directed so as to have a positive component relative to some preferred direction, and the number of bonds directed away from each site will be denoted by k. The polymer networks considered are constructed from chains of equal length s (having $s+1$ monomers), and we denote the topology of the network by G. The simplest topology is just a single chain; a number of chains connected at a single point is known as a *star* [see Fig. 1(a)], and a network of chains connected in parallel between two points is known as a *watermelon* [see Fig. 1(b)]. A directed polymer network is one which is embedded on the lattice, so that its chains form directed walks on the lattice and no site is occupied by more than one monomer. We consider the chains to be distinguishable to avoid trivial symmetry factors. The number of embeddings of a single chain with one end fixed is just k^s, but for other networks the topological constraint and self-avoidance condition make the problem more difficult. The number of configurations of a network of p chains will be denoted by $W_s(p,G)$, and for all cases considered it turns out that

$$W_s(p,G)\sim k^{sp}s^{-\psi(G)}\ , \tag{4}$$

with logarithmic corrections for $d=3$. This implies that the corresponding generating function

$$W_G(p,u)\equiv\sum_{s=0}^{\infty}W_s(p,G)u^s \tag{5}$$

has a singular part with asymptotic form

$$W_G(p,u)\sim(1-k^pu)^{-\gamma(G)} \tag{6}$$

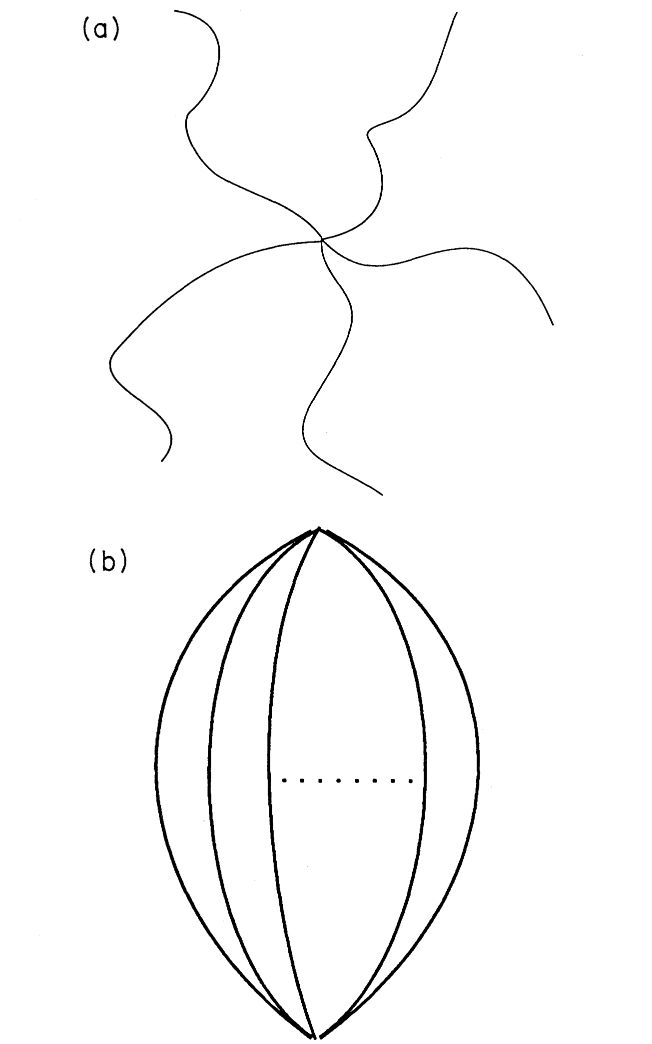

FIG. 1. (a) A polymer with star topology. (b) A polymer with watermelon topology.

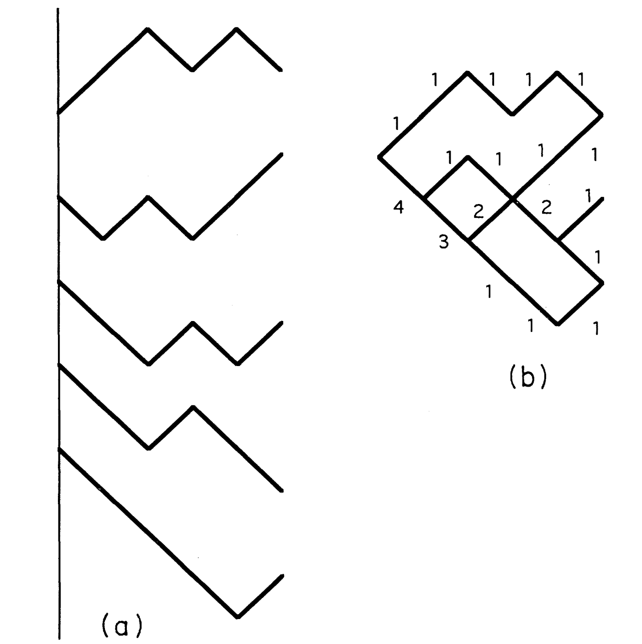

FIG. 2. (a) A polymer brush with five branches embedded on the directed square lattice or the space-time trajectories of five one-dimensional vicious walkers. (b) The noncrossing embedding of a star with five branches, which corresponds to the brush in (a). The number on each bond is the number of branches which pass through that bond.

as $u \rightarrow u_c = k^{-p}$, where the critical exponent

$$\gamma(G) = 1 - \psi(G) \ . \tag{7}$$

The vicious walker problem is equivalent to a directed polymer network on a lattice of dimension $d = D+1$. For example, consider walkers which start from adjacent points on the even sublattice of a linear chain and survive for s steps. If the trajectories are plotted in space-time, the points visited will be on a directed square lattice, and in polymer terminology the corresponding network is known as a *brush* [see Fig. 2(a)]. The self-avoidance condition of the network is equivalent to the vicious nature of the walkers. The survival probability is given by

$$P_S(s) = W_s(p, \text{brush}) / k^{sp} \ , \tag{8}$$

and comparing with (1) shows that $\psi(\text{brush}) = \psi$. If the walkers all start from the same point a star network is formed which is expected to have the same critical exponent as the brush, except that there can be no more than k chains. Similarly, vicious walkers which start at adjacent points and reunite at adjacent points after s steps give rise to a pair of brushes which have had their hairs joined in pairs (see Fig. 3). The critical exponent for this network will be the same as for a watermelon and hence $\psi(\text{watermelon}) = \Psi$. If the vicious walkers start from the same sublattice of a D-dimensional body-centered-hypercubic (bchc) lattice, the corresponding polymer network is on a similar lattice of one higher dimension.

Recently Guttmann and Prellberg [4] gave some exact combinatorial results for staircase polygons on the hypercubic lattice. These polygons are a subset of the self-avoiding polygons, and their enumeration is equivalent (apart from a symmetry factor of 2) to counting directed watermelons with two chains. This polymer problem corresponds to the reunion of vicious walkers on a cyclically directed D-dimensional hypertriangular lattice. For $D = 2$ this lattice has been considered by Blease [5] in the context of directed percolation.

The number of watermelons with two chains, $W_s(2, \text{watermelon})$, will be denoted by R_s. The generating function $R(u)$ for R_s on a general directed lattice may be written [6] in the form

$$R(u) = 1 - ku - Z(u)^{-1} \ , \tag{9}$$

where $Z(u)$ is the generating function for two-chain watermelons when the avoidance condition is relaxed. By convention we take $Z(0) = 1$ and $R(0) = 0$. This for-

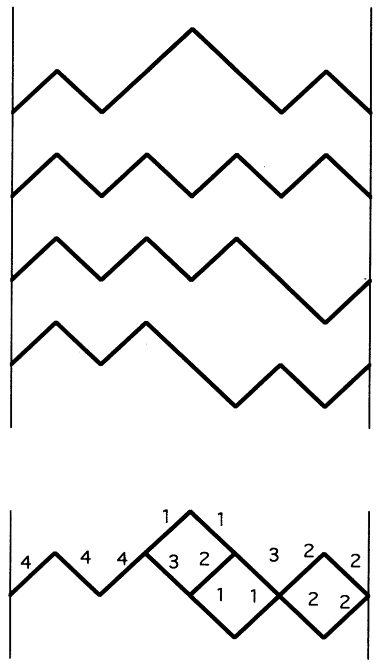

FIG. 3. A noncrossing embedding of a watermelon with four chains, and the corresponding joined pair of brushes. The number on each bond is the number of branches which pass through that bond.

mula was given by Guttmann and Prellberg [4] for the hypercubic lattice, and they also showed that, for $d=3$ and 4, $Z(u)$ is expressible in terms of Heun functions which satisfy a second order differential equation. For $d=2$, $Z(u)$ is algebraic, and for $d\geq 5$ it satisfies a differential equation the order of which increases with dimension.

For both the hypercubic (hc) and body-centered-hypercubic (bchc) lattice $Z(u)$ is expressible [6] in terms of the standard Green's function $G(D,u)$ which enumerates random walks which return to the origin on an undirected D-dimensional lattice. For the body-centered case the result is simply $Z_{\rm bchc}(d,u) = G_{\rm bchc}(D,u)$, but it is not so obvious that $Z_{\rm hc}(d,u)=G_{\rm hd}(D,u)$, where $G_{\rm hd}(D,u)$ is the generating function for returns to the origin on the D-dimensional hyperdiamond lattice (honeycomb for $D=2$).

In Sec. II we see that $Z_{\rm bchc}(d,u)$ is a generalized hypergeometric function. It is also shown that the generating function $S(u)\equiv W_{\rm star}(2,u)$ for the number of directed stars with two chains on a general directed lattice is expressible in terms of $Z(u)$. We note that $d=3$ ($D=2$) is the critical dimension for both watermelons and stars, in agreement with the continuum result of Mukherji and Bhattacharjee [2,3]. However, we differ on the exponent of the logarithmic factors which occur at the critical dimension [7].

In Sec. III, we consider the problem of the number of directed watermelons, $Z_s(p,d)$, with $p\geq 2$ chains of length s on a d-dimensional body-centered-cubic lattice, ignoring the avoidance condition. For $d=2$, the square lattice, this number is expressible as the sum of the pth powers of the binomial coefficients of order s. As a function of s these sums are known [8] to satisfy linear recurrence relations with polynomial coefficients. The relations for $p=2$, 3, and 4 were found by Franel [9,10] on the basis of which he conjectured that the order n of the relation is $p/2$ for even p and $(p+1)/2$ for odd p, and that the coefficients are of degree $p-1$. Perlstadt [11] verified the first part of the conjecture for $p=5$ and 6, but found that the coefficients had degrees 6 and 9 respectively. Cusick [12] established Franel's conjecture for the order of the recurrence in general, and gave a computational scheme for the coefficients involving the solution of $n(n-1)$ simultaneous linear equations.

The fact that $Z_s(p,d)$ satisfies a linear recurrence relation with polynomial coefficients implies that its generating function satisfies a differential equation of order equal to the degree of the coefficients in the recurrence relation. By examination of the differential equations for three and four chains the generating functions are identified as Heun functions. For any value of p the generating function is expressed in terms of the standard Green's function, $G(u)$, for the undirected p-dimensional body-centered-hypercubic lattice projected along its $(1,1,1,\ldots)$ direction. For $p=3$ the projected lattice is the triangular lattice, whereas for $p=4$ it is the body-centered-cubic lattice with first and second neighbor bonds. For general dimension $Z_s(p,d)=Z_s(p,2)^{(d-1)}$ and according to the theory of Zeilberger [13], the number of watermelons will still satisfy a linear recurrence relation with polynomial coefficients. We find such relations for $d=3$ and $p=3$ and 4. The relation for $p=3$ has the same order and degree as that for $d=2$ and $p=5$.

In Sec. IV, the avoidance condition is imposed, and only the directed square lattice is considered. An exact expression is derived for the number of brushes in which the hairs have fixed ends. This is used to express the number of noncrossing watermelons with p branches as a finite sum having a hypergeometric summand. The theory [13] which we quoted for sums of powers of binomial coefficients includes these more general sums, and we use the algorithm of Zeilberger [7] to derive the recurrence relations for noncrossing watermelons with $p\leq 6$. It is found that the order of these relations is the same as when the chains are allowed to cross, but the degrees of the coefficients for $p=2,\ldots,6$ are 1, 2, 4, 8, and 11. The exact expression for brushes with fixed ends is used to generate the number of noncrossing stars with $s\leq n$, where n is sufficiently large to enable the recurrence relations, which exist by Zeilberger's theory, to be found by computer search. On the basis of the relations for $p\leq 7$ a simple formula is conjectured for the number of such stars, for any value of p, and the corresponding generating functions are related to generalized hypergeometric functions.

Finally in Sec. V we consider the extension of the previous results to general polymer networks and obtain results similar to those of Duplantier and Saleur (see Ref. [14] and references therein) for undirected networks. For $d=2$ their decomposition of critical exponents into vertex contributions has been extended to directed networks [15] using the vicious walker results of Fisher [1]. At the critical dimension $d=3$, there is a logarithmic factor whose exponent is a sum of vertex parts.

II. DIRECTED WATERMELONS AND STARS WITH TWO CHAINS IN d DIMENSIONS

In Sec. I we recalled the known relation (9) between the number of watermelons having two chains, with and without the avoidance condition. We now show that in the case $p=2$ the number of stars S_s whose chains are nonintersecting may also be expressed in terms of the number of watermelons with no exclusion. The number of configurations of two chains, each of length s, which start at the same lattice point is k^{2s}, since each bond may be in one of k directions. Hence the generating function for these configurations is $(1-k^2u)^{-1}$, where we have included the term $s=0$. The configurations counted by this formula will include both nonintersecting and intersecting stars. Each intersecting star may be uniquely decomposed into a watermelon each chain of which has t bonds, for some t in the range 1 to s, followed by a nonintersecting star of length $s-t$. Thus, with $S_0=Z_0=1$,

$$k^{2s}=S_s+\sum_{t=1}^{s} Z_t S_{s-t}=\sum_{t=0}^{s} Z_t S_{s-t} , \tag{10}$$

or, in terms of generating functions,

$$S(u)=(1-k^2u)^{-1}Z(u)^{-1} . \tag{11}$$

The generating function $Z(u)$ for the body-centered-hypercubic lattice is easily seen to be a generalized hypergeometric function. Thus, for $d=2$ the square lattice

$$Z_s(\text{square})=\begin{bmatrix} 2s \\ s \end{bmatrix} , \tag{12}$$

since in this case the watermelon configurations correspond to random walks in one dimension which return to the origin in $2s$ steps. The generating function is therefore

$$Z_{\text{sq}}(u)=G_{\text{chain}}(u)=(1-4u)^{-1/2} , \tag{13}$$

which, using Eqs. (11) and (9), leads to the results

$$S_{\text{sq}}(u)=(1-4u)^{-1/2} \tag{14}$$

and

$$R_{\text{sq}}(u)=1-2u-(1-4u)^{1/2} . \tag{15}$$

Comparison with (6) shows that the γ exponents of these functions are $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and using (7) gives $\psi=\frac{1}{2}$ and $\Psi=\frac{3}{2}$. The asymptotic form of the coefficients S_s and R_s is therefore in agreement with the case $p=2$ of Eq. (1).

In higher dimension the watermelon configurations on a d-dimensional body-centered-cubic lattice project onto random walks which return to the origin on a $(d-1)$-dimensional body-centered-cubic lattice (see [6] for details). A given step vector of such a walk has a positive or negative component in each of $(d-1)$ dimensions. Since these components may be chosen independently with the only constraint being that the walk must return to the origin in every dimension, we find

$$Z_s(\text{bchc},d)=G_s(\text{bchc},d-1)$$

$$=\begin{bmatrix} 2s \\ s \end{bmatrix}^{d-1} \tag{16}$$

$$\sim(\pi^{-1}16^s s^{-1})^{(d-1)/2} \quad \text{for } s\to\infty . \tag{17}$$

Clearly, $Z_s(\text{bchc},d)$ is generated by a generalized hypergeometric function, since it may be written

$$Z_s(\text{bchc},d)=4^{(d-1)s}\left[\frac{(\frac{1}{2})_s}{(1)_s}\right]^{d-1} , \tag{18}$$

where $(a)_s$ is Pochhammer's symbol (see Appendix A) and, comparing with Eq. (A3) Appendix A,

$$Z_{\text{bchc}}(d,u)={}_{d-1}F_{d-2}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\ldots;1,1,\ldots;4^{d-1}u) . \tag{19}$$

This is a generalization of the formula given by Joyce [16] for the standard body-centered-cubic lattice Green's function $G(\text{bcc},u)$. For $d\geq 3, Z_{\text{bchc}}(d,u)$ has a singularity at $u=u_c=4^{-(d-1)}$ with the asymptotic form

$$Z_{\text{bchc}}(d,u)\sim\begin{cases} Z_c(d)-A(d)(1-4^{d-1}u)^{(d-3)/2} & \text{for } d \text{ even} \\ Z_c(d)-A(d)(1-4^{d-1}u)^{(d-3)/2}\ln(1-4^{d-1}u) & \text{for } d \text{ odd} . \end{cases} \tag{20}$$

For $d>3$, expanding $Z_{\text{bchc}}(d,u)^{-1}$ about u_c and substituting in Eqs. (11) and (9) gives

$$S_s(\text{bchc},d)\sim\frac{4^{(d-1)s}}{Z_c(d)} \tag{21}$$

and

$$R_s(\text{bchc},d)\sim\frac{Z_s(\text{bchc},d)}{Z_c(d)^2}\sim\frac{4^{(d-1)s}}{(\pi s)^{(d-1)/2}Z_c(d)^2} . \tag{22}$$

Notice that except for the amplitude factors, $Z_c(d)^{-1}$ and $Z_c(d)^{-2}$, these forms are the same as when there is no mutual exclusion; that is, $4^{(d-1)s}$ and Z_s, respectively. In particular the critical exponents ψ and Ψ are given by Eq. (3). We therefore say that the directed polymer problem has a critical dimension $d_c=3$, as found for the continuum model [2,3].

As usual the critical dimension is marked by the occurrence of logarithmic factors in the asymptotic forms. Thus for $d=3$ the second term in (20) diverges at u_c and dominates the constant $Z_c(3)$. Also, in this case, $Z(u)$ may be expressed as

$$Z_{\mathrm{bcc}}(u)=\sum_{s=0}^{\infty}\left[\binom{2s}{s}\right]^2 u^s=(2/\pi)K(16u)\ , \qquad (23)$$

and using the asymptotic form of the elliptic integral $K(m)$ for $m\to 1$ gives

$$Z_{\mathrm{bcc}}(u)\sim -(1/\pi)\ln(1-16u) \quad \text{for } u\to u_c\ , \qquad (24)$$

which may also be deduced from (17) with $d=3$. Substituting in (11) and (9) we find that the asymptotic forms of the numbers of stars and watermelons are given by

$$S_s(\mathrm{bcc})\sim \pi 16^s/\ln(s) \qquad (25)$$

and

$$R_s(\mathrm{bcc})\sim \pi 16^s/(2s\ln^2(s))\ . \qquad (26)$$

As expected, apart from amplitude factors, the watermelon results above have the same asymptotic form as the staircase polygon results [4] for the hypercubic lattice.

The results of Mukherji and Bhattacharjee [2,3] for the continuum model at the critical dimension differ from ours by a factor of 2 in the exponent of the logarithm. We have therefore examined their calculation and find that solving their renormalization-group (RG) differential equation gives results which are in agreement with ours [8]. The details are given in Sec. V where we generalize the formula to an arbitrary network.

III. WATERMELONS WITH MORE THAN TWO CHAINS BUT NO MUTUAL AVOIDANCE

This problem is equivalent to the reunions of more than two friendly walkers. The number of star configurations with no mutual avoidance is trivially given by $W_s(p,\mathrm{star})=k^{ps}$, but the watermelon constraint leads to results which are not so simple.

As in Sec. II only the body-centered-hypercubic lattice will be considered. There it was found that in the case of two chains the generating function was hypergeometric and the coefficients therefore satisfy a first order recurrence relation. We denote the number of watermelon configurations $W_s(p,\mathrm{watermelon})$ by $Z_s(p,d)$, and the recurrence relation for $p=2$ is

$$s^{d-1}Z_s(2,d)-2^{d-1}(2s-1)^{d-1}Z_{s-1}(2,d)=0\ . \qquad (27)$$

For three or more chains we shall find that higher order recurrence relations are satisfied and that the polynomial coefficients have higher degrees. A similar result is found on introducing mutual exclusion in Sec. III A.

For $d=2$ and $p\geq 3$ it turns out that it is still possible to relate the generating function to that for returns to the origin of a single random walker, but for $p>3$ the lattice has second and higher neighbor bonds.

A. Recurrence relations and differential equations

Just as for $p=2$ an explicit formula for the number of directed watermelons with no mutual exclusion may be obtained by considering the projection onto an undirected $(d-1)$-dimensional body-centered-cubic lattice. Since the enumeration may be carried out independently in each dimension,

$$Z_s(p,d)=Z_s(p,2)^{(d-1)}\ , \qquad (28)$$

and for $d=2$ the undirected one-dimensional problem may be solved by partitioning according to the distance r of the endpoint of each walk from the origin, yielding

$$Z_s(p,2)=\sum_{q=0}^{s}\binom{s}{q}^p\ . \qquad (29)$$

For two walkers this reduces to Eq. (16). Using (29) it may be verified directly that, asymptotically, for $s\to\infty$, $Z_s(p,d)\sim 2^{(d-1)ps}s^{\Psi}$, where Ψ is given by (3).

In Sec. I we discussed previous work on sums of the type appearing in Eq. (29). Zeilberger [13,7] considered a more general type of summation, which will include the formulas for nonintersecting watermelons in Sec. III B. He showed that, if $F_s(q)$ is such that both $F_s(q)/F_{s-1}(q)$ and $F_s(q)/F_s(q-1)$ are rational functions of the integers s and q, then

$$f_s=\sum_q F_s(q) \qquad (30)$$

satisfies a linear recurrence relation with polynomial coefficients. He also gave an algorithm which determines the relation together with the MAPLE code. In Sec. III B we use an extended MATHEMATICA version of this code written by Paule and Schorn [17].

The code rapidly enables the recurrence relations of Perlstadt [11], for $Z_s(p,2)$ with $p\leq 6$, to be rederived. It also provides a certificate which allows a simple verification of the results. An excellent discussion of verification and certificates is given in the lecture notes of Wilf [18], and is summarized in Appendix B. The relations for $p=3$ and 4, which were also given by Franel [9,10] some 90 years before Perlstadt, are

$$\begin{aligned} &s^2Z_s(3,2)-(7s^2-7s+2)Z_{s-1}(3,2)\\ &\qquad\qquad -8(s-1)^2Z_{s-2}(3,2)=0\ ,\\ &s^3Z_s(4,2)-2(2s-1)(3s^2-3s+1)Z_{s-1}(4,2)\\ &\qquad\qquad -4(s-1)(4s-5)(4s-3)Z_{s-2}(4,2)=0\ . \end{aligned} \qquad (31)$$

The corresponding generating functions $Z(p,d;u)\equiv W_{\mathrm{watermelon}}(p,u)$ consequently satisfy linear differential equations with polynomial coefficients. For $p=3$, with $y=Z(3,2;u)$, the equation is

$$u(1-7u-8u^2)y''+(1-14u-24u^2)y'-2(1+4u)y=0\ , \qquad (32)$$

which has singular points at $u=\frac{1}{8}$ and -1, with γ exponent 0 at both points corresponding to logarithmic singularities in the generating function. For $p=4$, where now $y=Z(4,2;u)$, we find

$$\begin{aligned} &u^2(1-12u-64u^2)y'''+u(3-54u-384u^2)y''\\ &\qquad +(1-40u-444u^2)y'-2(1+30u)y=0\ , \end{aligned} \qquad (33)$$

which has singular points at $u=\frac{1}{16}$ and $u=-\frac{1}{4}$, with exponents $-\frac{1}{2}$ at both points. For $p=5$ we give only the differential equation which is, where now $y=Z(5,2;u)$,

$$\begin{aligned}&55u^5(1-32u)(1+11u-u^2)y^{(6)}+22u^4(29-819u-17\,297u^2+1888u^3)y^{(5)}\\&\quad+u^3(1865-81\,273u-2\,317\,838u^2+318\,048u^3)y^{(4)}+u^2(1235-116\,168u-5\,106\,304u^2+943\,520u^3)y^{(3)}\\&\quad+4u(5-9744u-900\,682u^2+254\,200u^3)y''+4(7-164u-128\,052u^2+74\,464u^3)y'-8(7+230u-1168u^2)y=0\end{aligned} \tag{34}$$

having singular points at $u=\frac{1}{32}$ and $u=11\pm5^{3/2}$, with exponents -1 at all three points. The exponents agree with Eq. (3) in all cases.

We note that for $p=3$ the differential equation is of second order with four regular singular points and the generating function is therefore a Heun function. In the notation of Snow [19],

$$Z(3,2;u)=F(-\tfrac{1}{8},-\tfrac{1}{4};1,1,1,1;-u)\ . \tag{35}$$

Applying transformations (VII.15) and (VII.13) from Snow to the above Heun function gives

$$Z(3,2;u)=(1-8u)^{-1}\times F[\tfrac{9}{8},-\tfrac{3}{4};1,1,1,1;-9u/(1-8u)]\ , \tag{36}$$

and hence, following the same procedure as Guttmann and Prellberg [4], we find

$$Z(3,2;u)^2=(1-8u)^{-1}(2/\pi)^2K(k_+)K(k_-)\ , \tag{37}$$

where k_+ and k_- are given by Eq. (26) of Guttmann and Prellberg.

For $p=4$ the differential equation is third order but of a special type which may be reduced to a second order equation for $Y(u)\equiv[Z(4,2;u)]^{1/2}$. A similar reduction was found by Guttmann and Prellberg for staircase polygons on the hypercubic lattice with $d=4$. We find that the coefficients in the expansion of $Y(u)$ satisfy the recurrence relation

$$s^2Y_s-(7-18s+12s^2)Y_{s-1}-(143-192s+64s^2)Y_{s-2}=0\ , \tag{38}$$

and so $Y(u)$ satisfies the differential equation

$$u(1+4u)(1-16u)Y''+(1-18u-128u^2)Y'-(1+15u)Y=0\ , \tag{39}$$

and hence $Y(u)$ is a Heun function. In this case the Heun function is

$$Y(u)=F(-4,\tfrac{1}{4};\tfrac{3}{8},\tfrac{5}{8},1,\tfrac{1}{2};16u)\ , \tag{40}$$

which appears to afford no obvious simplification by the various transformations given in Snow [19].

Functions which satisfy linear difference or differential equations with polynomial coefficients are said to be holonomic. Zeilberger [13] shows that the product of two holonomic functions is holonomic. It follows from (28) that $Z_s(p,d)$ is holonomic. For example, Eqs. (31) are of the form

$$U(s)a_s+V(s)a_{s-1}+W(s)a_{s-2}=0\ , \tag{41}$$

and if $b_s=a_s^2$ a straightforward but lengthy calculation shows that

$$\bar{u}\bar{v}u^2b_s+\bar{u}v(\bar{u}w-\bar{v}v)b_{s-1}+\bar{v}w(\bar{v}v-\bar{u}w)b_{s-2}-\bar{w}^2vwb_{s-3}=0\ , \tag{42}$$

where

$$\begin{aligned}&u=U(s),\quad v=V(s),\quad w=W(s)\ ,\\&\bar{u}=U(s-1),\quad \bar{v}=V(s-1),\quad \bar{w}=W(s-1)\ .\end{aligned} \tag{43}$$

Using (28) and (42), we find that for $d=3$, and $p=3$

$$s^4(16-21s+7s^2)Z_s(3,3)+(2-7s+7s^2)f(s)Z_{s-1}(3,3)+8(16-21s+7s^2)f(s)Z_{s-2}(3,3)+512(s-2)^4(2-7s+7s^2)Z_{s-3}(3,3)=0\ , \tag{44}$$

where

$$f(s)=-40+186s-321s^2+228s^3-57s^4\ . \tag{45}$$

We notice that the relation is of the same order and the coefficients are of the same degree as for five chains in dimension $d=2$. The corresponding differential equation for $y=Z_s(3,3)$ is

$$\begin{aligned}&7u^5(1-64u)(1-u)(1+8u)y^{(6)}+84u^4(1-76u-760u^2+1024u^3)y^{(5)}+3u^3(87-9964u-132\,912u^2+223\,744u^3)y^{(4)}\\&\quad+3u^2(67-15\,180u-302\,752u^2+678\,400u^3)y^{(3)}+2u(7-8834u-336\,928u^2+1\,125\,632u^3)y''\\&\quad+2(1-316u-52\,928u^2+340\,992u^3)y'-8(1+80u-2816u^2)y=0\end{aligned} \tag{46}$$

which has an exponent -1 at all of the singular points $u=\frac{1}{64}$, 1, and $-\frac{1}{8}$ which is consistent with (1).

Again, using (42), the recurrence relation for $p=4$ and $d=3$ is

$$2s^6(1-s)^2(3-2s)(7-9s+3s^2)Z_s(4,3)+8(1-s)^2(1-2s)(1-3s+3s^2)g(s)Z_{s-1}(4,3)$$
$$+32(3-4s)(3-2s)(5-4s)(7-9s+3s^2)g(s)Z_{s-2}(4,3)$$
$$-128(3-4s)(2-s)^2(1-2s)(9-4s)^2(7-4s)^2(5-4s)(1-3s+3s^2)Z_{s-3}(4,3)=0\ , \quad (47)$$

where

$$g(s)=36-238s+655s^2-952s^3+758s^4-312s^5+52s^6\ . \quad (48)$$

The singular points of the corresponding differential equation, in addition to 0 and ∞, are at $\frac{1}{16}$, $\frac{1}{256}$, and $\frac{-1}{64}$, all with exponent -2 in agreement with (1).

The relations which are known to exist for higher values of p and d are expected to become progressively more complicated.

B. Exact result for the directed square lattice

The problem of enumerating directed watermelons with p chains on the square lattice is the same as that of p random walkers in $D=1$ dimension which must all end at the same point after s steps. Suppose that the coordinates of the walkers are $x_1, x_2, \ldots, x_p$, then an equivalent problem [1] is to consider a single walker on a p-dimensional body-centered-cubic lattice whose position vector has the ith component x_i. The condition that the walkers make a reunion after s steps is that the equivalent single walker is positioned on the line $L: x_1=x_2=\cdots=x_p$. For $p=2$ this gives yet another way of finding the generating function for $D=1$ ($d=2$).

In this case the step vectors for the single walker are $\mathbf{b}_1=(1,1)$, $\mathbf{b}_2=(-1,1)$, $\mathbf{b}_3=(1,-1)$, and $\mathbf{b}_4=(-1,-1)$. Now steps in the directions $\mathbf{b}_1$ and $\mathbf{b}_4$ are parallel to L and are therefore irrelevant in determining the distance from L. The distance from L is n_2-n_3, where n_i is the number of steps in direction i. The number of walks which end on L may therefore be obtained by considering walks perpendicular to L which return to the origin, and inserting at each point of the walk any number of steps parallel to L. Now $G_{\text{chain}}(u^2)=(1-4u^2)^{-1/2}$, which counts the walks perpendicular to L which return to the origin, giving weight u^s to a walk of s edges. Insertion of a walk of arbitrary length at each vertex of the walk is achieved by giving a further weight factor $1/(1-2u)^{s+1}$ to each perpendicular walk. This is done by replacing u by $u/(1-2u)$ in $G_{\text{chain}}(u^2)$ and then multiplying by $1/(1-u)$. We find in this way that

$$Z_{\text{sq}}(p=2,u)=G_{\text{chain}}(u^2/(1-2u)^2)/(1-2u)=(1-4u)^{-1/2} \quad (49)$$

in agreement with Eq. (13).

The argument extends immediately to p walkers, where again just two of the step vectors for the single equivalent walker are parallel to L. It is therefore sufficient to obtain the generating function for configurations of a single walker which return to the origin on a $p-1$ dimensional lattice with the $k-2$ step vectors ($k=2^p$) which are the components of $\mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_{k-1}$ perpendicular to L. This must then be renormalized by the factors $1/(1-2u)$ as for $p=2$.

For $p=3$ the six step vectors define a triangular lattice and hence, using the result of Horiguchi [20], for $0 \le u < \frac{1}{8}$,

$$Z_{\text{sq}}(p=3,u)=G_{\text{tri}}(u/(1-2u))/(1-2u)=g(u)h(u)(2/\pi)K(m(u))\ , \quad (50)$$

where

$$g(u)=[(1+u)^{1/2}-u^{1/2}]^{-3/2}\ ,\qquad h(u)=[(1+u)^{1/2}+3u^{1/2}]^{-1/2}\ , \quad (51)$$

and

$$m(u)=16u^{3/2}(1+u)^{1/2}g(u)^2h(u)^2\ . \quad (52)$$

The solution found in Sec. III A had singular points at $u=\frac{1}{8}$ and $u=-1$ with exponents corresponding to a logarithmic singularity. It is of interest to see how these singularities arise in Eq. (50). The point $u=\frac{1}{8}$ corresponds to $m=1$, which is where $K(m)$ has a logarithmic divergence. However, $u=-1$ corresponds to $m=0$, where $K(m)$ is nonsingular. The second singularity may be found by considering m to be complex and making an analytic continuation of K onto the second Riemann sheet (see [20]).

Note that we define $G_{\text{tri}}(x)$ as the generating function for random walks on the triangular lattice giving weight x^s to a walk of s steps, rather than $2s$ steps as in the case of a bipartite lattice.

This solution shows that the Heun function found for the same problem, in Sec. III A, must be expressible in terms of an elliptic integral. This is a further occurrence of a similar relation which was first observed for staircase polygons on the simple cubic lattice [4,6]. For $p=4$ the walks perpendicular to L are on a body-centered cubic with first and second neighbor bonds.

IV. STAR AND WATERMELON NETWORKS WITH NONCROSSING CHAINS ON THE DIRECTED SQUARE LATTICE

In this section we suppose that the p chains of a star or watermelon network are embedded on the directed square lattice in such a way that no two chains cross one another. It is, however, allowed that the chains have one or more lattice points in common. Suppose now that all chains except the first are translated such that the jth chain is moved through a distance $(j-1)a\sqrt{2}$ in the direction $(-1,1)$, where a is the lattice parameter. The chains then become mutually avoiding and a star configuration [Fig. 2(b)] becomes a brush with the points of attachment of the hairs of the brush equally spaced [Fig. 2(a)]. A watermelon becomes a pair of brushes with corresponding hairs joined (Fig. 3).

A. Exact formulas for networks of chains with fixed endpoints

An explicit formula may be obtained for the case where the polymer chains have fixed endpoints. Brush embeddings with fixed endpoints correspond to one-dimensional vicious walker configurations in which the ith walker starts at $x_{i0}=2(i-1)$ and arrives at x_i after s steps, where $x_i<x_j$ when $i<j$. Let $W_s(\mathbf{x}_0,\mathbf{x})$ be the number of configurations of such walkers, where $\mathbf{x}=\{x_1,x_2,\ldots,x_p\}$, then [1,21]

$$W_s(\mathbf{x}_0,\mathbf{x})=\begin{vmatrix} W_s(x_{10},x_1) & W_s(x_{10},x_2) & \cdots & W_s(x_{10},x_p) \\ W_s(x_{20},x_1) & W_s(x_{20},x_2) & \cdots & W_s(x_{20},x_p) \\ \vdots & \vdots & \vdots & \vdots \\ W_s(x_{p0},x_1) & W_s(x_{p0},x_2) & \cdots & W_s(x_{p0},x_p) \end{vmatrix}, \tag{53}$$

where $W_s(x_{i0},x_j)$ is the number of configurations of a single walker starting at x_{i0} and ending at x_j, i.e.,

$$W_s(x_{i0},x_j)=\begin{bmatrix} s \\ q_j+j-i \end{bmatrix}, \tag{54}$$

where $q_j=\frac{1}{2}(s+x_j-x_{j0})$, the number of positive steps made by the jth walker, which ranges from 0 to s. Also $q_i\le q_j$ when $i<j$. It is shown in Appendix C that (53) can be reduced to a single product, thus

$$W_s(\mathbf{x}_0,\mathbf{x})\equiv w_s(q_1,q_2,\ldots,q_p)=\prod_{1\le i<j\le p}(q_j-q_i+j-i)\prod_{j=1}^{p}\frac{(s+p-j)!}{(q_j+j-1)!(s-q_j+p-j)!}. \tag{55}$$

Suppose now that $s=2n+m$ and $q_i=n+m_i$, so that for fixed m and m_i the numbers of positive and negative steps are both of order n. The number of configurations can then be written in the form

$$w_s(q_1,q_2,\ldots,q_p)$$
$$=w_m(m_1,m_2,\ldots,m_p)4^{np}$$
$$\times\prod_{j=1}^{p}\frac{\left[\frac{j+m+1}{2}\right]_n\left[\frac{j+m}{2}\right]_n}{(m_j+j)_n(m-m_j+p-j+1)_n}, \tag{56}$$

and if $m_1=0$ the n-dependent factor, as a function of n, is a hypergeometric coefficient with asymptotic form $n^{-(1/2)p^2}$. Notice that the exponent is independent of m and m_i.

We note that if $m_i=0$ for $i=1$–p then the above formula counts the number of noncrossing watermelon configurations, the chains of which join the origin to the point with coordinates $(n+m,n)$. The formula then reduces to

$$w_{m+2n}(n,n,\ldots,n)=4^{np}\prod_{j=1}^{p}\frac{\left[\frac{j+m+1}{2}\right]_n\left[\frac{j+m}{2}\right]_n}{(j)_n(j+m)_n}. \tag{57}$$

B. Recurrence relation and differential equations for the number of noncrossing watermelons configurations

The number of noncrossing watermelon configurations $R_s(p)$ with p chains which join at any lattice point which is s steps from the origin may be written as

$$R_s(p)=\sum_{q=0}^{s}w_s(q), \tag{58}$$

where $w_s(q)=w_s(q,q,\ldots,q)$, and we have partitioned the number of configurations according to the vertical distance q of the endpoint from the origin.

With $q_j=q$ in (55), we obtain $w_s(s-q)=w_s(q), w_s(0)=1$, and, after some manipulation for $1\le q\le\frac{1}{2}s$,

$$w_s(q)=\prod_{i=1}^{q}\frac{(p+i)_{s-2i+1}}{(i)_{s-2i+1}}, \tag{59}$$

which is the polynomial form in p given by Arrowsmith, Mason, and Essam [22]. Hence $R_s(p)$ is a polynomial in p.

Inspection of (55) shows that the summand in (58) is of the form required for the application of Zeilberger's algorithm. We have used Paule and Schorn's code to prove that for $p=2$, 3, 4, and 5, $R_s(p)$ satisfies the following recurrence relations:

$$(2+s)R_s(2)-(2+4s)R_{s-1}(2)=0\ ,$$
$$(3+s)(4+s)R_s(3)-(12+21s+7s^2)R_{s-1}(3)+8s(1-s)R_{s-2}(3)=0\ , \tag{60}$$
$$(3+s)(4+s)(5+s)(6+s)R_s(4)-6(1+s)(3+s)(4+s)(5+2s)R_{s-1}(4)+4s(1-s)(7+4s)(9+4s)R_{s-2}(4)=0$$
$$(4+s)(5+s)^2(6+s)(7+s)(8+s)(252+253s+55s^2)R_s(5)-3(4+s)(5+s)(141\,120+362\,152s+373\,054s^2+192\,647s^3$$
$$+52\,441s^4+7161s^5+385s^6)R_{s-1}(5)+s(1-s)(5\,738\,880+14\,311\,976s+14\,466\,242s^2+7\,579\,175s^3$$
$$+2\,170\,343s^4+322\,289s^5+19\,415s^6)R_{s-2}(5)-32(2-s)(1-s)^2s^2(1+s)(560+363s+55s^2)R_{s-3}(5)=0\ .$$

The corresponding "certificates" (see Appendix B), which serve as proof of these relations, are given by

$$C_s^{(p)}(q)=\frac{(s-q)_p}{\prod_{t=0}^{\lfloor\frac{1}{2}(p-1)\rfloor}(s+t)_{p-2t}}\bar{C}_s^{(p)}(q)\ , \tag{61}$$

$$\bar{C}_s^{(2)}(q)=2q-3s\ ,$$

$$\bar{C}_s^{(3)}(q)=2q-6q^2+4q^3+(-6+21q-18q^2)s+(-20+27q)s^2-14s^3\ ,$$

$$\bar{C}_s^{(4)}(q)=60q-198q^2+264q^3-162q^4+36q^5+(-240+852q-1386q^2+1040q^3-306q^4+16q^5)s$$
$$+(-1050+2570q-2594q^2+1032q^3-104q^4)s^2+(-1755+3004q-1740q^2+276q^3)s^3$$
$$+(-1395+1486q-374q^2)s^4+(-525+260q)s^5-75s^6\ ,$$

$$\bar{C}_s^{(5)}=5\,080\,320q-18\,157\,440q^2+29\,680\,000q^3-28\,291\,200q^4$$
$$+15\,908\,480q^5-4\,654\,720q^6+224\,000q^7+291\,200q^8-89\,600q^9+8960q^{10}$$
$$+(-25\,401\,600+98\,123\,424q-198\,902\,592q^2+235\,063\,000q^3-166\,796\,160q^4$$
$$+66\,893\,496q^5-10\,829\,256q^6-1\,971\,600q^7+1\,163\,160q^8-170\,080q^9+5808q^{10})s$$
$$+(-165\,758\,880+494\,395\,800q-774\,091\,640q^2+709\,462\,270q^3-377\,009\,090q^4$$
$$+100\,370\,030q^5-2\,242\,210q^6-5\,723\,740q^7+1\,276\,220q^8-81\,400q^9+880q^{10})s^2$$
$$+(-482\,408\,536+1\,226\,863\,484q-1\,557\,504\,990q^2+1\,112\,218\,015q^3-426\,667\,365q^4$$
$$+61\,044\,471q^5+11\,404\,515q^6-4\,974\,930q^7+495\,000q^8-11\,000q^9)s^3$$
$$+(-824\,989\,364+1\,806\,775\,622q-1\,858\,197\,805q^2+1\,008\,793\,955q^3-256\,644\,820q^4$$
$$+5\,314\,163q^5+10\,572\,245q^6-1\,681\,900q^7+60\,500q^8)s^4$$
$$+(-919\,066\,922+1\,708\,993\,427q-1\,392\,526\,560q^2+546\,582\,990q^3$$
$$-76\,178\,835q^4-9\,847\,557q^5+3\,409\,725q^6-189\,750q^7)s^5$$
$$+(-698\,466\,721+1\,073\,483\,343q-666\,154\,305q^2+173\,224\,365q^3-6\,358\,185q^4-3\,923\,073q^5+367\,125q^6)s^6$$
$$+(-368\,411\,226+450\,221\,100q-200\,113\,500q^2+29\,238\,000q^3+1\,610\,400q^4-435\,930q^5)s^7$$
$$+(-134\,440\,101+123\,674\,640q-35\,686\,765q^2+1\,946\,450q^3+276\,375q^4)s^8$$
$$+(-33\,150\,638+21\,139\,615q-3\,318\,150q^2-15\,125q^3)s^9$$
$$+(-5\,241\,379+2\,011\,075q-114\,125q^2)s^{10}+(-476\,278+79\,750q)s^{11}-18\,755s^{12}\ . \tag{62}$$

A similar recurrence relation exists for any value of p. For $p=6$ the relation is of order 3 with coefficients of degree 11, so that the order of the relation appears to be $\lfloor\frac{1}{2}(p+1)\rfloor$, where $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

For $p=2$, 3, and 4 the differential equations satisfied by the generating function $R(p,u)$, which follow from the above relations, are, where $y=R(2,u)$, $R(3,u)$, and $R(4,u)$, respectively,

$$u(1-4u)y'+(2-6u)y=2\ ,$$

$$u^2(1+u)(1-8u)y''+u(8-42u-32u^2)y'+(12-40u-16u^2)y=12\ ,$$

$$u^4(1+4u)(1-16u)y''''+u^3(24-246u-1088u^2)y'''+u^2(180-1524u-5244u^2)y''+u(480-3180u-7536u^2)y'+(360-1680u-2040u^2)y=360\ . \quad (63)$$

The solution for $p=2$ which satisfies the initial condition $R(2,0)=1$ is

$$R(2,u)=[1-2u-(1-4u)^{1/2}]/(2u^2)\ . \quad (64)$$

However, for $p\geq 3$ the equations become increasingly complex. For $p=3$, a suitable substitution to remove the inhomogeneous part leads to

$$R(3,u)=[-1+u-3u^2+F(-\tfrac{1}{8},\tfrac{1}{4};-1,-2,2,-2;-u)]/(8u^2)\ , \quad (65)$$

where F is a Heun function satisfying the equation

$$u(1+u)(1-8u)F''+2(1+8u^2)F'+2(1-8u)F=0\ . \quad (66)$$

For $p=4$, removal of the inhomogeneous part gives

$$R(4,u)=[-2+u-2u^2+z(u)]/(2u^3) \quad (67)$$

where $z(u)$ is a solution of the homogeneous equation

$$u^3(1+4u)(1-16u)z''''+u^2(12-102u+320u^2)z'''+u(36+174u-60u^2)z''+(24-12u+120u^2)z'+(12-120u)z=0\ . \quad (68)$$

The indices for $p=2$, 3, and 4 corresponding to the singular points $u=\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ yield $\Psi=\frac{3}{2}$, 4, and $\frac{15}{2}$, respectively, in agreement with (2).

C. Conjectured exact expression for the number of noncrossing stars

The number of noncrossing stars $S_s(p)$ with p chains (or brushes with p mutually avoiding hairs) may be obtained by summing (55) over q_i:

$$S_s(p)=\sum_{0\leq q_1\leq q_2\cdots\leq q_p\leq s} w_s(q_1,q_2,\ldots,q_p)\ . \quad (69)$$

We have computed sequences of values of $S_s(p)$ for each p from 0 to 7, and find that they satisfy simple first order recurrence relations. The first four such relations are

$$S_s(2)=4\frac{(\frac{1}{2}+2)}{(1+s)}S_{s-1}(2)\ ,$$
$$S_s(3)=8\frac{(\frac{1}{2}+s)}{(2+s)}S_{s-1}(3)\ ,$$
$$S_s(4)=16\frac{(\frac{1}{2}+s)(\frac{3}{2}+s)}{(2+s)(3+s)}S_{s-1}(4)\ ,$$
$$S_s(5)=32\frac{(\frac{1}{2}+s)(\frac{3}{2}+s)}{(3+s)(4+s)}S_{s-1}(5)\ . \quad (70)$$

A clear pattern emerges, and we conjecture the following formula:

$$S_s(p)=2^{ps}\prod_{j=1}^{p}\frac{\left(\left\lfloor\frac{j+1}{2}\right\rfloor\right)_s}{(j)_s}\ . \quad (71)$$

At first sight it appears from the formula that the polynomial coefficients have degree p, and that the sequence is of generalized hypergeometric type. However, cancellation of factors takes place giving rise to the quoted results, which generally only become hypergeometric by redefining the origin of s. Alternatively, the theory of Appendix A may be extended by replacing $k!$ in (A1) by $(b_{n+1})_k$ which modifies the value of g in (A6) to

$$g=\sum_{i=1}^{n+1}(a_i-b_i)\ . \quad (72)$$

This implies that for the noncrossing stars, $\psi=-g=\frac{1}{4}p(p-1)$, in agreement with (2). The differential equation satisfied by the generating function is also modified. The leading D in (A3) is replaced by $D+b_{n+1}-1$, which makes the differential equation inhomogeneous with the constant $\prod_{i=1}^{n+1}(b_i-1)$ on the right-hand side. This is consistent with the differential equations for the generating function $S(p,u)$ with $p=2$, 3, 4, and 5 obtained from (70), i.e.,

$$u(1-4u)y'+(1-6u)y=1\ ,$$
$$u(1-8u)y'+2(1-6u)y=2\ ,$$
$$u^2(1-16u)y''+u(6-80u)y'+6(1-10u)y=6\ ,$$
$$u^2(1-32u)y''+u(8-160u)y'+12(1-10u)=12\ . \quad (73)$$

The generating functions may be expressed in terms of generalized hypergeometric functions; thus

$$S(2,u)=[(1-4u)^{-1/2}-1]/(2u)\ ,$$
$$S(3,u)=[1-4u-(1-8u)^{1/2}]/(8u^2)\ ,$$
$$S(4,u)=[1-2u-{}_2F_1(-\tfrac{1}{2},\tfrac{1}{2};2;16u)]/(4u^2)\ ,$$
$$S(5,u)=[1+12u-16u^2-{}_2F_1(-\tfrac{3}{2},-\tfrac{1}{2};2;32u)]/(32u^3)\ ,$$
$$S(6,u)=[1+4u-4u^2-{}_3F_2(-\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2};2,3;64u)]/(8u^3)\ . \quad (74)$$

In general a number of leading terms must be dropped from ${}_{n+1}F_n$, where $n=\lfloor p/2\rfloor-1$, and a normalizing factor must be included to make the first term equal to unity. The number of terms to be dropped depends on the required shift in the origin of s referred to above.

An equation which is equivalent to (71) is

$$S_s(p)=\prod_{j=1}^{\lfloor \frac{1}{2}(s+1)\rfloor} \frac{(p+2j-1)_{2s-4j+3}}{(2j-1)_{2s-4j+3}} . \tag{75}$$

This formula was given without proof by Arrowsmith, Mason, and Essam [22]. Notice the polynomial dependence on p which is a general property of networks which can be mapped onto integer flows (see Ref. 22). The formula was obtained by observing the form of the polynomials for increasing values of s.

V. CRITICAL EXPONENTS FOR DIRECTED POLYMER NETWORKS IN THE CONTINUUM LIMIT

Our results so far have shown that at least for stars and watermelons the critical exponents for networks on a lattice are the same as for the continuum. It has been shown by Duplantier [14] that for undirected networks the critical exponent $\psi(G)$ may be decomposed into a global part, depending on the number of independent cycles $c(G)$ in the network and the correlation exponent ν, and a contribution from each vertex. The result may be written

$$\psi(G)-1-\gamma(G)=dc(G)\nu-\sum_{L\geq 1} n_L(G)\sigma_L . \tag{76}$$

Here $n_L(G)$ is the number of L-leg vertices in G. The vertex factor σ_L depends only on L and d and arises from the interaction between the chains incident at the vertex; it vanishes for $d\geq d_c=4$. At $d=d_c$, a power of $\ln(s)$ must be included as a factor in the formula for $W_s(G,p)$. The exponent of the logarithm may be broken down in the same way as $\psi(G)$.

For directed networks a formula similar to (76) has been derived for $d=2$ by Zhao, Lookman, and Essam [15]. In this case each vertex contributes two terms to the sum, one depending on the number of chains directed into the vertex and one depending on the number directed outwards. Thus $n_L(G)$ is the number of inward or outward fans having L legs, and the formula becomes

$$\psi(G)=1-\gamma(G)=c(G)/2+\tfrac{1}{4}\sum_{L\geq 1} n_L(G)L(L-1) , \tag{77}$$

which agrees with (2) in the case of the star and watermelon with $c(\text{star})=0, n_p(\text{star})=1$, $c(\text{watermelon})=p-1$, and $n_p(\text{watermelon})=2$, other values of $n_L(G)$ being zero. Generalizing the work of Zhao, Lookman, and Essam [15], it is easily shown that for $d\geq d_c$, ignoring the mutual avoidance condition,

$$\psi(G)=1-\gamma(G)=Dc(G)/2 , \tag{78}$$

which generalizes (3). This is the same as Duplantier's result for undirected networks, except that d is replaced by the number of transverse dimensions D ($\nu=\frac{1}{2}$ above the critical dimension). This was to be expected since, ignoring the avoidance condition, the directed network problem is isomorphic to the undirected problem on the D-dimensional lattice obtained by projecting onto a plane perpendicular to the special direction.

Finally, at the critical dimension, we summarize the calculation of Mukherji and Bhattacharjee [2,3] and extend it to a general directed network. In their continuum model our combinatorial function $W_s(p,G)$ is replaced by a partition function $Z_{G,p}(s)$ which is a multiple integral over the p paths of length s defining the polymer configuration subject to the topological constraints of the network G. The integrand is a Boltzmann factor with Edward's Hamiltonian [23] having a repulsive δ function interaction of strength v_0 whenever two chains cross one another. The model is renormalized to an arbitrary length scale L at which the renormalized interaction and partition function are denoted by $u(L)$ and $Z_{G,p|r}(L,u,s)$, respectively. Mukherji and Bhattacharjee [3] show that this partition function satisfies a partial differential equation which on setting $d=3$ becomes

$$\left[L\frac{\partial}{\partial L}-\frac{u^2}{2\pi}\frac{\partial}{\partial u}-2\gamma_{G,p}(u)\right]Z_{G,p|r}(L,u,s)=0 . \tag{79}$$

Solution by the method of characteristics shows that the trajectory through $L=L_0$ and $u=u_0$ has parametric form

$$L(t)=L_0e^t,\quad u(t)=\left[\frac{1}{u_0}+\frac{t}{2\pi}\right]^{-1} \tag{80}$$

and, along this trajectory,

$$Z_{G,p|r}(L(t),u(t),s)=Z_{G,p|r}(L_0,u_0,s)e^{\int_0^t 2\gamma_{G,p}(u(\tau))d\tau} . \tag{81}$$

From (80), as $L\to\infty$ and $u\to 0$, in this limit $Z_{G,p|r}(L,u,s)$ may be replaced by the Gaussian approximation $Z^0_{G,p}(s)$ in which the interactions are ignored.

Also the correct asymptotic form may be obtained by approximating $\gamma_{G,p}(u)$ to first order in u, and from Eq. (3.12) of [3], for the watermelon with p chains, we obtain

$$2\gamma_{\text{watermelon},p}(u)=\frac{u}{\pi}\begin{bmatrix}p\\2\end{bmatrix}+O(u^2) . \tag{82}$$

In Appendix D this formula is generalized to an arbitrary network, with the result

$$2\gamma_{G,p}(u)=\frac{u}{2\pi}\lambda(G)+O(u^2) , \tag{83}$$

where

$$\lambda(G)=\sum_L n_L(G)\begin{bmatrix}L\\2\end{bmatrix} . \tag{84}$$

As above, $n_L(G)$ counts inward and outward fans separately.

Carrying out the integration in (81) yields

$$Z_{G,p|r}(L_0,u_0,s) \sim Z^0_{G,p}(s)\left[1+\frac{u_0}{2\pi}\ln\left[\frac{L(t)}{L_0}\right]\right]^{-\lambda(G)} \quad (85)$$

and taking $u_0=v_0$ and $L(t)=s^{1/2}$ gives the asymptotic form for $s\rightarrow\infty$:

$$Z_{G,p}(s)\sim Z^0_{G,p}(s)\left[\frac{u_0}{4\pi}\ln\left[\frac{s}{L_0^2}\right]\right]^{-\lambda(G)} . \quad (86)$$

Assuming that for long chains the discrete network and the continuum have the same asymptotic behavior, we find

$$W_s(G,p)\sim k^{ps}s^{-c(G)}(\ln s)^{-\lambda(G)} , \quad (87)$$

where the factor $s^{-c(G)}$ comes from $Z^0_{G,p}(s)$.

For networks with only two chains, $\lambda(G)=2$ for the watermelon, in agreement with (26); and $\lambda(G)=1$ for the star configuration, in agreement with (25).

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APPENDIX A: GENERALIZED HYPERGEOMETRIC FUNCTIONS

The hypergeometric function ${}_mF_n$ is defined by

$${}_mF_n(a_1,a_2,\ldots,a_m;b_1,b_2,\ldots,b_n;z)=\sum_{k=0}^{\infty} f_k z^k , \quad (A1)$$

where, in terms of Pochhammer's symbol $(a)_k=a(a+1)(a+2)\cdots(a+k-1)$,

$$f_k=\frac{(a_1)_k(a_2)_k\cdots(a_m)_k}{(b_1)_k(b_2)_k\cdots(b_n)_k k!} . \quad (A2)$$

The coefficients satisfy the first order recurrence relation

$$f_k=\frac{(k+a_1-1)(k+a_2-1)\cdots(k+a_m-1)}{(k+b_1-1)(k+b_2-1)\cdots(k+b_n-1)k}f_{k-1} \quad (A3)$$

and the function ${}_mF_n$ satisfies the mth order linear differential equation

$$D(D+b_1-1)(D+b_2-1)\cdots(D+b_n-1)y - z(D+a_1)(D+a_2)\cdots(D+a_m)y=0 , \quad (A4)$$

where $D=z(d/dz)$.

In the case $m=n+1$, which arises in this paper, the coefficients have the asymptotic form

$$f_k\sim k^g , \quad (A5)$$

where

$$g=a_1+a_2+\cdots+a_{n+1}-b_1-b_2-\cdots-b_n-1 , \quad (A6)$$

and the differential equation is of the form

$$z^n(1-z)y^{(n+1)}+z^{n-1}(c_n-d_nz)y^{(n)} +z^{n-2}(c_{n-1}-d_{n-1}z)y^{(n-1)} +\cdots+(c_1-d_1z)y'+c_0y=0 . \quad (A7)$$

Near the singular point $z=1$, the asymptotic form is

$${}_{n+1}F_n(a_1,a_2,\ldots,a_{n+1};b_1,b_2,\ldots,b_n;z)\sim(1-z)^{-\gamma} , \quad (A8)$$

where

$$\gamma=g+1 . \quad (A9)$$

APPENDIX B: CERTIFICATES AND VERIFICATION OF RECURRENCE RELATIONS

In this appendix we summarize the method of proof described fully in the lecture notes of Wilf [18]. Given a recurrence relation

$$a_0(s)f_s+a_1(s)f_{s-1}+\cdots+a_n(s)f_{s-n}=g_s , \quad (B1)$$

satisfied by the sum

$$f_s=\sum_q F_s(q) , \quad (B2)$$

it is not a trivial matter to verify the relation by direct substitution. However, if $F_s(q)$ is hypergeometric in both variables s and q, then [13] the coefficients $a_i(s)$ are polynomial in s. If, further, we are told that

$$a_0(s)F_s(q)+a_1(s)F_{s-1}(q)+\cdots+a_n(s)F_{s-n}(q) =G_s(q)-G_s(q-1) , \quad (B3)$$

where

$$G_s(q)=C_s(q)F_s(q) , \quad (B4)$$

with $C_s(q)$ a rational function of s and q, then verification becomes simple. Dividing (B3) through by $F_s(q)$, both sides of the equation become rational functions of both variables, the equality of which may be routinely checked by a computer algebra program. Summing (B8) over q from $q_{\min}$ to $q_{\max}$ gives (B1) with

$$g_s=G_s(q_{\max})-G_s(q_{\min}-1) . \quad (B5)$$

If $F_s(q)$ has a natural range of q outside which it is zero, then taking the sum in (B2) to be over this range and using (B4) and (B5) gives $g_s=0$.

Giving the coefficients $a_i(s)$ together with the certificate $C_s(q)$ may therefore be taken as a proof of (B1). The MATHEMATICA program of Paule and Schorn [17] inputs $F_s(q)$ and outputs (B1) with the variable s replaced by $s+n$. The certificate may be obtained from their variable GOSOL with the recurrence variable replaced by $s-n$ and the range variable replaced by $q+1$, thus

$$C_s(q)=\text{gosol}(s-n,q+1)F_{s-n}(q+1)/F_s(q)\ . \tag{B6}$$

APPENDIX C: EVALUATION OF THE NONCROSSING WALK DETERMINANT

Equation (53) may be written explicitly as

$$w_s(q_1,q_2,\ldots,q_p)=\begin{vmatrix} \begin{bmatrix} s \\ q_1 \end{bmatrix} & \begin{bmatrix} s \\ q_2+1 \end{bmatrix} & \cdots & \begin{bmatrix} s \\ q_{p+p-1} \end{bmatrix} \\ \begin{bmatrix} s \\ q_1-1 \end{bmatrix} & \begin{bmatrix} s \\ q_2 \end{bmatrix} & \cdots & \begin{bmatrix} s \\ q_p+p-2 \end{bmatrix} \\ \vdots & \vdots & \vdots & \vdots \\ \begin{bmatrix} s \\ q_1-p+1 \end{bmatrix} & \begin{bmatrix} s \\ q_2-p+2 \end{bmatrix} & \cdots & \begin{bmatrix} s \\ q_p \end{bmatrix} \end{vmatrix}\ . \tag{C1}$$

Similar binomial determinants have been evaluated by Gessel and Viennot [24]. Carrying out a sequence of row operations, the determinant becomes

$$w_s(q_1,q_2,\ldots,q_p)=\begin{vmatrix} \begin{bmatrix} s \\ q_1 \end{bmatrix} & \begin{bmatrix} s \\ q_2+1 \end{bmatrix} & \cdots & \begin{bmatrix} s \\ q_p+p-1 \end{bmatrix} \\ \begin{bmatrix} s+1 \\ q_1 \end{bmatrix} & \begin{bmatrix} s+1 \\ q_2+1 \end{bmatrix} & \cdots & \begin{bmatrix} s+1 \\ q_p+p-1 \end{bmatrix} \\ \vdots & \vdots & \vdots & \vdots \\ \begin{bmatrix} s+p-1 \\ q_1 \end{bmatrix} & \begin{bmatrix} s+p-1 \\ q_2+1 \end{bmatrix} & \cdots & \begin{bmatrix} s+p-1 \\ q_p+p-1 \end{bmatrix} \end{vmatrix}\ . \tag{C2}$$

Removing common factors from both rows and columns gives

$$w_s(q_1,q_2,\ldots,q_p)=D_p\prod_{j=1}^{p}\frac{(s+p-j)!}{(q_j+j-1)!(s-q_j+p-j)!}\ , \tag{C3}$$

where, writing $q_i=r_i-i+1$,

$$D_p=\begin{vmatrix} (s-r_1+1)_{p-1} & (s-r_2+1)_{p-1} & \cdots & (s-r_p+1)_{p-1} \\ (s-r_1+2)_{p-2} & (s-r_2+2)_{p-2} & \cdots & (s-r_p+2)_{p-2} \\ \vdots & \vdots & \vdots & \vdots \\ (s-r_1+p-1) & (s-r_2+p-1) & \cdots & (s-r_p+p-1) \\ 1 & 1 & \cdots & 1 \end{vmatrix}\ . \tag{C4}$$

D_p is polynomial, of degree $p-1$ in each of the r_i variables, and is zero if any two of these variables are equal. It follows that

$$D_p=f_p(s)\prod_{1\le i<j\le p}(r_j-r_i)\ . \tag{C5}$$

Setting $r_i=i-1$ gives

$$f_p(s)=\frac{\hat{D}_p}{\prod_{i=2}^{p-1} i!} \tag{C6}$$

where $\hat{D}_p$ is the determinant with $r_i=i-1$. It is easy to show that $\hat{D}_p=(p-1)!\hat{D}_{p-1}$, and hence $f_p(s)=1$, from which (55) follows.

APPENDIX D: EVALUATION OF $\gamma_{G,p}(u)$

From the work of Mukherji and Bhattacharjee [3], it follows that

$$2\gamma_{G,p}(u)=\frac{u}{Z_{G,p}^{(0)}}\sum_{[i,j]}\lim_{\epsilon\to 0}\epsilon Z_{G,p}^{(1)}(i,j)+O(u^2)\ , \tag{D1}$$

where $\epsilon=2-D$ and the sum is over all pairs of chains. $Z_{G,p}^{(0)}$ is the partition function with the interactions set to zero and $Z_{G,p}^{(1)}(i,j)$ is the first order perturbation coming from the interaction of chains i and j. These may be expressed in terms of the Brownian motion propagator for a single chain

$$G(\mathbf{r},z)=(2\pi z)^{-(1/2)D}e^{-r^2/2z} , \qquad (D2)$$

where $\mathbf{r}$ is a D-dimensional vector giving the perpendicular displacement of a point distant z along the chain. $Z^{(0)}_{G,p}$ is an integral over the nodes $N(G)$ of the network with an integrand which is a product of propagators over all chains $C(G)$:

$$Z^{(0)}_{G,p}=\int \prod_{n\in N(G)} d\mathbf{r}_n \prod_{c\in C(G)} G(\mathbf{r}_c^+-\mathbf{r}_c^-,s) , \qquad (D3)$$

where $\mathbf{r}_c^-$ and $\mathbf{r}_c^+$ are the positions of the beginning and end of the chain c. $Z^{(1)}_{G,p}(i,j)$ is defined by replacing the propagators for the chains i and j by two propagators into the interaction which takes place at $(\mathbf{r},z)$ and two outward propagators:

$$Z^{(1)}_{G,p}(i,j)=\int \prod_{n\in N(G)} d\mathbf{r}_n Y(\mathbf{r}_i^-,\mathbf{r}_i^+,\mathbf{r}_j^-,\mathbf{r}_j^+,s) \times \prod_{c\in C(G)\setminus\{i,j\}} G(\mathbf{r}_c^+-\mathbf{r}_c^-,s) , \qquad (D4)$$

where

$$Y(\mathbf{r}_i^-,\mathbf{r}_i^+,\mathbf{r}_j^-,\mathbf{r}_j^+,s) =\int_0^s dz \int d\mathbf{r}\, G(\mathbf{r}-\mathbf{r}_i^-,z)G(\mathbf{r}-\mathbf{r}_j^-,z) \times G(\mathbf{r}_i^+-\mathbf{r},s-z)G(\mathbf{r}_j^+-\mathbf{r},s-z) . \qquad (D5)$$

Using (D2) and carrying out the integral over $\mathbf{r}$,

$$Y(\mathbf{r}_i^-,\mathbf{r}_i^+,\mathbf{r}_j^-,\mathbf{r}_j^+,s)=G(\mathbf{r}_i^+-\mathbf{r}_i^-,s)G(\mathbf{r}_j^+-\mathbf{r}_j^-,s)I(s) , \qquad (D6)$$

where, defining a normalized integration variable $y=z/s$ and mean position vector $\bar{\mathbf{r}}_i(y)=y\mathbf{r}_i^++(1-y)\mathbf{r}_i^-$,

$$I(s)=\frac{s^{1-(1/2)D}}{(4\pi)^{(1/2)D}}\int_0^1 dy \frac{\exp\left[-\dfrac{[\bar{\mathbf{r}}_i(y)-\bar{\mathbf{r}}_j(y)]^2}{4sy(1-y}\right]}{[y(1-y)]^{1-(1/2)\epsilon}} . \qquad (D7)$$

The integral is convergent provided that $\mathbf{r}_i^-\neq\mathbf{r}_j^-$ and $\mathbf{r}_i^+\neq\mathbf{r}_j^+$. If $\mathbf{r}_i^-=\mathbf{r}_j^-$, then for $\epsilon\geq 0$ the integral diverges at $y=0$, and if $\mathbf{r}_i^+=\mathbf{r}_j^+$ it diverges at $y=1$. If either or both of these conditions occurs, then carrying out the integral for $\epsilon<0$, multiplying by ϵ, and taking the limit $\epsilon\to 0$ gives

$$\lim_{\epsilon\to 0}\epsilon I(s)=\frac{1}{2\pi}[\delta(\mathbf{r}_i^-,\mathbf{r}_j^-)+\delta(\mathbf{r}_i^+,\mathbf{r}_j^+)] , \qquad (D8)$$

where δ is the Kronecker delta function. Combining equations (D1), (D3), (D4), (D6), and (D8),

$$2\gamma_{G,p}(u)=\frac{u}{2\pi}\sum_{[i,j]}[\delta(\mathbf{r}_i^-,\mathbf{r}_j^-)+\delta(\mathbf{r}_i^+,\mathbf{r}_j^+)]+O(u^2) , \qquad (D9)$$

and carrying out the sum gives (83).

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