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Simple proof of Page's conjecture on the average entropy of a subsystem

Jorge Sánchez-Ruiz
Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain
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It is shown that Page's formula for the average entropy $S_{m,n}$ of a subsystem of dimension $m \leq n$ of a quantum system of Hilbert space dimension mn in a pure state [Phys. Rev. Lett. **71**, 1291 (1993)] can be written in terms of the one-point correlation function of a Laguerre ensemble of random matrices. This leads to a proof of Page's conjecture, $S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}$, which is simpler than that given by Foong and Kanno [Phys. Rev. Lett. **72**, 1148 (1994)].

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A quantum system AB with Hilbert space dimension mn in a pure state ($\rho_{AB} = |\psi\rangle\langle\psi|$) has entropy $S_{AB} = 0$. However, if AB is divided into two subsystems A and B, of dimension m and n, respectively (without loss of generality, we can take $m \leq n$), the entropy of the subsystems, $S_A = S_B$, is greater than zero unless A and B are uncorrelated in the quantum sense ($\rho_{AB} = \rho_A \otimes \rho_B$) [1,2]. A convenient measure of the amount of entropy that arises from this coarse graining is the average $\langle S_A \rangle \equiv S_{m,n}$ of the entropy S_A over all pure states of the total system AB, the average being defined with respect to the unitarily invariant Haar measure on the space of unit vectors $|\psi\rangle$ in the mn-dimensional Hilbert space of the total system [1,2]. In a recent work, Page [2] obtained for $S_{m,n}$ the formula

$$S_{m,n} = \psi(mn+1) - \frac{\int \left(\sum_{i=1}^{m} x_i \ln x_i\right) |\Delta_m(x)|^2 \prod_{i=1}^{m} \left(e^{-x_i} x_i^{n-m}\right) dx_1 \cdots dx_m}{mn \int |\Delta_m(x)|^2 \prod_{i=1}^{m} \left(e^{-x_i} x_i^{n-m}\right) dx_1 \cdots dx_m} , \tag{1}$$

where $x_i \geq 0$, $\Delta_m(x)$ is the Vandermonde determinant of m variables,

$$\Delta_m(x) \equiv \prod_{1 \leq i < j \leq m} (x_i - x_j) , \tag{2}$$

and, for positive integer z,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{k=1}^{z-1} \frac{1}{k} , \tag{3}$$

where γ is Euler's constant. As conjectured by Page [2], Eq. (1) is equivalent to

$$S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n} . \tag{4}$$

The first proof of this conjecture was given by Foong and Kanno [3]. Here we show that a simpler proof can be achieved by noting that the second term in the right-hand side of (1) can be written as a one-dimensional integral in terms of the one-point correlation function of a Laguerre ensemble of complex Hermitian random matrices (see, e.g., Ref. [4]), whose explicit expression readily follows from a well-known result of random matrix theory [5,6].

Taking into account the symmetry between the m variables x_i, Eq. (1) can be written as

$$S_{m,n} = \psi(mn+1) - \frac{\int dx_1 x_1 \ln(x_1) \int |\Delta_m(x)|^2 \prod_{i=1}^{m} \left(e^{-x_i} x_i^{n-m}\right) dx_2 \cdots dx_m}{n \int |\Delta_m(x)|^2 \prod_{i=1}^{m} \left(e^{-x_i} x_i^{n-m}\right) dx_1 \cdots dx_m} . \tag{5}$$

On the other hand, the n-point correlation function for the eigenvalues of an ensemble of complex Hermitian random matrices is defined as [4–6]

$$X_n(x_1,\ldots,x_n) \equiv Z^{-1}\frac{m!}{(m-n)!}\int |\Delta_m(x)|^2 \times \left[\prod_{k=1}^{m}\mu(x_k)\right] dx_{n+1}\cdots dx_m \;, \qquad (6)$$

where $\mu(x)$ is a positive weight function with all its moments finite, and the normalization constant Z is the partition function,

$$Z \equiv \int |\Delta_m(x)|^2 \left[\prod_{k=1}^{m}\mu(x_k)\right] dx_1\cdots dx_m \;. \qquad (7)$$

Using this notation, Eq. (5) reads

$$S_{m,n} = \psi(mn+1) - \frac{1}{mn}\int_0^\infty X_1(x) x \ln x \, dx \;, \qquad (8)$$

where $X_1(x)$ is the one-point correlation function corresponding to the so-called Laguerre ensemble of (complex Hermitian) random matrices, with weight function $\mu(x) = x^{n-m}e^{-x}$ [4].

Let $\{C_k(x)\}$ denote a sequence of monic polynomials of degree k, $C_k(x) = x^k + O(x^{k-1})$, satisfying the orthogonality relations

$$\int C_k(x)C_l(x)\mu(x)\,dx = \delta_{kl}h_k \;. \qquad (9)$$

Then it can be shown [5,6] that the correlation functions (6) are given by

$$\begin{aligned} X_n(x_1,\ldots,x_n) &= \det\left[f(x_i,x_j)\right]_n \;, \\ f(x,y) \equiv \sqrt{\mu(x)\mu(y)}\sum_{k=0}^{m-1}\frac{C_k(x)C_k(y)}{h_k} \\ &= \sqrt{\mu(x)\mu(y)}\frac{C_{m-1}(x)C_m(y) - C_{m-1}(y)C_m(x)}{(y-x)h_{m-1}} \;, \end{aligned} \qquad (10)$$

where the last equality follows from the Christoffel-Darboux formula for orthogonal polynomials. In the particular case $n=1$, Eq. (10) simplifies to

$$\begin{aligned} X_1(x) &= f(x,x) \\ &= \mu(x)\sum_{k=0}^{m-1}\frac{[C_k(x)]^2}{h_k} \\ &= \mu(x)\frac{C_{m-1}(x)C_m'(x) - C_{m-1}'(x)C_m(x)}{h_{m-1}} \;. \end{aligned} \qquad (11)$$

The orthogonal polynomials corresponding to the weight function $\mu(x) = x^{n-m}e^{-x}$ are the associated Laguerre polynomials $L_k^{(n-m)}(x)$. From the explicit formula and the orthogonality relation for these polynomials [7],

$$L_k^{(n-m)}(x) = \sum_{t=0}^{k}(-1)^t \binom{n-m+k}{k-t}\frac{x^t}{t!} \;,$$

$$\int_0^\infty x^{n-m}e^{-x}L_k^{(n-m)}(x)L_l^{(n-m)}(x)\,dx = \frac{(n-m+k)!}{k!}\delta_{kl} \;, \qquad (12)$$

we see that $X_1(x)$ in (8) is given by (11), with

$$\begin{aligned} \mu(x) &= x^{n-m}e^{-x} \;, \\ C_k(x) &= (-1)^k k! L_k^{(n-m)}(x) \;, \\ h_k &= k!(n-m+k)! \;. \end{aligned} \qquad (13)$$

Using the functional relations [7]

$$\begin{aligned} L_n^{(\alpha)\prime}(x) &= -L_n^{(\alpha+1)}(x) \;, \\ L_n^{(\alpha-1)}(x) &= L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) \;, \end{aligned} \qquad (14)$$

the Christoffel-Darboux expression for $X_1(x)$ in (11) can be cast into the more convenient form

$$X_1(x) = \frac{m!}{(n-1)!}x^{n-m}e^{-x}\left\{\left[L_{m-1}^{(n-m+1)}(x)\right]^2 - L_{m-2}^{(n-m+1)}(x)L_m^{(n-m+1)}(x)\right\} \;, \qquad (15)$$

so that Eq. (8) then reads

$$\begin{aligned} S_{m,n} &= \psi(mn+1) - \frac{(m-1)!}{n!}\left(I_{m-1,m-1}^{(n-m+1)} - I_{m-2,m}^{(n-m+1)}\right) \;, \\ I_{r,s}^{(\alpha)} &\equiv \int_0^\infty x^\alpha e^{-x}\ln(x)L_r^{(\alpha)}(x)L_s^{(\alpha)}(x)\,dx \;. \end{aligned} \qquad (16)$$

The integrals $I_{r,s}^{(\alpha)}$ can be evaluated by taking advantage of the following result, which appears in the study of quantum-mechanical systems such as N-dimensional hydrogen atom and Morse oscillator [8–10],

$$\int_0^\infty x^q e^{-x}L_r^{(\alpha)}(x)L_s^{(\beta)}(x)\,dx = \sum_{k=0}^{\min(r,s)}(-1)^{r+s}\binom{q-\alpha}{r-k}\binom{q-\beta}{s-k}\frac{\Gamma(q+k+1)}{k!} \;, \qquad (17)$$

where $q > -1$, α, and β are real parameters. On differentiating both sides of this equation with respect to q, we get [10]

$$\int_0^\infty x^q e^{-x} L_r^{(\alpha)}(x) L_s^{(\beta)}(x) \ln x \, dx = \sum_{k=0}^{\min(r,s)} (-1)^{r+s} \binom{q-\alpha}{r-k} \binom{q-\beta}{s-k} \frac{\Gamma(q+k+1)}{k!} \times \left[\psi(q-\alpha+1) + \psi(q-\beta+1) + \psi(q+k+1) - \psi(q-\alpha-r+k+1) - \psi(q-\beta-s+k+1)\right] . \tag{18}$$

Both $\Gamma(z)$ and $\psi(z)$ have simple poles for $z = -n$, $n = 0, 1, 2, \ldots$, with residues $(-1)^n/n!$ and -1, respectively [7]. Therefore, in the case when $q = \alpha = \beta$, the only nonvanishing term in the summation is that corresponding to $k = \min(r, s)$, and a simple calculation yields

$$I_{r,r+t}^{(\alpha)} = -\frac{\Gamma(\alpha+r+1)}{r!\,t} \quad (t > 0) ,$$
$$I_{r,r}^{(\alpha)} = \frac{\Gamma(\alpha+r+1)}{r!} \psi(\alpha+r+1) , \tag{19}$$

so that we have

$$I_{m-2,m}^{(n-m+1)} = -\frac{(n-1)!}{2(m-2)!} ,$$
$$I_{m-1,m-1}^{(n-m+1)} = \frac{n!}{(m-1)!} \psi(n+1) . \tag{20}$$

Substituting these results in (16) and using (3), we complete our proof of Page's conjecture, Eq. (4).

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