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Analytic criterion for soliton instability in a nonlinear fiber array

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The stability problem for the so-called continuous-discrete solitons in a nonlinear fiber array is examined. We prove that the ground states are unstable if the first derivative of the "energy" integral $P = \sum_n \int |\Psi_n|^2 dt$ with respect to the soliton parameter λ^2 is negative.

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Nonlinear systems with coexisting stable and unstable solitons have been investigated in different physical problems (see, e.g., [1–4] and references therein). Nonlinear optical media with such properties are of interest for future photonic switching devices. The soliton dynamics in materials with large nonlinear coefficients is determined by a combined action of dispersion (diffraction) and higher-order nonlinear effects. A generic model is the nonlinear Schrödinger (NLS) equation with a saturating nonlinearity. It is well-known that a saturation of the nonlinearity plays a crucial role in preventing a blow-up of field distributions (self-focusing). Recent studies of discrete models have shown many interesting new features caused by discreteness [5–9]. Discreteness contributes to the dispersion of a system but it may simultaneously play a role as a saturation mechanism for nonlinearity. In this paper we demonstrate for the case of a two-dimensional NLS equation how the discreteness changes stability of solitons. We study here stability of solitons in a continuous-discrete nonlinear system describing optical pulse evolution in a nonlinear fiber array (NFA). Nonlinear fiber arrays exhibit rich propagation phenomena that can be of interest for all-optical data processing [10–13]. We present an analytical criterion of the instability for the continuous-discrete solitons. The basic equation describing short optical pulse evolution in a system of coupled nonlinear fibers reads [12]

$$i\partial_z \Psi_n + \Psi_{n+1} - 2\Psi_n + \Psi_{n-1} + \partial_t^2 \Psi_n + 2|\Psi_n|^2 \Psi_n = 0, \quad n = 0, \pm 1, \dots . \tag{1}$$

We study solutions localized in t and periodic or localized in n. Two types of the boundary conditions in n correspond to the two different variants of the NFA fabrication. For the fibers organized in a circle, the boundary conditions are periodic $\Psi_{-N} = \Psi_N$, where $2N$ is the number of fibers in the array, and for the aligned array we require that $|\Psi_{\pm N}| = 0$ in the case of a finite array or $|\Psi_n| \to 0$ as $|n| \to \infty$ in the case of the infinite array. The main result of our paper, the instability theorem, is applied to any of these three types of the boundary conditions.

Equation (1) can be written in the Hamiltonian form

$$i\frac{\partial \Psi_n}{\partial z} = \frac{\delta H}{\delta \Psi_n^*}, \tag{2}$$

with the Hamiltonian

$$H = \sum_n \int |\Psi_n - \Psi_{n-1}|^2 dt + \sum_n \int |\partial_t \Psi_n|^2 dt - \sum_n \int |\Psi_n|^4 dt \equiv I_1 + I_2 - I_3. \tag{3}$$

An additional conserved quantity is $P = \sum \int |\Psi_n|^2 dt$.

The continuum limit of Eq. (1) can be used to describe the evolution of broad field distributions involving many modes. Introducing a coordinate x in the "n direction" we have in the continuum approximation of Eq. (1),

$$iU_z + U_{xx} + U_{tt} + 2|U|^2 U = 0. \tag{4}$$

This equation is a well-known two-dimensional NLS equation that has been studied in a number of applications. The integrals of motion mentioned above have obvious continuous analogs: $P = \int |U|^2 dxdt$ and $H = \int (|U_x|^2 + |U_t|^2) dxdt - \int |U|^4 dxdt$, respectively.

For Eq. (4) the so-called virial theorem has been proved [15]:

$$\partial_z^2 \int (x^2 + t^2)|U|^2 dxdt = 8H. \tag{5}$$

Because H is a conserved quantity this equation can be integrated: $\langle R^2 \rangle \equiv \int (x^2 + t^2)|U|^2 dxdt = 4Hz^2 + Az + B$, where $A = \frac{d\langle R^2 \rangle}{dz}|_{z=0}$ and $B = \langle R^2 \rangle|_{z=0}$. It can be easily shown that if the integral H is negative for some initial field distribution, then $U(z,x,t)$ develops a singularity at a finite distance. Two-dimensional soliton solutions of Eq. (1) have $H = 0$. Thus, in the continuous system (5) the stability of solitons will be determined by small perturbations of the marginal case. It is very interesting to find out now in which direction discreteness effects, inherent in Eq. (1) [in comparison to Eq. (4)],

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will shift the stability boundary. It is important to notice that in contrast to the continuous limit (with an unbounded Hamiltonian), in the discrete case the Hamiltonian H is bounded from below for fixed integral P, as was shown in [12]. Thus, if this minimum is attained on some steady-state solution it is stable. In the continuous-discrete system collapse is prevented by the discreteness. Explosive concentration of the energy leads to the formation of narrow states. The process of self-localization of energy, initially being dispersed in nonlinear discrete systems, through a collapse mechanism, has been intensively investigated recently [8,9,14].

Consider the steady-state solutions $\Psi(t,z) \equiv \{\Psi_n(t,z), n = 0, \pm 1, \ldots\}$ of Eq. (1) of the form $\Psi_n(t,z) = F_n(t)\exp(i\lambda^2 z)$, where the shape of $F(t) \equiv \{F_n(t), n = 0, \pm 1, \ldots\}$ is determined by

$$F_{n+1} + F_{n-1} + \partial_t^2 F_n - (2+\lambda^2)F_n + 2|F_n|^2 F_n = 0. \quad (6)$$

These equations can be viewed as a nonlinear eigenvalue problem for λ^2 and F_n. We analyze in this paper solutions of Eq. (6) being localized in t and n. Solitons arise from a balance between nonlinearity, dispersion, and discreteness. Ground solutions of Eq. (6) have been studied first in [12]. Recently, a very detailed classification of the different types of the stationary solutions of Eq. (6), with a finite number of modes and the periodic boundary conditions in n, has been presented in [16]. Typical numerical solutions of Eq. (6), with the boundary conditions $|F_n(t)| \to 0$ as $|t| \to \infty$ and $F_{\pm N} = 0$, are plotted in Fig. 1 for different values of the parameter λ^2 .

To investigate the stability of the ground solution, we linearize Eq. (6) with respect to the soliton solution $\Psi_n = (F_n + f_n + ig_n)\exp(i\lambda^2 z)$. By decomposing into real and imaginary parts, we obtain equations for the evolution of the real functions f_n and g_n:

$$-\frac{dg_n}{dz} = -f_{n+1} - f_{n-1} - f_{ntt} + (2+\lambda^2)f_n - 6F_n^2 f_n \equiv (H_- f)_n, \quad (7)$$

$$\frac{df_n}{dz} = -g_{n+1} - g_{n-1} - g_{ntt} + (2+\lambda^2)g_n - 2F_n^2 g_n \equiv (H_+ g)_n. \quad (8)$$

The stability of a stationary ground state solution F is determined by the properties of the self-adjoint operators H_+ and H_-. Equations (7) and (8) can be rewritten in the form

$$-\frac{\partial^2 f_n}{\partial z^2} = [H_+(H_- f)]_n. \quad (9)$$

We will use the notation $(f,g) \equiv \sum_n \int f_n g_n dt$. Next, an instability criterion will be derived, making use of the following properties of the operators H_+ and H_- (see below).

(i) $(f, H_+ f) \geq 0$, and $(f, H_+ f) = 0$ only if $f = 0$ or $f = F_n$.

(ii) There exists some $\tilde{f}$ for which $(\tilde{f}, H_- \tilde{f}) < 0$ under the constraint $(\tilde{f}, F) = 0$.

Now, for a perturbation $f \equiv \{f_n(t,z), n = 0, \pm 1, \ldots\}$ obeying Eq. (9) we can prove $(f, H_+ f) \geq A\exp(2\gamma z)$ with constant A and

$$\gamma^2 = \sup\Big|_{(\varphi,F)=0} \frac{-(\varphi, H_-\varphi)}{(\varphi, H_+^{-1}\varphi)}. \quad (10)$$

In the proof of this theorem, we follow a procedure developed by Laedke and Spatschek [2] for continuous systems. A sketch of the proof for the continuous-discrete systems is as follows. Using the above mentioned properties of the operator H_+, Eq. (9) can be rewritten [for distributions $f(t)$ satisfying $(f,F) = 0$] in the form

$$\frac{\partial^2 (H_+^{-1} f)_n}{\partial z^2} = -(H_- f)_n. \quad (11)$$

Multiplying Eq. (11) by $\partial_z f_n$, summarizing over n, and integrating over t, we obtain an integral of motion for the evolution equation (11)

$$(\partial_z f, H_+^{-1}\partial_z f) + (f, H_- f) = C = \text{const}. \quad (12)$$

Equation (11) represents second order ordinary differential equations for $f_n(t)$. Using assumption (2) we can consider solutions with $C = 0$. Let at $z = 0$, the distribution f satisfy $(f_0, H_- f_0) < 0$ under the additional constraint $(f_0, F) = 0$, and $f|_{z=0} = f_0$, $\partial_z f|_{z=0} = \tilde{\gamma} f_0$, where $\tilde{\gamma} > 0$ is defined by

$$\tilde{\gamma}^2 = -\frac{(f_0, H_- f_0)}{(f_0, H_+^{-1} f_0)}.$$

It is easy to check that for such a solution $C = 0$. Multiplying Eq. (11) by f_n and taking into account Eq. (12), after straightforward algebra we obtain

$$\frac{\partial^2}{\partial z^2}(f, H_+^{-1} f) = 4(\partial_z f, H_+^{-1}\partial_z f) \geq \frac{4(\partial_z f, H_+^{-1} f)^2}{(f, H_+^{-1} f)} = \frac{[\frac{\partial}{\partial z}(f, H_+^{-1} f)]^2}{(f, H_+^{-1} f)}. \quad (13)$$

Integration of this inequality yields

$$\frac{\frac{\partial}{\partial z}(f, H_+^{-1} f)}{(f, H_+^{-1} f)} \geq 2\tilde{\gamma} = \left.\frac{\frac{\partial}{\partial z}(f, H_+^{-1} f)}{(f, H_+^{-1} f)}\right|_{z=0}. \quad (14)$$

Thus, $(f, H_+^{-1} f) \geq (f_0, H_+^{-1} f_0)\exp(2\tilde{\gamma} z)$, and the perturbation of the ground state $F_n(t)$ grows with z. The maximum of the increment $\tilde{\gamma}$ is given by the γ defined above.

The instability is determined by the existence of a negative eigenvalue of the operator H_- under the additional constraint $(f,F) = 0$. Now we prove that the operators H_- and H_+ defined above can satisfy all requirements of the theorem, and thus instability of a ground state takes place when $\frac{\partial}{\partial \lambda^2}(F,F) = \frac{\partial}{\partial \lambda^2} P < 0$.

Note that $H_+ F_n = 0$ and $H_- \frac{\partial F_n}{\partial \lambda^2} = -F_n$. It is easy to see that the operator H_+ is non-negative. Indeed, H_+ can be presented in a form

$$(H_+f)_n = -\frac{1}{F_n}\frac{\partial}{\partial t}F_n^2\frac{\partial}{\partial t}\frac{1}{F_n}f_n - f_{n+1} - f_{n-1} + \frac{F_{n+1}+F_{n-1}}{F_n}f_n. \quad (15)$$

Thus,

$$(f,H_+f) = \sum_n \int \left[F_n^2\left(\frac{\partial}{\partial t}\frac{f_n}{F_n}\right)^2 + \left(\sqrt{\frac{F_{n-1}}{F_n}}f_n - \sqrt{\frac{F_n}{F_{n-1}}}f_{n-1}\right)^2\right]dt \geq 0.$$

To demonstrate that there exists some $\tilde{f}$ satisfying $(\tilde{f},F)=0$ and $(\tilde{f},H_-\tilde{f})<0$, we consider $s=\frac{\partial F}{\partial \lambda^2}-\alpha F$, where $\alpha=\frac{1}{2}\frac{d}{d\lambda^2}\ln(F,F)$. The constraint $(s,F)=0$ is satisfied by construction of s. Using the evident equalities $H_-\frac{\partial F_n}{\partial \lambda^2}=-F_n$ and $H_-F=-4F^3$ we obtain

$$\begin{aligned}(s,H_-s) &= -\left(F,\frac{\partial F}{\partial\lambda^2}\right) - 2\alpha\left(F,H_-\frac{\partial F}{\partial\lambda^2}\right) + \alpha^2(F,H_-F)\\ &= -(F,F^3)\left[\frac{d}{d\lambda^2}\ln(F,F)\right]^2 + \frac{d}{d\lambda^2}(F,F).\end{aligned} \quad (16)$$

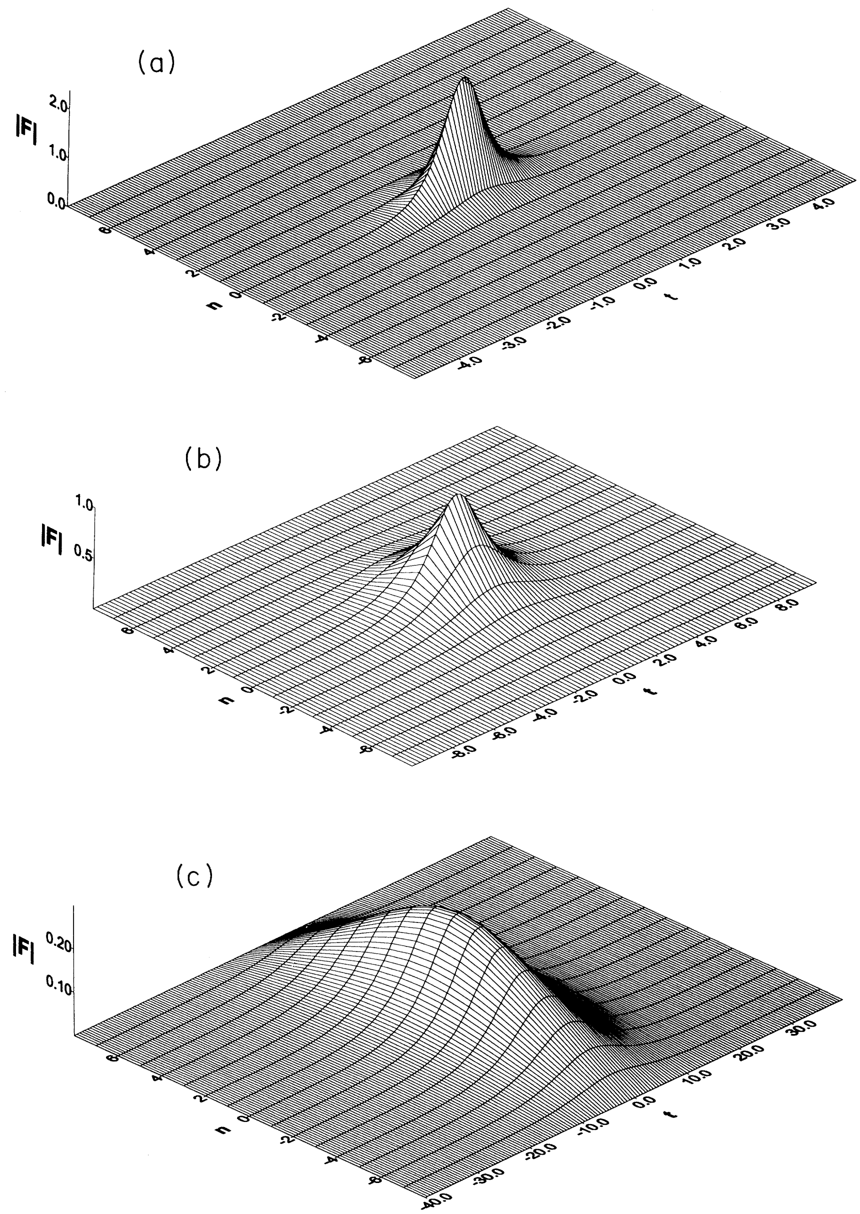

FIG. 1. Structure of continuous-discrete solitons for different values of parameter λ. (a) The narrow soliton with $\lambda = 2$. (b) Unstable soliton with $\lambda = 0.6$. (c) Broad state with $\lambda = 0.01$ resembles the two-dimensional soliton of the continuum limit.

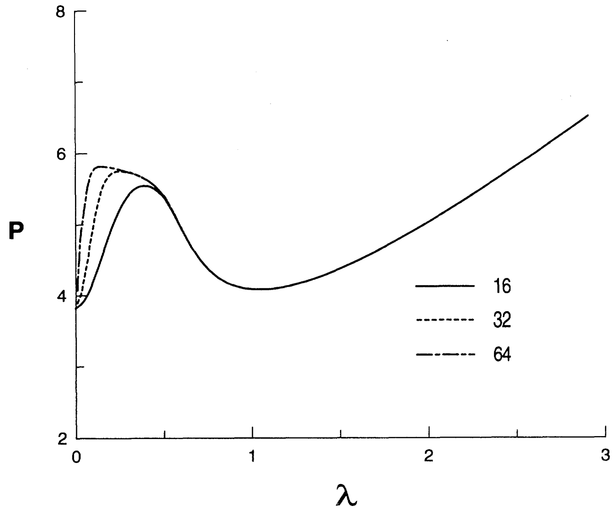

FIG. 2. Integral P versus λ. It was found numerically that the solitons satisfying condition $\frac{dP}{d\lambda^2} > 0$ are stable and solutions with $\frac{dP}{d\lambda^2} < 0$ are unstable. It is shown also how the curve changes by increasing the number of the fibers in array. Solid line is for $2N = 16$, dashed line corresponds to $2N = 32$, and long-dashed line to $2N = 64$.

From this it is clear that a sufficient criterion of the instability can be formulated as $\frac{d}{d\lambda^2}(F, F) = \frac{d}{d\lambda^2}P < 0$.

Thus, we proved that the instability of the stationary solutions of Eq. (1) depends on the sign of $\frac{dP}{d\lambda^2}$. In Fig. 2 the energy P is shown as a function of λ^2 for the case $N = 8$ (15 fibers in the array). Stationary solutions that correspond to the negative slope on this curve are unstable. An important feature caused by the discreteness is a coexistence of the stable and unstable solitons. It is interesting to discuss solutions corresponding to the left wing of the curve shown in Fig. 2. These broad solitons plotted in Fig. 1(c) are a discrete analog of the two-dimensional continuum soliton known in the theory of self-focusing as Town's mode. The 2D soliton of the corresponding continuum problem is known to be weakly (not exponentially) unstable in the continuum NLS equation. The discreteness can stabilize the instability of the broad ground state in the case of a finite number of fibers in array. It is important to mention that the left part of the curve presented in Fig. 2 degenerates into the point in the case of infinite array. $P(\lambda^2)$ in this case consists of two branches only. Contrary to the case of a finite array, an unstable branch for the infinite array starts from $\lambda = 0$, but not from a finite value of λ, as in Fig. 2 for $2N = 16$. This tendency is illustrated in Fig. 2 where evolution of the curve is shown with an increase in the number of fibers in array. First, this effect was mentioned in [18] in the context of another physical problem. It should be pointed out that the solutions in the form of the broad solitons corresponding to this left part of the curve are not artifacts of the numerical simulations, because any real array obviously consists of a finite number of fibers.

In Fig. 3 the dependence $H(P)$ is plotted for numerically calculated soliton solutions. Our criterion for instability coincides with the predictions by the "catastrophe theory." The occurrence of a Whitney gather (Whitney surface) corresponds to the existence of unstable solitons. The latter realize saddle points of the Hamiltonian for fixed P. The cuspidal edge corresponds to a degenerate critical point (see, e.g., the review [17]).

Results of numerical simulations presented in Fig. 4 confirm that the stability depends on the sign of $\frac{dP}{d\lambda^2}$. Solitons with a positive sign of $\frac{dP}{d\lambda^2}$ from the right wing of the curve plotted in Fig. 2 show rather stable behavior

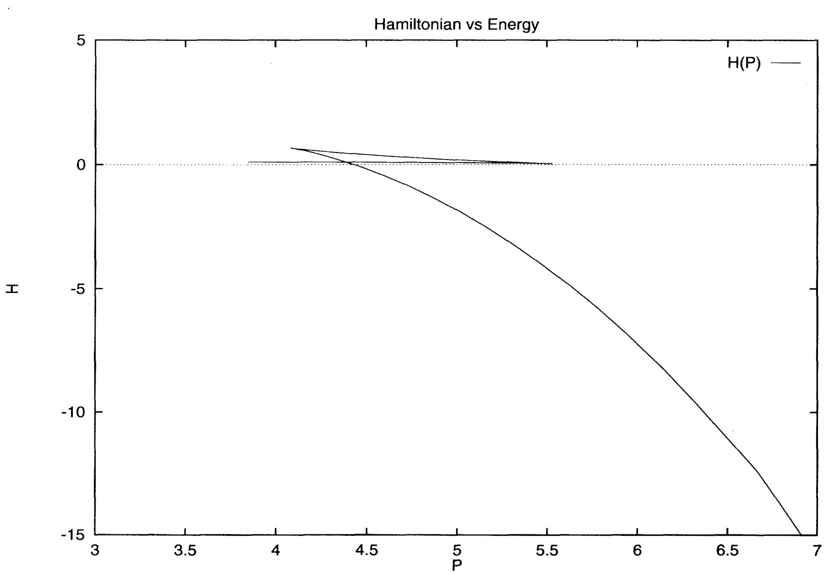

FIG. 3. Hamiltonian H as a function of the energy P. The occurrence of a Whitney gather (Whitney surface) corresponds to the existence of unstable solitons.

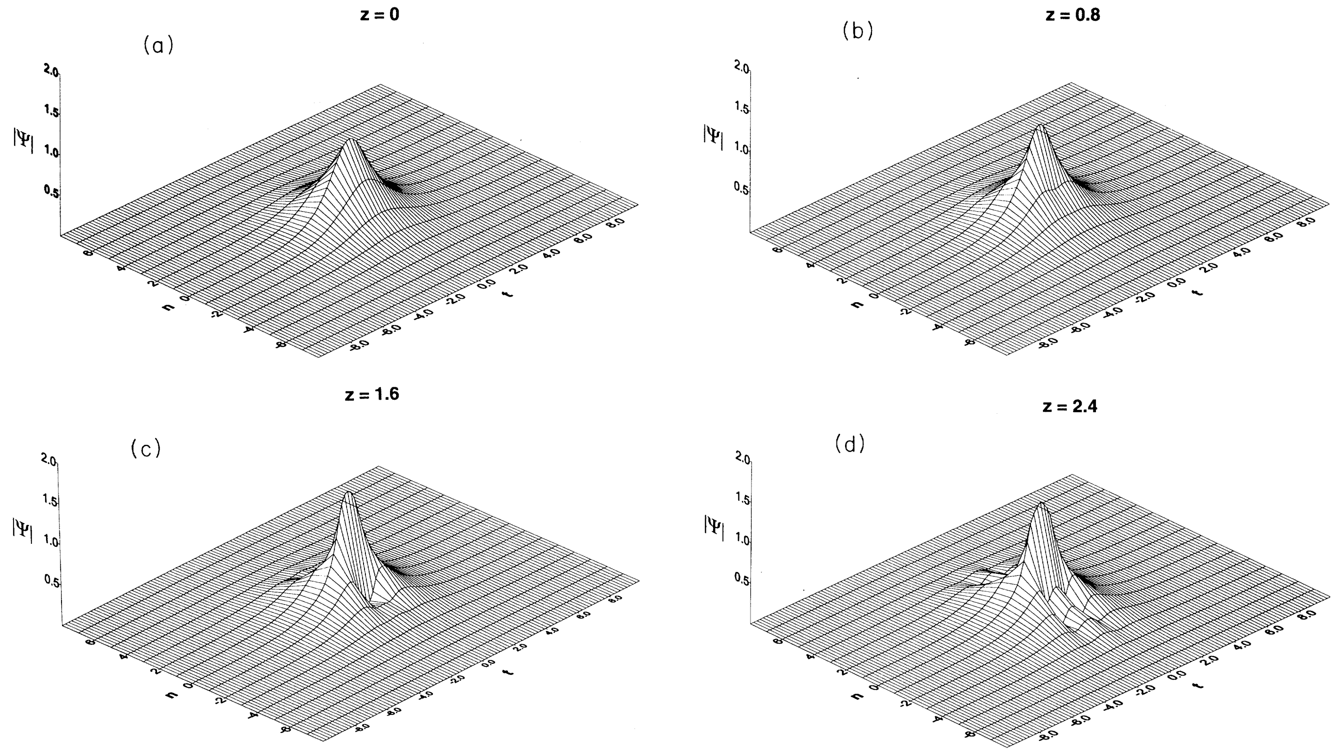

FIG. 4. Nonstationary evolution of the solitons from unstable branch $\frac{dP}{d\lambda^2} < 0$ ($\lambda = 0.6$). (a)–(d) show a development of the instability.

in numerical simulations. Contrary to that, stationary states with a negative sign of this derivative are unstable, as is shown in Fig. 4. Nonlinear development of the instability of the discrete soliton from an unstable branch leads to the formation of the breatherlike solution.

In fact, it is clear from Eq. (16) that we can obtain an even more sharp criterion of the instability. We can rewrite this criterion in terms of the conserved quantities P and H. Indeed, stationary solutions satisfying Eq. (6) realize extremum of the Hamiltoniam H for the fixed value of the integral P: $\delta(H+\lambda^2 P) = 0$. Therefore, using it as a trial function in the later variational equation $g_n(t) = (1+a)F_n(t)$ yields a condition $\frac{\partial}{\partial a}(H + \lambda^2 P)|_{a=0}$. After a trivial calculation we get

$$I_1 + I_2 - 2I_3 + \lambda^2 P = 0. \tag{17}$$

Here all integrals are calculated based on the stationary solution of Eq. (6). Substituting now $I_3 = (F, F^3) = H + \lambda^2 P$ into the final part of Eq. (16) we obtain a more sharp criterion of the instability, namely $(s, H_{-}s)$ is negative if $\frac{d}{d\lambda^2}P - (H + \lambda^2 P)\left[\frac{d}{d\lambda^2}\ln P\right]^2 < 0$.

We restrict the consideration in this paper to the proof of the instability criterion. A comprehensive analysis of the stability of the ground states and a proof of the stability criterion will be presented elsewhere.

In conclusion, we examined the stability of the continuous-discrete solitons in a NFA. An exact criterion of the instability is obtained. Symmetrical ground states, which are functions of a spectral parameter λ, are unstable if the first derivative of the integral $P = \sum_n \int |\Psi_n|^2 dt$ with respect to a λ^2 is negative.

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[1] A. I. Kaplan, Phys. Rev. Lett. **55**, 1291 (1985).
[2] E. W. Laedke and K. H. Spatschek, in *Differential Geometry, Calculus of Variations, and Their Applications*, edited by G. M. Rassias and T. M. Rassias (Marcel Dekker, Inc., New York, 1985).
[3] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Phys. Rep. **142**, 105 (1986).
[4] R. A. Enns, S. S. Rangnekar, and A. E. Kaplan, Phys. Rev. A **35** 466 (1987).
[5] M. J. Ablowitz and J. F. Ladik, Stud. Appl. Math. **55**, 213 (1976).
[6] A. C. Scott and L. Macneil, Phys. Lett. A **98**, 87 (1983).
[7] Yu. S. Kivshar and M. Peyrard, Phys. Rev. A **46**, 3198 (1992).
[8] O. Bang, J. J. Rasmussen, and P. L. Christiansen, Nonlinearity **7**, 205 (1993).

[9] E. W. Laedke, K. H. Spatschek, and S. K. Turitsyn, Phys. Rev. Lett. **73**, 1055 (1994).
[10] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 794 (1988).
[11] C. Schmidt-Hattenberger, U. Trutschel, R. Muschall, and F. Lederer, Opt. Commun. **89**, 473 (1992); Opt. Quantum Electron. **25**, 185 (1993).
[12] A. B. Aceves, C. De Angelis, A. M. Rubenchik, and S. K. Turitsyn, Opt. Lett. **19**, 329 (1994).
[13] A. B. Aceves, G. G. Luther, C. De Angelis, A. M. Rubenchik, and S. K. Turitsyn, Opt. Fiber Technol. **2**, 1 (1995).
[14] Yu. S. Kivshar, Phys. Rev. E **48**, 4132 (1993).
[15] S. N. Vlasov, V. A. Petrishchev, and V. I. Talanov, Radiophys. Quantum Electron. **14**, 1062 (1971).
[16] A. V. Buryak and N. N. Akhmediev, IEEE J. Quantum Electron. **31**, 682 (1995) .
[17] F. V. Kustmartsev, Phys. Rep. **183**, 1 (1989).
[18] V. K. Mezentsev, S. L. Musher, I. V. Ryuzhenkova, and S. K. Turitsyn, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 11 (1994) [JETP Lett. **60**, 829 (1994)].