Sequential partitioning: An alternative to understanding size distributions of avalanches in first-order phase transitions

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We study the problem of the partition of a system of initial size V into a sequence of fragments $s_1, s_2, s_3 \ldots$. By assuming a scaling hypothesis for the probability $p(s;V)$ of obtaining a fragment of a given size, we deduce that the final distribution of fragment sizes exhibits power-law behavior. This minimal model is useful to understanding the distribution of avalanche sizes in first-order phase transitions at low temperatures.

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The existence in nature of magnitudes that exhibit a scale-free (power-law) statistical distribution has always been an intringuing phenomenon. In equilibrium systems, such distributions are usually associated with the existence of a critical point where the divergence of the correlations in time and space masks any dependence of the macroscopic magnitudes with intrinsic microscopic scales of the system. Thus, universality may arise and the systems can be classified by a small set of exponents characterizing the power-law distributions of few magnitudes. In systems out of equilibrium, the landscape is far from being well understood. The experimental examples range from condensed matter to earth sciences and biophysics: earthquakes, volcanic activity, evolution of species, sandpiles, fracture processes, avalanche phenomena in magnetic and structural first-order transitions, etc.

Different theories have been proposed for the explanation of such a lack of temporal and spatial scales in nature. Besides the famous self-organized criticality theory (SOC) [1], one should also mention the sweeping of an instability theory [2], the existence of a mechanism of multiplicative nature giving log-normal distributions [3] and the extremal dynamics theory [4], etc. Quite recently it has been realized that some of the experiments exhibiting avalanches with power-law distributions have in common the fact that they are first-order phase transitions with intrinsic disorder and with thermal fluctuations playing a secondary role (much smaller than the energy barriers between metastable states). After this, they have been cataloged as "athermal" [5] or "fluctuationless" [6] first-order phase transitions (FLFO). Some experimental examples of FLFO systems are (i) magnetic phase transitions at low temperatures induced by sweeping an external field [7,8]; (ii) martensitic transformations induced by an applied stress or temperature [9]; (iii) precipitation of gases on substrates [10,11]; and (iv) the superconductivity transition in granular Al films driven by a parallel magnetic field [12]. All these systems share the two characteristics (having intrinsic disorder and negligible temperature fluctuations) necessary for the avalanche phenomena to occur [5]. The prototypical models for such systems are the random field (RFIM) [5] and random bond (RBIM) [6] Ising models driven by external fields at zero temperature. The first-order phase transition separating the up phase from the down phase is crossed by sweeping the external field. The magnetization (or amount of transformed system) increases by avalanche steps. For a particular value of the amount of quenched disorder in the system, the avalanches exhibit a power-law distribution. This suggests the existence of a "critical point" induced by disorder [5]. The fact that such "criticality" appears only for a particular value of the disorder is not satisfactory: experimentally power-law distributions of avalanche sizes are seen without clear tuning of the amount of disorder [7–12].

The main motivation for this Brief Report is to understand the apparition of the power-law distribution of avalanches in the FLFO systems from a more general point of view using a minimal model without leaning on the RFIM or RBIM. Nevertheless, the model might also account for the understanding of other phenomena yielding $1/s$ distributions. The FLFO systems exhibit, in common with the typical SOC models, a separation of time scales, which allows the definition of avalanches when smoothly driven by the external field. What makes the FLFO systems different from the SOC systems is that the scalar parameter exhibiting avalanches (amount of transformed material, magnetization in the direction of the field, resistivity, etc.) is bounded from above and below. Such a global constraint does not apply, for instance, to sandpile models in which the total amount of sand is unlimited.

Assuming that such a bounded scalar parameter changes monotonously (there are no retransforming avalanches [13]) one can treat this phenomenon as a sequential fragmentation process: from a system with initial size V one sequentially extracts fragments (or avalanches) of sizes s_1, s_2, s_3, ... satisfying $\sum_{k=1}^{\infty} s_k = V$. This problem has previously been formulated as a particular case of multivalley structure [14,15]. Here we focus on its sequential character and show that such a simple and general model exhibits a power-law $1/s$ distribution of fragments. The unique hypothesis defining the model is that the probability law for extracting a fragment (or having an avalanche) of size s from a system of size V satisfies the self-similar scaling relation [14] $p(s;V) =$

$\frac{1}{V}g\left(\frac{s}{V}\right)$, where $g(x)$ is normalized between 0 and 1. This easy looking hypothesis, when applied recurrently over a system, has many implications. Given a succession of fragments $s_1, s_2, \ldots, s_k$, the conditional probability of extracting s_{k+1} is given by $p(s_{k+1}|s_k, s_{k-1}, \ldots, s_1; V) = p(s_{k+1}; V - s_k - s_{k-1} - \cdots - s_1)$. Thus, the sequence of fragments is of non-Markovian nature, although a connection to a Markov chain will be shown later. In the language of FLFO phase transitions, this last equation implies that the only interaction between transformed and untransformed domains is of "excluded volume" nature; i.e., the maximum size of an avalanche is what is left untransformed from the system. This means that our model neglects any long-range force (elastic, electric, magnetic, etc.) that could exist in the system and any interfacial energy that would depend on the domain geometry. Nevertheless, we will see that such a minimal excluded volume interaction is enough to demonstrate the existence of a power-law distribution for the fragment size. It can be easilly deduced that the probability $p_k(s; V)$ of extracting a fragment of size s in the k-step satisfies the recurrence

$$p_k(s;V) = \int_0^{V-s} d\tilde{s}\, p(\tilde{s};V) p_{k-1}(s;V-\tilde{s}) \;, \qquad (1)$$

given $p_1(s;V) = p(s;V)$. If we perform M extractions in a system with initial size V, the expected value of the number of fragments with size between s and $s+ds$, $n_M(s;V)$ is given by $n_M(s;V) = \sum_{k=1}^{M} p_k(s;V)$, with $0 \le s \le V$. Using (1) one gets the following integral relation:

$$n_M(s;V) = p(s;V) + \int_0^{V-s} d\tilde{s}\, p(\tilde{s};V) n_{M-1}(s;V-\tilde{s}) \;. \qquad (2)$$

This equation can be physically understood since the expected number of fragments of size s after M extractions is the sum of the expected number of fragments of size s in the first extractions $[n_1(s;V) = p(s;V)]$ plus the expected number of fragments of size s from the rest of the system, which has any size $V - \tilde{s}$ (with $V - \tilde{s} \ge s$) after the next $M-1$ extractions. If the limit $n(s;V) \equiv n_{M\to\infty}(s;V)$ exists it will satisfy the following integral equation [14]

$$n(s;V) = p(s;V) + \int_0^{V-s} d\tilde{s}\, p(\tilde{s};V) n(s;V-\tilde{s}) \;. \qquad (3)$$

Although $n(s;V)$ is an unnormalizable distribution, the integral of $sn(s;V)$ from 0 to V should be the total volume V. A general solution of Eq. (3) is difficult to obtain, but can be easily found in the two following cases: (i) uniform fragmentation probability: $p(s;V) = \frac{1}{V}$ exactly giving $n(s;V) = \frac{1}{s}$; and (ii) restricted beta fragmentation probability:

$$p(s;V) = \frac{\beta+1}{V}\left(1 - \frac{s}{V}\right)^{\beta} \;, \qquad (4)$$

where $\beta > -1$. The solution is [14]

$$n(s;V) = \frac{\beta+1}{s}\left(1 - \frac{s}{V}\right)^{\beta} \;. \qquad (5)$$

These $n(s;V)$ solutions, for small enough values of s, exhibit a $1/s$ behavior. For other particular cases of beta distributions $\left[p(s;V) = \frac{1}{V}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left(\frac{s}{V}\right)^{\alpha}\left(1-\frac{s}{V}\right)^{\beta}\right]$ solutions of (3) can be obtained [16]. These results point towards the possibility of demonstrating that for all $p(s;V)$ the distribution $n(s;V)$ exhibits $1/s$ behavior. Such a theorem can be demonstrated by noting that, from the scaling hypothesis for $p(s;V)$, $n(s;V)$ should satisfy the same scale-free dependence: $n(s;V) = \frac{1}{V}h(s/V)$. Substituting in (3) one obtains

$$h(x) = g(x) + \int_0^{1-x} d\tilde{x}\, g(\tilde{x}) \frac{1}{1-\tilde{x}} h\left(\frac{x}{1-\tilde{x}}\right) \;. \qquad (6)$$

From this equation we can calculate the "moments" of the distribution $n(s;V)$:

$$I_r \equiv \int_0^V ds \left(\frac{s}{V}\right)^r n(s;V) \;. \qquad (7)$$

Note that I_r cannot be negative. On the other hand, if I_r converges, from Eq. (6) it can be deduced that [17]

$$I_r = \frac{\langle (s/V)^r \rangle}{1 - \langle (1-s/V)^r \rangle} \;. \qquad (8)$$

The symbol $\langle\cdots\rangle$ indicates averages calculated with the probability distribution $p(s;V)$. Since $0 < s < V$ such averages $[\langle (s/V)^r\rangle$ and $\langle (1-s/V)^r\rangle]$ are always real and positive for $r \ge 0$, but might diverge for $r < 0$. For $r = 0$ the right-hand side of (8) diverges and for $r < 0$, if the moments of the distribution $p(s;V)$ exist, it has a negative value, contradicting the definition of I_r. Thus, I_r should diverge for $r \le 0$ and converge for $r > 0$ [18]. Thus, the behavior for small values of s is $1/s$, although logarithmic corrections cannot be excluded. In order to test the range of validity of the $1/s$ behavior we have performed simulations using different distribution probabilities.

Figure 1 shows the obtained $n_M(s;V)$ ($M = 10^3$) in log-log plot for the following probability distributions: uniform (a), triangular (b), restricted beta for $\beta = 3$ (c), inverse square root $[g(x) = \frac{1}{2\sqrt{x}}]$ (d), beta distribution with $\alpha = 1$ and $\beta = 5$ (e), and $\alpha = 3$ and $\beta = 2$ (f). Data have been averaged over 10^4 realizations. For all the cases, the behavior is compatible with $1/s$ for $s/V \lesssim 0.1$, while for $s/V \sim 1$ the behavior differs from one case to the other. A theoretical estimation of the "critical zone" (the region $s \ll s_{\max}$ where the behavior is $1/s$) can be obtained discarding logarithmic corrections and assuming an expansion $n(s;V) = \frac{a}{s} + \frac{b}{V} + \cdots \sim \frac{a}{s}\exp\left(\frac{b}{a}\frac{s}{V}\right)$. Substituting in Eq. (8) one gets $\frac{s_{\max}}{V} \sim \frac{a}{b} = \left(\frac{2}{3} - I_2\right)/(2I_2 - 1)$. For the uniform distribution $I_2 = 1/2$, giving $s_{\max} \to \infty$ as expected. The values $0.1 s_{\max}/V$ for the different studied probability functions are indicated by arrows in Fig. 1.

The description of the succession of fragments s_1, s_2, s_3,... can be related to the succesion of remainders

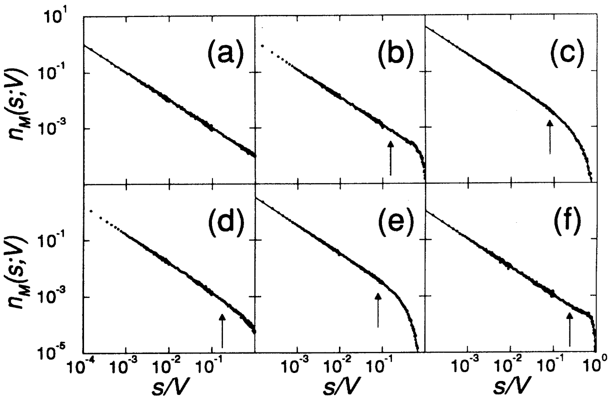

FIG. 1. $n_M(s;V)$ for $M = 1000$ and $V = 1$ in the cases: (a), uniform (b), triangular restricted (c) beta distribution for $\beta = 3$, (d), inverse square root (e), full beta distribution with $\alpha = 1$ and $\beta = 5$, and (f) $\alpha = 3$ and $\beta = 2$. In cases (a), (c), (e), and (f) we have also plotted with a continuous line the analytical solution. The arrows indicate the estimation of the "critical zone" as explained in the text.

$t_1 = V - s_1$, $t_2 = V - s_1 - s_2, \ldots$. The probability that after an extraction from a system of size V, the remainder is t is given by $\hat{p}(t;V) = p(V-t;V)$. The probability that a remainder t_k is left given the succession of remainders $t_1, t_2, \ldots t_{k-1}$, is $\hat{p}(t_k|t_{k-1},\ldots,t_1;V) = \hat{p}(t_k;t_{k-1})$. Thus, the succession of remainders is a Markov chain, and a master equation can straightforwardly be written. Nevertheless, the study of this chain does not render new insights on the problem and it will be developed elsewhere. Moreover, for the case of a uniform $\hat{p}(t;V)$, the succession of remainders is known to tend to a log-normal distribution [19].

Despite the big experimental efforts, few measurements of the distribution of avalanche sizes in FLFO systems are available: (i) For the magnetic materials, a power-law distribution of the Barkhausen noise amplitude has been found [7], but it is difficult to relate it with the real size of the avalanches or any bounded magnitude in the system. Something similar happens with the experiments on reversal of magnetic domains [8]. (ii) For martensitic materials [9], the acoustic emission amplitude distribution is related, in a not completely known way, to the advance of the interfaces. It shows power-law behavior that suggests that the distribution of avalanche sizes is also power-law but does not give information about its exponent. (iii) For the case of H on Nb [10] the measured quantity is also the acoustic emission generated by the propagating cracks when H precipitates into the substrate. This is also a too indirect measurement to obtain the distribution of avalanche sizes. (iv) For the superconducting Al films [12], the authors report a distribution of avalanches (resistance jumps) with power-law behavior with an exponent close to 2. In this case the resistance is probably proportional to the transformed fraction, or at least is a bounded quantity having two determined values at each side of the transition. Nevertheless a RBIM-like interaction between domains might be very important [12]. In the RFIM [5], RBIM [6], and similar models [20], the exponent for the avalanche distribution has been found to be universal but dependent on the space dimensionality: ~ 1.3 (2D) and ~ 1.8 (3D). This is a clear fingerprint that, besides the excluded volume interaction of our model, interfacial interactions between domains play a relevant role. Most previous theoretical studies of fragmenting have not considered the sequential partitioning problem, but only the final distribution of fragments. It is worth mentioning the work by Cheng and Redner [21] in which the temporal evolution of the distribution of fragments is analyzed. Their hypothesis of a homogeneous kernel is equivalent to the self-similar hypothesis. Nevertheless such theory cannot be applied for a sequential case. It should also be remarked that a general theory for the statistics of fragments has been published by Mekjian [22]. The author proposes different weights of each possible partition of a system made of A discrete units in n_1 fragments of size 1 unit, n_2 fragments of size 2 units, etc. so that $\sum_{k=1} kn_k = A$. By choosing the weights according to $p(n_1,n_2,n_3,\ldots) = \left(\prod_k k^{n_k}\, n_k!\right)^{-1}$, he obtains a $1/s$ distribution of sizes. Following our model, we can provide a physical interpretation of such weights: they correspond to the probabilities of a sequential partition of the system of initial size A with uniform distribution, i.e., in the first extraction one chooses uniformly a fragment s_1 of size $1,2,\ldots$ or A, in the second extraction one chooses a fragment s_2 of size $1,2,\ldots$ or $A - s_1$, and so on. Mekjian's formula can be easily deduced by induction, and the fact that such a sequential partition implies a final $1/s$ distribution of fragments comes naturally from the discrete version of our model, which will be presented elsewhere.

We should also mention some experiments of fragmentation [23]. The observed distribution of fragment sizes has been found to be a power law in some cases. Nevertheless the applicability of our model to such experiments is not straightforward, since the objects are fragmented in a very short time scale where the sequentiality of the process is doubtful and the excluded volume interaction may not apply (multifragmentation can exist).

In summary, we have shown that a sequential partition of a system renders a $1/s$ distribution of fragments. This can be applied to the case of a FLFO phase transition, justifying the appearance of a $1/s$ distribution of avalanches, as a consequence of the minimal excluded volume interaction between the transformed and untransformed domains.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 36 (1988).
[2] D. Sornette, J. Phys. I **4**, 209 (1994).
[3] B. J. West and M. F. Shlesinger, Int. J. Mod. Phys. B **3**, 795 (1989); Am. Scientist **78**, 40 (1989).
[4] S. L. Miller, W. M. Miller, and P. J. McWorter, J. Appl. Phys. **73**, 2617 (1993).
[5] J. P. Sethna, K. Dahmen, S. Kartha, J. A. Krumhansl, B. W. Roberts, and J. D. Shore, Phys. Rev. Lett. **70**, 3347 (1993); K. Dahmen and J. P. Sethna, *ibid.* **71**, 3222 (1993).
[6] E. Vives and A. Planes, Phys. Rev. B **50**, 3839 (1994).
[7] P. J. Cote and L. V. Meisel, Phys. Rev. Lett. **91**, 1334 (1991); L. V. Meisel and P. J. Cote, Phys. Rev. B **46**, 10 822 (1992).
[8] K. L. Babcock and R. M. Westervelt, Phys. Rev. Lett. **64**, 2168 (1990); P. Bak and H. Flyvbjerg, Phys. Rev. A **45**, 2192 (1992).
[9] E. Vives, J. Ortín, Ll. Mañosa, I. Ràfols, R. Pérez-Magrané, and A. Planes, Phys. Rev. Lett. **72**, 1694 (1994).
[10] G. Cannelli, R. Cantelli, and F. Cordero, Phys. Rev. Lett. **70**, 3923 (1993).
[11] M. P. Lilly, P. T. Finley, and R. B. Hallock, Phys. Rev. Lett. **71**, 4186 (1993).
[12] W. Wu and P. W. Adams, Phys. Rev. Lett. **74**, 610 (1995).
[13] For the RFIM the absence of reverse avalanches has been proved and in fact is necessary for the return point memory effect to appear (Ref. [5]). Contrarily for the RBIM such avalanches may appear but always represent a tiny fraction of all the avalanches during the transformation process (Ref. [6]). Experimentally such reverse avalanches have also been seen in Martensitic transformation but also representing a very small fraction; J. Ortín (private communication).
[14] B. Derrida and H. Flyvbjerg, J. Phys. A **20**, 5273 (1987).
[15] B. Derrida and H. Flyvbjerg focused on the analysis of the distribution of the biggest fragment, but they did not analyze the assymptotic $1/s$ behavior.
[16] A general formula for integer α and beta requires some mathematical effort. Nevertheless, we have solved the particular beta distributions for $\alpha = 1$, $\beta = 5$, and $\alpha = 3$, $\beta = 2$. The solutions are presented in Figs. 1(e) and 1(f).
[17] These relations have previously been calculated without a careful analysis of its convergence for different values of r; H.T. Davis, Chem. Eng. Sci. **44**, 1799 (1989).
[18] The arguments given in the paper demonstrate the convergence of I_r for $r \geq 1$. For $0 < r < 1$, the convergence can be demonstrated by continuity of I_r on r.
[19] J. C. Kapteyn, Astron. Lab. Groningen (Noordoff. Groningen, 1903) (as cited in B. J. West and M. F. Schlesinger (Ref. [3])).
[20] E. Vives, J. Goicoechea, J. Ortín, and A. Planes, Phys. Rev. E **52**, R5 (1995).
[21] Z. Cheng and S. Redner, Phys. Rev. Lett. **60**, 2450 (1988).
[22] A. Z. Mekjian, Phys. Rev. Lett. **64**, 2125 (1990), and references therein.
[23] T. Ishii and M. Matsushita, J. Phys. Soc. Jpn. **61**, 3474 (1992); L. Oddershede, P. Dimon, and J. Bohr, Phys. Rev. Lett. **71**, 3107 (1993).