Normal and anomalous scaling of the fourth-order correlation function of a randomly advected passive scalar

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(Received 22 March 1995; revised manuscript received 3 August 1995)

For a short-correlated velocity field, simultaneous correlation functions of a passive scalar satisfy closed equations. We analyze the equation for the four-point function. To describe a solution completely, one has to solve the matching problems at the scale of the source and at the diffusion scale. We solve both the matching problems and thus find the dependence of the four-point correlation function on the diffusion and pumping scale for large space dimensionality d. It is shown that anomalous scaling appears in the first order of $1/d$ perturbation theory. Anomalous dimensions are found analytically both for the scalar field and for its derivatives, in particular, for the dissipation field.

PACS number(s): 47.10.+g, 47.27.−i, 05.40.+j

INTRODUCTION

It is already commonplace to talk about an anomalous scaling of the high-order correlation functions in developed turbulence. By this, the deviation of the scaling exponents from their "naive" values taken from dimensional estimate or perturbation theory is usually meant. Another meaning ascribed to that term is related to the cases where the exponent of the $2n$th correlation function is not n times the exponent of the second one so that the degree of non-Gaussianity depends on scale. The experimental evidence in favor of anomalous scaling of a scalar field advected by turbulence has existed for some time [1–4], while attempts at a consistent derivation (starting from the equations of fluid mechanics) of correlation functions of order higher than 2 started only recently [5–10]. The problem of a passive scalar advection, being of physical importance by itself, may serve also as a starting point in studying anomalous scaling in turbulence [6,11]. At a first step, we found [12] the whole set of simultaneous correlation functions for the Batchelor-Kraichnan problem of a scalar advected by a large-scale random velocity field [13,14]. It has been shown that, whatever the (finite) temporal correlations of the velocity field are, all correlation functions of the scalar are integer powers of a logarithm for all the distances in the convective interval of scales and no anomalous scaling thus appears at the leading terms. The present paper is an account of the next step: we consider a multiscale velocity field with power spectrum. Following Kraichnan [15], we restrict ourselves by the simplest possible temporal behavior assuming both velocity field and scalar source to be white in time. That leads to a substantial simplification of the analytical description, since any simultaneous correlation function of a scalar satisfies a closed linear differential equation of second order (see [6] and below).

In an isotropic turbulence, an n-point correlation function depends on $n(n-1)/2$ distances for a dimensionality of space $d > n-2$. For the pair correlation function, the respective ordinary differential equation could be readily solved for any distance between the points — see [15] and (1.20) below. The solution is expressed via the flux of a squared scalar P_2 and molecular and eddy diffusivity, the scaling exponent ζ_2 in the inertial interval being fixed by the condition of flux constancy. If the scaling exponent of the $2n$th correlation function is $n\zeta_2$, that is called a normal scaling. An anomalous scaling would mean that the true answer has additional factors like $(l/r)^\Delta$, where Δ is anomalous dimension, r is the distance between points, and l is some length parameter. One may imagine three reasons for anomalous scaling according to the three lengths that may be relevant: (i) l is the pumping scale L so that the anomalous scaling appears due to infrared nonlocality, (ii) l is the diffusion scale r_d so that the anomalous scaling appears due to ultraviolet nonlocality [16,11], and (iii) anomalous scaling appears due to the existence of the high-order integrals of motion [17] so that the length parameter appears from the ratio of the different fluxes. The first nontrivial object that may reveal anomalous scaling is the fourth-order correlation function. One may be interested in two-point objects like $\langle(\theta_1-\theta_2)^4\rangle$ or the pair correlation function $\langle\epsilon_1\epsilon_2\rangle$ of the dissipation field $\epsilon(t,\mathbf{r}) = \kappa[\nabla\theta(t,\mathbf{r})]^2$, yet they do not satisfy any closed equation. To find such two-point–fourth-order objects, one should solve the complete equation for the four-point function and then fuse some points. Considering $d > 2$, one has to deal with the space of six variables, which makes the direct solution at arbitrary parameters quite difficult. Gawedski and Kupiainen [10] recently developed a perturbation theory that describes the effect of the weak advection on the fourth-order correlation function for a noncascade diffusionlike regime and found the respective anomalous exponents.

Our main target here is the description of the cascade in the convective interval of scales. We solve the equation for the four-point correlation function assuming the space dimensionality d to be large. The non-Gaussian part of the correlation function is small in parameter $1/d$, which makes it possible to develop a regular perturbation theory. In the present paper, we obtain the scaling exponents analytically in the first order in $1/d$ (assuming $1/d$ to be the smallest parameter, in particular, $1/d \ll \zeta_2$) and show that the main contribution into the general fourth-order correlation function has L-related anomalous scaling with

$$\Delta = \frac{4(2-\zeta_2)}{d} \; .$$

Note that Δ turns into zero when $\zeta_2 \to 2$, which contradicts Kraichnan's closure [6,7] and is in qualitative agreement with the result of Gawedski and Kupiainen [10], even though they considered a different limit.

For the correlation functions between the points separated by some distances from the convective interval and by others from a diffusion interval of scales, we show that the main term does not depend on r_d. We also show that the subleading terms in the case of some distances being much smaller than others necessarily have another anomalous exponent with respect to small distances. Such subleading terms determine, in particular, correlations of the spatial derivatives of the scalar field, which thus have an r_d-related anomalous scaling (in addition to a general L-related scaling). Note that the results on the r_d dependence are obtained without $1/d$ expansion so they are exact at any dimension. For example, the dissipation field has an r_d-related anomalous scaling with an exponent that is equal to the scaling exponent of the eddy diffusivity so that the irreducible correlator $\langle\langle \epsilon_1\epsilon_2\rangle\rangle \propto (L/r_{12})^{\Delta} r_d^0$ does not depend on the diffusion scale r_d when the distance r_{12} is in the convective interval. Another consequence of our results is the statement that $\langle\epsilon^2\rangle/\langle\epsilon\rangle^2$ tends to infinity with the Péclet number increasing. On the contrary, the one-point statistics of θ (say, the flatness $\langle\theta^4\rangle/\langle\theta^2\rangle^2$) is independent of the Péclet number Pe$= L/r_d$ at the limit of large Pe.

Besides, we find the nontrivial r_d-related anomalous scaling describing the traceless tensor $\nabla_\alpha\theta\nabla_\beta\theta - d^{-1}\delta_{\alpha\beta}(\nabla\theta)^2$. We also argue that high-order integrals of motion may influence scaling if the pumping is non-Gaussian and that the different kinds of L-related anomalous scaling may be the case starting from the sixth correlation function.

The anomalous L-related scaling of the fourth-order correlation function of the scalar field confronted with a diagrammatic analysis that shows no infrared divergences at any finite order of a Wyld diagram technique [11] evidently means that such an anomalous scaling is a nonperturbative phenomenon.

The structure of this paper is as follows. We formulate the problem and find the pair correlation function in Sec. I. The second section is devoted to L dependence while the third section is devoted to r_d dependence of the fourth-order correlation functions. The correlation functions of the scalar derivatives are also considered in the third section. Section IV describes possible generalizations and the Conclusion summarizes the results.

I. FORMULATION OF THE PROBLEM

We consider the advection of a passive scalar field $\theta(t,\mathbf{r})$ by an incompressible turbulent flow. The advection is governed by the following equations:

$$(\partial_t - \hat{P})\theta = \phi, \tag{1.1}$$

$$\hat{P}(t) = -u^\alpha\nabla^\alpha + \kappa\triangle, \qquad \nabla^\alpha u^\alpha = 0, \tag{1.2}$$

where both convective and diffusive terms are included in the operator $\hat{P}(t)$; $\triangle$ designates Laplacian here. The external velocity $\mathbf{u}(t,\mathbf{r})$ and the external source $\phi(t,\mathbf{r})$ are random functions of t and $\mathbf{r}$. We regard the statistics of the velocity and of the source to be independent. Therefore all correlation functions of θ are to be treated as averages over both statistics. Averaging over pumping will be designated by an overbar and averaging over velocity will be designated by angular brackets.

A. Basic relations

We assume that the source ϕ is δ-function-correlated in time and spatially correlated on a scale L. The latter means, e.g., that the pair correlation function of the source

$$\overline{\phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)} = \delta(t_1-t_2)\chi(r_{12}), \tag{1.3}$$

as a function of the argument $r_{12} \equiv |\mathbf{r}_1 - \mathbf{r}_2|$, decays on the scale L. The value $\chi(0) = P_2$ is the production rate of θ^2. Since the pumping is Gaussian, high-order correlation functions are determined by $\chi(r)$. For example,

$$\overline{\phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\phi(t_3,\mathbf{r}_3)\phi(t_4,\mathbf{r}_4)}$$

$$\begin{aligned} &= \chi(r_{12})\chi(r_{34})\delta(t_1-t_2)\delta(t_3-t_4) \\ &\quad+\chi(r_{13})\chi(r_{24})\delta(t_1-t_3)\delta(t_2-t_4) \\ &\quad+\chi(r_{14})\chi(r_{23})\delta(t_1-t_4)\delta(t_2-t_3). \end{aligned} \tag{1.4}$$

A formal solution of (1.1) is

$$\theta(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\hat{U}(t,t')\phi(t',\mathbf{r}), \tag{1.5}$$

$$\begin{aligned} \hat{U}(t,t') &= T\exp\left(\int_{t'}^{t}\hat{P}(\tau)d\tau\right) \\ &\equiv \sum_{n=0}^{\infty}\int_{t'}^{t} d\tau_1\int_{t'}^{\tau_1} d\tau_2\cdots\int_{t'}^{\tau_{n-1}} d\tau_n \\ &\quad\times\hat{P}(\tau_1)\cdots\hat{P}(\tau_n). \end{aligned} \tag{1.6}$$

Since θ and ϕ are related linearly, the pair product of the passive scalar field averaged over the statistics of pumping is expressed via the pair correlation function of the pumping

$$\overline{\theta(t,\mathbf{r}_1)\theta(t,\mathbf{r}_2)} = \int_{-\infty}^{t} d\tau \hat{U}_1(t,\tau)\hat{U}_2(t,\tau)\chi(r_{12}) = \int_{-\infty}^{t} d\tau T\exp\Big(\int_\tau^t [\hat{P}_1(\tau) + \hat{P}_2(\tau)]d\tau\Big)\chi(r_{12}). \quad (1.7)$$

Here and below $\hat{U}_i, \hat{P}_i$ designate operators acting on variables $\mathbf{r}_i$. Time-ordered exponents on the right-hand side of (1.7) commute with each other (because they have different space arguments), which allows us to rewrite in (1.7) their product as a single $T\exp$ factor.

The next step is to average $T\exp$ in (1.7) with respect to the statistics of the velocity field $\mathbf{u}$. Following Kraichnan, we consider the case of a velocity δ-function-correlated in time but *multiscale* in space. Velocity statistics is completely determined by the pair correlation function

$$\langle u^\alpha(t_1,\mathbf{r}_1)u^\beta(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)V^{\alpha\beta}, \quad V^{\alpha\beta} = V_0\delta^{\alpha\beta} - \mathcal{K}^{\alpha\beta}(\mathbf{r}_{12}), \quad \mathcal{K}^{\alpha\beta}(0) = 0. \quad (1.8)$$

Here the so-called eddy diffusivity is as follows:

$$\mathcal{K}^{\alpha\beta} = \frac{D}{r^\gamma}(r^2\delta^{\alpha\beta} - r^\alpha r^\beta) + \frac{D(d-1)}{2-\gamma}\delta^{\alpha\beta}r^{2-\gamma}, \quad (1.9)$$

where $0<\gamma<2$ is supposed and $\langle\,\rangle$ stands for an average over the velocity field statistics, isotropy being assumed. The representation (1.9) is valid for the scales less than velocity infrared cutoff L_u, which is supposed to be the largest scale of the problem. The marginal "logarithmic" $\gamma=0$ and "diffusive" $\gamma=2$ regimes require special care.

B. Simultaneous pair correlator

Let us calculate the simultaneous pair correlation function of the passive scalar

$$f(r_{12}) = \langle\overline{\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)}\rangle. \quad (1.10)$$

Since we assume the statistics of the velocity field to be δ-function-correlated and Gaussian, then the nth order correlator of the velocity field [which appears in the expansion of the $T\exp$ from the integrand of (1.7)] can be found explicitly by the Wick theorem which prescribes to reduce any average to a product of pair correlation functions. Those terms are summed up into a usual operator exponent

$$\Big\langle T\exp\Big(\int_\tau^t [\hat{P}_1(\tau)+\hat{P}_2(\tau)]d\tau\Big)\Big\rangle = e^{(t-\tau)\hat{\mathcal{L}}_{12}}, \quad (1.11)$$

$$\hat{\mathcal{L}}_{ij} \equiv V^{\alpha\beta}(\mathbf{r}_{ij})\nabla_i^\alpha\nabla_j^\beta + \Big(\frac{V_0}{2}+\kappa\Big)(\triangle_i+\triangle_j), \quad (1.12)$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and $\triangle_i = \nabla_i^2$. We find from (1.7)

$$f(r_{12}) = \int_{-\infty}^t d\tau \exp\Big((t-\tau)\hat{\mathcal{L}}_{12}\Big)\chi(r_{12}). \quad (1.13)$$

Integrating the right-hand side of (1.13) with respect to time ($\hat{\mathcal{L}}_{ij}^{-1}$ is a well defined integral operator) we get $\hat{\mathcal{L}}_{12}f(r_{12}) = -\chi(r_{12})$, which is an ordinary second-order differential equation

$$\hat{\mathcal{L}}^{(p)}f = -\chi, \quad \text{where} \quad \partial_r \equiv \partial/\partial r, \quad (1.14a)$$

$$\hat{\mathcal{L}}^{(p)} = \frac{(d-1)D}{2-\gamma}r^{1-d}\partial_r\big(r^{2-\gamma}+r_d^{2-\gamma}\big)r^{d-1}\partial_r, \quad (1.14b)$$

previously derived by Kraichnan [15]. In (1.14b) the diffusion scale r_d was introduced,

$$r_d^{2-\gamma} = \frac{2\kappa(2-\gamma)}{D(d-1)}. \quad (1.15)$$

Equation (1.14) is integrated explicitly. The solution is completely determined by two physical boundary conditions: zero at $r=\infty$ and absence of a singularity at $r=0$. We consider the pumping correlation function close to a step function $\chi(r) = P_2$ at $r<L$ and zero otherwise:

$$f(r) = P_2\int_r^\infty g(r')dr', \quad (1.16)$$

$$g(r) = \begin{cases} \frac{L^d}{2\kappa d}\frac{r^{1-d}}{1+(r/r_d)^{2-\gamma}}, & r>L \\ \frac{r}{2\kappa d}\frac{1}{1+(r/r_d)^{2-\gamma}}, & r<L. \end{cases} \quad (1.17)$$

For a general γ, the pair correlation function is expressed via the confluent hypergeometric function

$$f(r) = \frac{P_2 r_d^\gamma}{d(d-1)}\Big\{\frac{\pi}{\sin[2\pi/(2-\gamma)]} - \Phi\Big(-\big(r/r_d\big)^{2-\gamma}, 1, 2/(2-\gamma)\Big)\Big\}, \quad (1.18)$$

see [18] p. 29 and [19] p. 27. In the Richardson-Kolmogorov case, $d=3$, $\gamma=2/3$, and one gets

$$f(0)-f(r) = \frac{P_2}{3}\Big[r^{2/3} - r_d^{2/3}\arctan(r/r_d)^{2/3}\Big]. \quad (1.19)$$

Both (1.18) and (1.19) are valid at any $r\le L$ for the steplike pumping. At $r\ll L$, those formulas describe the pair correlation function for the case of an arbitrary pumping χ with the correlation scale L. One can see that the transition from convective to diffusive interval is described by a universal function.

If the Péclet number Pe$=L/r_d$ is large, the pumping and diffusion scales separate three intervals of scales with a different scaling behavior:

$$f(r) \approx \begin{cases} P_2\Big(\frac{L^\gamma(2-\gamma)}{\gamma(d-\gamma)(d-1)D} - \frac{r^2}{4\kappa d}\Big), & r\ll r_d\ll L, \\ P_2\frac{2-\gamma}{\gamma(d-1)D}\Big(\frac{L^\gamma}{d-\gamma} - \frac{r^\gamma}{d}\Big), & r_d\ll r<L \\ P_2\frac{L^d(2-\gamma)r^{\gamma-d}}{d(d-1)(d-\gamma)D}, & r_d\ll L<r. \end{cases} \quad (1.20a)$$

Those expressions fully determine the behavior of the pair correlation function.

C. Different-time pair correlator

The dynamical analog of (1.7) looks as follows ($t_2 > t_1$):

$$\overline{\theta(t_1,\mathbf{r}_1)\theta(t_2,\mathbf{r}_2)}$$

$$= \hat{U}_2(t_2,t_1)\int_{-\infty}^{t} d\tau \hat{U}_1(t_1,\tau)\hat{U}_2(t_1,\tau)\chi(r_{12})$$
$$= \hat{U}_2(t_2,t_1)\overline{\theta(t_1,\mathbf{r}_1)\theta(t_1,\mathbf{r}_2)}\,. \qquad (1.21)$$

Since the velocity field is δ-function-correlated, one may reduce the average of the product from the right-hand side of (1.21) to the product of the averages

$$\langle\overline{\theta(t_1,\mathbf{r}_1)\theta(t_2,\mathbf{r}_2)}\rangle = \langle\hat{U}_2(t_2,t_1)\rangle f(r_{12})$$
$$= \exp\Big(\mid t_2-t_1\mid(V_0+\kappa)\triangle\Big)f(r_{12}).$$

Let us stress that if $r_d \ll L$, then $V_0 \gg \kappa$. The point is that V_0 can be estimated as $V_0 \sim DL_u^{2-\gamma}$, where L_u is the scale of the largest vortices. Since $L_u > L$ is assumed, $V_0 \gg \kappa$. That means that the time dependence of correlation functions of the passive scalar θ is determined by the velocity of the largest vortices and is therefore fast. Note that in the comoving reference frame this dynamics is much slower [11].

II. FOUR-POINT CORRELATION FUNCTION OF THE PASSIVE SCALAR

We begin the analysis of the fourth-order correlation function F_{1234}. It will be based upon the equation (2.5), previously derived independently by Kraichnan [20], Sinai and Yakhot [21], and Shraiman and Siggia [22]. It is a second-order partial differential equation in the space of $4d$ variables. Isotropy allows one to diminish the number of variables to five at $d = 2$ and to six at $d > 2$, which is still too many to enable finding F explicitly at all possible distances r_{ij} ranging from zero to infinity. Our aim is modest: we are looking for the scaling exponents only. First, we shall find the overall scaling exponent ζ_4 which describes how Γ (which is the irreducible part of F_{1234}) scales if all the distances r_{ij} are multiplied by the same factor. This is the subject of this section where we employ $1/d$ perturbation theory assuming space dimensionality to be large. That will also allow us to find the anomalous exponents that describe the L dependence of the correlation function. Second, we shall consider the case with one or two distances being much less than other ones and find the scaling exponents with respect to small and large distances separately. This could be done at an arbitrary d. That will allow us to fuse some points and to find the r_d dependence of the cumulants of second powers of spatial derivatives. This is the subject of the next section.

A. The equation for the simultaneous four-point correlation function

Using the technique developed in the preceding section we derive the equation for the simultaneous four-point correlation function F_{1234}. Let us form first a four-point object averaged over the pumping only by analogy with (1.7),

$$\overline{\theta(t,\mathbf{r}_1)\theta(t,\mathbf{r}_2)\theta(t,\mathbf{r}_3)\theta(t,\mathbf{r}_4)} = \int_{-\infty}^{t} d\tau_1\int_{-\infty}^{t} d\tau_2\int_{-\infty}^{t} d\tau_3\int_{-\infty}^{t} d\tau_4 \hat{U}_1(t,\tau_1)\hat{U}_2(t,\tau_2)$$
$$\times\hat{U}_3(t,\tau_3)\hat{U}_4(t,\tau_4)\overline{\phi(\tau_1,\mathbf{r}_1)\phi(\tau_2,\mathbf{r}_2)\phi(\tau_3,\mathbf{r}_3)\phi(\tau_4,\mathbf{r}_4)}. \qquad (2.1)$$

Due to (1.4) the object separates onto three parts $Q_{12;34}+Q_{13;24}+Q_{14;23}$, where

$$Q_{ij;kl} = \int_{-\infty}^{t}\int d\tau d\tilde{\tau}\hat{U}_i(t,\tau)\hat{U}_j(t,\tau)\hat{U}_k(t,\tilde{\tau})\hat{U}_l(t,\tilde{\tau})\chi(r_{ij})\chi(r_{kl})$$
$$= \int_{-\infty}^{t} d\tau\, T\exp\left[\int_{\tau}^{t} d\tau'(\hat{P}_i+\hat{P}_j+\hat{P}_k+\hat{P}_l)\right]\int_{-\infty}^{\tau} d\tilde{\tau}\left[T\exp\left(\int_{\tilde{\tau}}^{\tau} d\tilde{\tau}'(\hat{P}_i+\hat{P}_j)\right)\right.$$
$$\left.+T\exp\left(\int_{\tilde{\tau}}^{\tau} d\tilde{\tau}'(\hat{P}_k+\hat{P}_l)\right)\right]\chi(r_{ij})\chi(r_{kl}). \qquad (2.2)$$

The commutativity of $\hat{P}_i$ and $\hat{P}_j$ at $i \neq j$ has been taken into account at (2.2).

For the δ-function-correlated velocity field, one may average the time-ordered exponents on the right-hand side of (2.2) explicitly, first expanding them into series, and second performing the Gaussian decomposition of the velocity correlators in the same manner as was done for the pair correlator of the passive scalar in (1.11). That results in the following expressions:

$$\langle Q_{ij;kl}\rangle = \int_{-\infty}^{t} d\tau e^{(t-\tau)\hat{\mathcal{L}}}$$
$$\times\int_{-\infty}^{\tau} d\tilde{\tau}\left(e^{(\tau-\tilde{\tau})\hat{\mathcal{L}}_{12}}+e^{(\tau-\tilde{\tau})\hat{\mathcal{L}}_{34}}\right)$$
$$\times\chi(r_{ij})\chi(r_{kl})\,, \qquad (2.3)$$

where $\hat{\mathcal{L}}_{ij}$ was defined in (1.12) and $\hat{\mathcal{L}}$ has the following form:

$$\hat{\mathcal{L}} \equiv \frac{1}{2}\sum_{i,j} V^{\alpha\beta}(\mathbf{r}_{ij})\nabla_i^\alpha\nabla_j^\beta + \kappa\sum \triangle_i\,, \tag{2.4}$$

where $\nabla_i^\alpha \equiv \partial/\partial r_i^\alpha$. The integration over $\tilde{\tau}$ in (2.3) gives

$$-(\hat{\mathcal{L}}_{ij}^{-1}+\hat{\mathcal{L}}_{kl}^{-1})\chi(r_{ij})\chi(r_{kl}) = f(r_{ij})\chi(r_{kl})+\chi(r_{ij})f(r_{kl})\ ,$$

where $\hat{\mathcal{L}}_{ij}f(r_{ij}) = -\chi(r_{ij})$ was used. Inserting the last expressions into (2.3) and collecting all the terms $\langle Q_{ij;kl}\rangle$ we find for the full four-point object $F_{1234} \equiv \langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\theta(\mathbf{r}_3)\theta(\mathbf{r}_4)\rangle$

$$F_{1234} = \int_{-\infty}^{t} d\tau\, \exp\left[(t-\tau)\hat{\mathcal{L}}\right] \sum_{\{ij\}\neq\{kl\}} f(r_{ij})\chi(r_{kl}),$$

where the notation $\{ij\}$ stands for a pair of noncoinciding ($i \neq j$) site indices. That leads to the following equation:

$$\begin{aligned}-\hat{\mathcal{L}}F_{1234} = {} & f(r_{12})\chi(r_{34}) + f(r_{34})\chi(r_{12}) + f(r_{13})\chi(r_{24}) \\ & + f(r_{24})\chi(r_{13}) \\ & + f(r_{14})\chi(r_{23}) + f(r_{23})\chi(r_{14}),\end{aligned} \tag{2.5}$$

with the right-hand side expressed in terms of the functions $f(r_{ij})$ already found: see (1.16), and (1.20). Note that at a confluence of the points $\mathbf{r}_i \to \mathbf{r}_j$ the right-hand side of (2.5) tends to a constant. Using (1.14) one can obtain an equation for the irreducible four-point correlator $\Gamma_{1234} = \langle\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\theta(\mathbf{r}_3)\theta(\mathbf{r}_4)\rangle\rangle \equiv F_{1234} - f_{12}f_{34} - f_{13}f_{24} - f_{14}f_{23}$:

$$-\hat{\mathcal{L}}\Gamma_{1234} = \Phi_{12;34} + \Phi_{13;24} + \Phi_{14;23}, \tag{2.6a}$$

$$\begin{aligned}\Phi_{ij;kl} = {} & \left[\mathcal{K}^{\alpha\beta}(\mathbf{r}_{il}) - \mathcal{K}^{\alpha\beta}(\mathbf{r}_{ik}) + \mathcal{K}^{\alpha\beta}(\mathbf{r}_{jk}) - \mathcal{K}^{\alpha\beta}(\mathbf{r}_{jl})\right] \\ & \times\nabla_{ij}^\alpha\nabla_{kl}^\beta f(\mathbf{r}_{ij})f(\mathbf{r}_{kl})\,,\end{aligned} \tag{2.6b}$$

where $\nabla_{ij}^\alpha \equiv \partial/\partial r_{ij}^\alpha$.

The operator $\hat{\mathcal{L}}$ (2.4) is negatively defined. To prove this we represent $\hat{\mathcal{L}} = \kappa\hat{\mathcal{L}}_{dif} + \hat{\mathcal{L}}_c$, where the convective part could be written as $\hat{\mathcal{L}}_c = \left\langle\left[\sum \mathbf{u}(\mathbf{r}_i)\nabla_i\right]^2\right\rangle_u$. Here we designate $\langle\,\rangle_u$ as an average over the velocity field $\mathbf{u}(\mathbf{r})$ random in space — from the viewpoint of initial $\mathbf{u}(\mathbf{r},t)$, the averaging is over the instant configuration. Incompressibility guarantees that $\hat{\mathcal{L}}_c$ is Hermitian and negatively defined as well as $\hat{\mathcal{L}}_{dif}$. Therefore, $\hat{\mathcal{L}}$ has a continuous nonpositive spectrum and the density of states regular at zero. The last statement follows from the inequality $\|\kappa\hat{\mathcal{L}}_{dif} + \hat{\mathcal{L}}_c\| \geq \|\kappa\hat{\mathcal{L}}_{dif}\|$ and the absence of singularity in the density of states for $\hat{\mathcal{L}}_{dif}$. We can thus conclude that the equation $-\hat{\mathcal{L}}\Gamma = \Phi$ is well defined for Γ and Φ, which do not grow at infinity.

The operator $\hat{\mathcal{L}}$ is scale invariant if all the distances r_{ij} are either much larger than r_d or much smaller. The right-hand side is scale invariant if all the distances are either larger or smaller than L. We thus could divide our space of r_{ij} into three domains where the scale invariance of $\Gamma(r_{ij})$ is to be expected. It is natural now to ask a simple question: what prevents us from making the statement that the scaling exponent of the solution is equal to the difference between the exponents of the right-hand side and of the operator $\hat{\mathcal{L}}$? Of course, the solutions with such "naive" scalings exist in all three regions (we call them forced solutions F_{forc}). The problem is that to match those particular solutions at pumping and diffusion scale, it may be necessary to include into the full solution the zero modes of the operator

$$F = F_{\text{forc}} + \mathcal{Z}\ . \tag{2.7}$$

Here the zero mode $\mathcal{Z}$ may have a scaling different from that of F_{forc}. To avoid misunderstanding, note that since the operator $\hat{\mathcal{L}}$ is nonpositive, it cannot have a global zero mode that satisfies boundary conditions. The parts of the solution with an anomalous scaling may nevertheless be considered as zero modes within separate domains. At $r_{ij} \ll L$, for instance, one should worry about zero modes that appear due to matching conditions at $r_{ij} \simeq L$, as such modes are allowed to grow with r_{ij}. At an oversimplified level, this is what happens when we obtain the constant L-dependent term in the expressions (1.20a) and (1.19b) for the pair correlation function. In a multidimensional space, operators may have zero modes much more complicated than a constant and, indeed, the operator $\hat{\mathcal{L}}$ does have an infinite number of zero modes.

We cannot yet find the zero modes and solve the matching problem analytically at arbitrary d. Fortunately, for $d > 2$ (generally, for $d > n-2$, where n is the number of points in the correlation function) we found the representation of $\hat{\mathcal{L}}$ that allows one to represent $\Gamma(r_{ij})$ (with all distances of the same order or the main term if some distances are smaller than others) as a power series with respect to some numerical parameter ($1/d$ in the convective interval). The coefficients in the series are functions of r_{ij} all having the same scaling exponents and logarithms that appear from expanding anomalous exponents in powers of $1/d$. By analyzing those functions in the next subsection, we establish the overall scaling properties of $\Gamma(r_{ij})$ in the convective and diffusion intervals. The possibility to sum the logarithms into a power function is provided by an explicit scale invariance of the equation that determines the zero mode in the convective interval. The consideration of the subleading terms in the case of some distances being small in comparison with others requires a special approach, which will be developed in Sec. III.

B. Representation of tetrahedron lengths

To establish overall scaling properties of the irreducible fourth-order correlator Γ it will be convenient for us to use a special representation of Eqs. (2.5) and (2.6). In an isotropic case, the four-point correlation function Γ is a function of six distances r_{ij} between the points, $d > 2$ being assumed. The case $d = 2$ where there are only five independent variables deserves special consideration. In the variables $r_{ij} = |\mathbf{r}_{ij}|$, the operator $\hat{\mathcal{L}}$ is a sum of two parts, $\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_1$:

$$\hat{\mathcal{L}}_0 = \frac{D(d-1)}{2-\gamma}\sum_{i>j} r_{ij}^{1-d}\partial_{r_{ij}}(r_{ij}^{2-\gamma}+r_d^{2-\gamma})r_{ij}^{d-1}\partial_{r_{ij}} ,$$

$$\hat{\mathcal{L}}_1 = -\frac{D(d-1)}{2(2-\gamma)}\sum(r_{in}^2-r_{ij}^2-r_{jn}^2)\frac{r_{ij}^{1-\gamma}}{r_{jn}}\frac{\partial^2}{\partial r_{ij}\partial r_{jn}} - \frac{D}{4}\sum\frac{1}{r_{ij}^{\gamma}r_{im}r_{jn}}\Big(\frac{d+1-\gamma}{2-\gamma}r_{ij}^2(r_{in}^2+r_{jm}^2-r_{ij}^2-r_{mn}^2)$$
$$+\frac{1}{2}(r_{ij}^2+r_{im}^2-r_{jm}^2)(r_{ij}^2+r_{jn}^2-r_{in}^2)\Big)\frac{\partial^2}{\partial r_{im}\partial r_{jn}} + \kappa\sum\frac{r_{ij}^2+r_{im}^2-r_{mj}^2}{2r_{ij}r_{im}}\frac{\partial^2}{\partial r_{ij}\partial r_{im}} . \tag{2.8}$$

Here the summation is performed over subscripts satisfying the conditions $i \neq j$ and $m \neq i,j,n \neq i,j$.

What is nice about that representation is that if we omit $\hat{\mathcal{L}}_1$ from $\hat{\mathcal{L}}$, then the reducible part $F_{1234}^{(0)} = f_{12}f_{34} + f_{13}f_{24} + f_{14}f_{23}$ of the fourth-order correlator appears to be a solution of (2.5). It is

$$F^{(0)} = \sum f(r_{ij})f(r_{kl}) = (\text{in the convective interval})$$
$$= \frac{3(2-\gamma)^2L^{2\gamma}}{\gamma^2(d-1)^2(d-\gamma)^2} + \frac{(2-\gamma)^2}{\gamma^2(d-1)^2d^2}\sum\mathcal{Z}_{ij,kl} + F_{\text{forc}},$$

$$\mathcal{Z}_{ij,kl} = r_{ij}^{\gamma}r_{kl}^{\gamma} - \frac{d}{2(d+\gamma)}(r_{ij}^{2\gamma}+r_{kl}^{2\gamma}), \tag{2.9}$$

$$F_{\text{forc}} = \frac{(2-\gamma)^2\sum r_{ij}^{2\gamma}}{2\gamma^2(d+\gamma)(d-1)^2d} - \frac{(2-\gamma)^2L^{\gamma}\sum r_{ij}^{\gamma}}{\gamma^2(d-1)^2d(d-\gamma)} , \tag{2.10}$$

which is the zero approximation of (2.7). Here $\mathcal{Z}_{ij,kl}$ is the zero mode of $\hat{\mathcal{L}}_0$. What is remarkable is that a direct check shows F_{forc} to be a partial solution in the convective interval of the full equation $-\hat{\mathcal{L}}F_{\text{forc}} = f(r_{12})\chi(r_{34}) + \cdots$. It means, particularly, that if we develop the iteration procedure in $\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1$, then all the terms appearing at higher steps will enter zero modes of the full operator $\hat{\mathcal{L}}$ but not the forced term found. Another remarkable feature of the zero step is an absence among the terms three-point zero modes of $\hat{\mathcal{L}}_0$, like $\mathcal{Z}_{ij,ik}$. They will appear in the next step only.

The basic fact is that $\hat{\mathcal{L}}_0 \propto d^2$ while $\hat{\mathcal{L}}_1 \propto d$ as $d \to \infty$. Assuming $1/d$ to be a formal small parameter, we shall implement the iteration procedure with respect to $\hat{\mathcal{L}}_1$,

$$\hat{\mathcal{L}}_0\Gamma^{(1)} = -\hat{\mathcal{L}}_1F^{(0)} , \quad \hat{\mathcal{L}}_0\Gamma^{(n)} = -\hat{\mathcal{L}}_1\Gamma^{(n-1)} . \tag{2.11}$$

This procedure leads to a representation of Γ as a series over the powers of $1/d$:

$$\Gamma = \sum_{n=1}^{\infty}(-\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1)^nF^{(0)} , \tag{2.12}$$

which is actually the series for the non-Gaussian part of the zero mode.

One may note that the four-point correlation function is defined not in the whole six-dimensional space of r_{ij} but rather in the physical subspace restricted by triangle inequalities $r_{ij} + r_{jk} \geq r_{ik}$. This creates no additional difficulties since the solution we shall find satisfies all boundary conditions in the physical subspace. In addition, the answer is expressed in terms of powers of $\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1$ which do not have singularities at the boundary of the subspace.

At each step of the iteration procedure we should solve an equation of the (2.11) type. Such an equation is much easier to analyze than, e.g., (2.6a). The point is that the operator $\hat{\mathcal{L}}_0$ is a sum of the six suboperators $\hat{\mathcal{L}}^{(p)}$ from (1.14b), each depending on the single variable r_{ij} only. This particular form of $\hat{\mathcal{L}}_0$ enables us to analyze the zero modes and establish the necessary properties of its resolvent. The solution of Eq. (2.11) is expressed via the corresponding integral kernel as follows:

$$\Gamma^{(n)}(\vec{r}) = \int dt\prod_{i>j}\int dr'_{ij}\mathcal{R}(t;\vec{r},\vec{r}')\hat{\mathcal{L}}_1\Gamma^{(n-1)}(\vec{r}') , \tag{2.13}$$

where $\vec{r}$ designates the set of six variables r_{ij} and the integration is performed over time t and six separations r'_{ij}. The resolvent $\mathcal{R}$ can be represented as a product,

$$\mathcal{R}(t;\vec{r},\vec{r}') = \prod_{i>j}\mathcal{R}^{(p)}(t;r_{ij},r'_{ij}) , \tag{2.14}$$

where the function $\mathcal{R}^{(p)}(t;r,r')$ satisfies the equation

$$(\partial_t - \hat{\mathcal{L}}_r^{(p)})\mathcal{R}^{(p)}(t;r,r') = \delta(t)\delta(r-r') , \tag{2.15}$$

with the condition $\mathcal{R}^{(p)}(t;r,r') = 0$ at $t < 0$. Note the essential property of $\mathcal{R}$ simplifying the subsequent analysis: it is independent of the pumping scale L. That means that there is only one characteristic length in $\mathcal{R}$: the diffusion scale introduced by (1.15).

Now we are going to establish the properties of the resolvent $\mathcal{R}^{(p)}(t;r,r')$. Let us remember that there are two terms in the operator $\hat{\mathcal{L}}^{(p)}$ (1.14b) which are of the same order at the diffusion scale r_d. Solving (2.15) at the diffusion and convection intervals, we get for the resolvent

$$\mathcal{R}^{(p)} \approx \frac{1}{4t\kappa}\exp\left(-\frac{r^2+r'^2}{8t\kappa}\right)I_{d/2-1}\left(\frac{rr'}{4t\kappa}\right)r^{1-d/2}r'^{d/2}, \quad \text{at} \quad r, r', \sqrt{t\kappa} \ll r_d$$
$$\mathcal{R}^{(p)} \approx \frac{(2-\gamma)}{\gamma(d-1)tD}r^{(\gamma-d)/2}r'^{(d+\gamma-2)/2} \qquad (2.16a)$$
$$\times\exp\left(-\frac{(r^\gamma+r'^\gamma)(2-\gamma)}{\gamma^2(d-1)tD}\right)I_{d/\gamma-1}\left(\frac{2(2-\gamma)(rr')^{\gamma/2}}{\gamma^2(d-1)tD}\right), \quad \text{at} \quad r, r', (Dt)^{1/\gamma} \gg r_d\,.$$

We shall need also the asymptotics in the mixed limit $r, r_d \ll r', (Dt)^{1/\gamma}$, for r and r' lying in the different intervals (diffusion and convective ones, respectively). This asymptotic has to match with one following from (2.16a) in the subinterval $r_d \ll r \ll r', (Dt)^{1/\gamma}$. It gives us an idea: to improve the expansion of (2.16a) with respect to r/r' introducing a function $y(r)$:

$$\mathcal{R}^{(p)} \approx \left[\frac{(2-\gamma)}{\gamma^2(d-1)tD}\right]^{d/\gamma}\frac{\gamma r'^{(d-1)}}{\Gamma(d/\gamma)}\exp\left(-\frac{r'^\gamma(2-\gamma)}{\gamma^2(d-1)tD}\right)$$
$$\times\left\{1+\left[-1+\frac{2(2-\gamma)r'^\gamma}{d(d-1)\gamma t}\right]\frac{2(2-\gamma)y(r)}{(d-1)\gamma t}\right\}, \quad \text{at} \quad r, r_d \ll r', (Dt)^{1/\gamma}\,. \qquad (2.17a)$$

In the limit $r_d \ll r$, the function $y(r)$ should pass to r^γ as follows from (2.16a). Substituting (2.17a) into (2.15) and solving the resulting equation for y one gets an expression with one unknown parameter:

$$y(r) = y(0) + (d-1)D\int_0^r \frac{x\,dx}{D(d-1)x^{2-\gamma}+2\kappa(2-\gamma)}\,. \qquad (2.17b)$$

Generally, the integration of the second-order differential equation produces two parameters, but here one of them has been already fixed by the condition of the finiteness of $y(r)$ at $r \to 0$. The only scale which can determine the dimensional parameter $y(0)$ is r_d. It gives the following estimate $y(0) \sim r_d^\gamma$ and, thus, closes our analysis of the mixed asymptotics of the resolvent at $r, r_d \ll r', (Dt)^{1/\gamma}$.

C. Overall scaling of the fourth-order correlator

The analysis of the series (2.12) enables us to establish the scaling behavior of the irreducible part of the fourth-order correlator Γ in different regions of scales. The crucial point is that all terms of the series (2.12) have the same scaling behavior as $\Gamma^{(1)}$ up to some logarithmic functions that are our main concern. Indeed, we should worry that some logarithms may appear at each step of the iteration procedure (2.11). Summation of powers of logarithms can produce anomalous exponents changing the index of the whole sum in comparison with the first term of (2.12).

We shall analyze in this section only logarithms $\ln(L/r_{ij})$. Besides, the logarithms of the ratios of separations $\ln(r_{ij}/r_{kl})$ may arise at any step of the iteration procedure (2.11) which could change the behavior at small ratios. This happens only in subleading terms where the coefficient at logarithms is proportional to a power of the small ratio. One may worry, nevertheless, whether the logarithms could be summed into the large exponent that compensates the small factors. That means that the case of small ratios needs a separate nonperturbative analysis, which is done in Sec. III. The results of the analysis show that after resummation the subleading terms remain the subleading ones. Therefore, the logarithms of the ratios can be neglected at the general investigation. Later in this section, we will speak about the *overall scaling* of the zero modes, which deals with their exponents in terms of the ratio L/r.

First, we show that the zero modes of $\hat{\mathcal{L}}_0$ cannot contain logarithms $\ln(L/r_{ij})$. It can be proved using an "angular" representation in r_{ij} space introduced as

$$r_{12}^{\gamma/2} = R^{\gamma/2}n_1\,, \quad r_{13}^{\gamma/2} = R^{\gamma/2}n_2\,, \dots\,, \qquad (2.18)$$

where R is a "radial" variable and $\vec{n}$ is a unit six-component vector. In terms of those variables the operator $\hat{\mathcal{L}}_0$ at $R \gg r_d$ is written as

$$\hat{\mathcal{L}}_0 = \frac{(d-1)D}{(2-\gamma)R^\gamma}\left(R^2\partial_R^2 + (6d+1-\gamma)R\partial_R + \hat{\Upsilon}\right),$$

where $\hat{\Upsilon}$ is an angular operator:

$$\hat{\Upsilon} = \frac{\gamma(1-\gamma+d)}{2}\sum_{m=1}^{6}\left((n_m^{-1}-6n_m)\frac{\partial}{\partial n_m}\right)$$
$$+(\gamma/2)^2\sum_{m=1}^{6}n_m^{2/\gamma-1}\frac{\partial^\perp}{\partial n_m}n_m^{1-2/\gamma}\frac{\partial^\perp}{\partial n_m}\,, \qquad (2.19)$$
$$\frac{\partial^\perp}{\partial n_m} \equiv \frac{\partial}{\partial n_m} - \sum_{l=1}^{6}n_m n_l\frac{\partial}{\partial n_l}\,.$$

The consequence of (2.19) is a possibility to look for the zero modes in the following form:

$$\mathcal{Z}_n = R^{a_n}Y_n(\vec{n})\,, \qquad (2.20)$$
$$\hat{\Upsilon}Y_n(\vec{n}) = \lambda_n Y_n(\vec{n})\,, \qquad (2.21)$$
$$a_n^2 + (6d-\gamma)a_n + \lambda_n = 0\,. \qquad (2.22)$$

The roots of Eq. (2.22) determine exponents in the expression (2.20) for the zero modes which appear to be power functions [like (2.9)]. Dangerous logarithms would occur if the roots of (2.22) coincided. That corresponds to the values $a_{n,cr} = -(6d-\gamma)/2$, $\lambda_n = (6d-\gamma)^2/4$, which is impossible since all eigenvalues λ_n of the operator $\hat{\Upsilon}$ are negative. The property follows from the fact that all eigenvalues of $\hat{\mathcal{L}}_0$ are negative and $\hat{\mathcal{L}}_0$ is reduced to $\hat{\Upsilon}$ for R-independent functions. Nondegeneracy of the eigenvalues guarantees the absence of the logarithms $\ln(L/r_{ij})$ in the zero modes of $\hat{\mathcal{L}}_0$. Note that the coincidence of a_n for two different λ_n (i.e., the existence of two zero modes with the same scaling yet different angular structures) apparently does not lead to the appearance of logarithms as well.

To establish the exponent of the overall scaling, let us analyze the integral (2.13). First, we show that at any step of the iteration the integral converges at $r' \gg L$ so that this region gives a negligible contribution if we consider separations $r_{ij} \lesssim L$. To do that, one should find the behavior of $\Gamma^{(n)}$ at large scales. The decay of Φ from (2.6b) is different for a different direction in the six-dimensional space $\{r_{ij}\}$. By analyzing the most slowly decaying case we get $\Phi \sim R^{-\gamma}$, where R is determined by (2.18). Now we can analyze (2.13), and using the resolvent which is determined for all scales one can directly show that the region $R' \gg L$ gives a negligible contribution. Then the analysis of $\Gamma^{(1)}$ in the region $R \gg L$ shows that $\Gamma^{(1)}$ decays like Φ. That means that the above scheme is reproduced at any step of the iteration procedure.

Consider the case where all r_{ij} are in the convective interval. Then we can use (2.16a) for $\mathcal{R}^{(p)}$. The contribution to $\Gamma^{(1)}$ associated with $r' < r$ is L independent. One can check using (2.16a) and (1.20a) that the integral in (2.13) converges at small r' already in the convective interval. That means that the contribution associated with r' from the convective interval is also r_d independent and it is thus given by the simple dimensional estimate: P_2^2/D^2 multiplied by a scale invariant function of r_{ij} with the exponent 2γ. That contribution could be neglected in comparison with the terms proportional to positive powers of L which will be found below. Using the expressions analogous to (2.17) for small r' one can also check that the contribution to the integral due to the diffusion region is finite and negligible. The main contribution to $\Gamma^{(n)}$ at the integration over r'_{ij} in (2.13) is given by the region $r'_{ij} \sim L$, the characteristic time being estimated as $t \sim L^\gamma/D$. It follows then from (2.16a) that for $r \ll L$ the corresponding expression for $\mathcal{R}^{(p)}$ is a regular expansion in r^γ, which means that the contribution to $\Gamma^{(1)}$ associated with the considered region $r'_{ij} \sim L$ is also a regular expansion in r_{ij}^γ/L^γ. Only the first terms, proportional to positive powers of L, are important; the subsequent terms proportional to negative powers of L can be neglected in the convective interval.

The zero term in the r^γ expansion of (2.13) is a constant which can be estimated as $P_2^2 L^{2\gamma}/D^2$. The first term (that would be proportional to $L^\gamma r^\gamma$) is actually equal to zero. This can be understood as follows. As is seen from (1.20a) and (2.8) in the convective interval, the right-hand side of (2.11) does not depend on L: (1.20a) contains only an L-dependent constant. This constant had to be removed in (2.13) since the operator (2.8) contains only cross derivatives. That L independence on the right-hand side means that an L-dependent term in $\Gamma^{(1)}$ should be treated as a zero mode (up to a possible logarithmic factor) of the operator $\hat{\mathcal{L}}_0$ (in the convective interval only). Zero modes with the scaling r^γ exist (e.g., $r_{12}^\gamma - r_{34}^\gamma$) but one cannot construct such a mode symmetric in permutations of the points $1, 2, 3, 4$. Thus the presence of such zero modes would violate the intrinsic $\mathbf{r}_i \leftrightarrow \mathbf{r}_j$ symmetry of the problem.

The next term behaves $\propto r^{2\gamma}$. If $\Gamma^{(1)}$ contains logarithmic factors multiplied by $r^{2\gamma}$, then the normal scaling is renormalized into an anomalous one. Let us consider the general expression for $\Gamma^{(1)}$,

$$\Gamma^{(1)}(\vec{r}) = \int^\infty dt \int d\vec{r}\,'\mathcal{R}(t;\vec{r},\vec{r}\,')\hat{\mathcal{L}}_1 F^{(0)}(\vec{r}\,') \ , \quad (2.23)$$

expand it in the series in r_{ij}^γ, and calculate (for instance) the terms proportional to $r_{12}^\gamma r_{34}^\gamma$, $r_{12}^\gamma r_{13}^\gamma$, and $r_{12}^{2\gamma}$, respectively:

$$\Delta\Gamma^{(1)}_{12,34} = r_{12}^\gamma r_{34}^\gamma \int \frac{dt}{t}\mathcal{A}_1(tD/L^\gamma), \quad \Delta\Gamma^{(1)}_{12,13} = r_{12}^\gamma r_{13}^\gamma \int \frac{dt}{t}\mathcal{A}_2(tD/L^\gamma), \quad \Delta\Gamma^{(1)}_{12,12} = r_{12}^{2\gamma} \int \frac{dt}{t}\mathcal{A}_3(tD/L^\gamma), \quad (2.24)$$

$$\mathcal{A}_1(t/L^\gamma) = \int^L dr'_{12}\tilde{\mathcal{R}}^{(p)}(t;0,r'_{12}) \int^L dr'_{34}\tilde{\mathcal{R}}^{(p)}(t;0,r'_{34}) \int^L dr'_{13}\mathcal{R}^{(p)}(t;0,r'_{13}) \int^L dr'_{14}\mathcal{R}^{(p)}(t;0,r'_{14})$$
$$\times \int^L dr'_{23}\mathcal{R}^{(p)}(t;0,r'_{23}) \int^L dr'_{24}\mathcal{R}^{(p)}(t;0,r'_{24})\hat{\mathcal{L}}_1(\vec{r}\,')F^{(0)}(\vec{r}\,'), \quad (2.25)$$

$$\mathcal{A}_2(t/L^\gamma) = \int^L dr'_{12}\tilde{\mathcal{R}}^{(p)}(t;0,r'_{12}) \int^L dr'_{13}\tilde{\mathcal{R}}^{(p)}(t;0,r'_{13}) \int^L dr'_{34}\mathcal{R}^{(p)}(t;0,r'_{34}) \int^L dr'_{14}\mathcal{R}^{(p)}(t;0,r'_{14})$$
$$\times \int^L dr'_{23}\mathcal{R}^{(p)}(t;0,r'_{23}) \int^L dr'_{24}\mathcal{R}^{(p)}(t;0,r'_{24})\hat{\mathcal{L}}_1(\vec{r}\,')F^{(0)}(\vec{r}\,'), \quad (2.26)$$

$$\mathcal{A}_3(t/L^\gamma) = \int^L dr'_{12}\tilde{\tilde{\mathcal{R}}}^{(p)}(t;0,r'_{12}) \int^L dr'_{34}\mathcal{R}^{(p)}(t;0,r'_{34}) \int^L dr'_{13}\mathcal{R}^{(p)}(t;0,r'_{13}) \int^L dr'_{14}\mathcal{R}^{(p)}(t;0,r'_{14})$$
$$\times \int^L dr'_{23}\mathcal{R}^{(p)}(t;0,r'_{23}) \int^L dr'_{24}\mathcal{R}^{(p)}(t;0,r'_{24})\hat{\mathcal{L}}_1(\vec{r}\,')F^{(0)}(\vec{r}\,'), \quad (2.27)$$

$$\mathcal{R}^{(p)}(t;0,x) = \left[\frac{2-\gamma}{\gamma^2(d-1)tD}\right]^{d/\gamma} \frac{\gamma}{\Gamma[d/\gamma]} \exp\left[-x^\gamma \frac{2-\gamma}{\gamma^2(d-1)tD}\right] x^{d-1}, \quad (2.28)$$

$$\tilde{\mathcal{R}}^{(p)}(t;0,x) = \mathcal{R}^{(p)}(t;0,x)\frac{(2-\gamma)}{(d-1)\gamma^2}\left[-1+\frac{(2-\gamma)x^\gamma}{d(d-1)\gamma tD}\right], \quad (2.29)$$

$$\tilde{\tilde{\mathcal{R}}}^{(p)}(t;0,x) = \mathcal{R}^{(p)}(t;0,x)\frac{(2-\gamma)^2}{2(d-1)^2\gamma^4}\left[1-\frac{2(2-\gamma)x^\gamma}{d(d-1)\gamma tD}+\frac{(2-\gamma)^2x^{2\gamma}}{d(d-1)^2(d+\gamma)\gamma^2t^2D^2}\right]. \quad (2.30)$$

While integrating over time at $tD \ll L^\gamma$ one can expand the functions $\mathcal{A}_i(tD/L^\gamma)$ in a regular series in tD/L^γ:

$$\mathcal{A}_i(tD/L^\gamma) = C_0^{(i)} + C_1^{(i)}tD/L^\gamma + \cdots. \quad (2.31)$$

The crucial point is the presence of the zero term in the expansion. Were $C_0^{(i)}$ equal to zero, we would immediately figure out the part of the well known zero mode from the divergent integral over t ($\sim C_1^{(i)}$) in (2.24), the respective term in (2.24) being L independent. As we show below, $C_0^{(i)} \neq 0$ so that one gets the logarithm $\ln[L/r]$ in the respective ($\sim C_0^{(i)}$) contribution to $\Delta\Gamma_i^{(1)}$. This term stems from the forced solution of the equation $\hat{\mathcal{L}}_0\Gamma^{(1)} = -\hat{\mathcal{L}}_1F^{(0)}$ (it is proved already that there are no logarithms in zero modes of $\hat{\mathcal{L}}_0$). From the viewpoint of the initial equation (2.5), those logarithms stem from the zero modes of the full operator $\mathcal{L}$ — see (2.10) and below.

Let us calculate $C_0^{(i)}$ directly. First of all one finds from (2.8)

$$-\hat{\mathcal{L}}_1F^{(0)} = -\left[\frac{2-\gamma}{\gamma(d-1)d}\right]^2 \hat{\mathcal{L}}_1\left[r_{12}^\gamma r_{34}^\gamma + r_{13}^\gamma r_{24}^\gamma + r_{14}^\gamma r_{23}^\gamma\right]$$
$$= \frac{(2-\gamma)^2}{8(d-1)^2d^2}\sum\left(2\frac{d+1-\gamma}{2-\gamma}r_{ij}^2(r_{in}^2+r_{jm}^2-r_{ij}^2-r_{mn}^2)\right.$$
$$\left.+(r_{ij}^2+r_{im}^2-r_{jm}^2)(r_{ij}^2+r_{jn}^2-r_{in}^2)\right)r_{ij}^{-\gamma}r_{im}^{\gamma-2}r_{jn}^{\gamma-2}, \quad (2.32)$$

where the summation is performed over all the sets of different subscripts $\{i,j,m,n\} = \{1,2,3,4\}$. $F^{(0)}(\vec{r})$ and, respectively, the integrands in (2.28)–(2.30) are sums of a huge number of very simple terms that have been calculated by MATHEMATICA. We do not present here the cumbersome expressions obtained formally at arbitrary γ and d since the physical meaning could be ascribed to them only when the $\hat{\mathcal{L}}_1/\hat{\mathcal{L}}_0$ expansion is a small-parameter expansion. The formulas below show that there are two cases where this is so: $C_0^{(i)}$ go to zero at $\gamma \to 2$ and at $d \to \infty$:

$$C_0^{(1)} \to \frac{4(2-\gamma)^3}{\gamma^3d^5}, \quad \text{at } d\to\infty \quad (2.33)$$

$$C_0^{(2)} \to -\frac{2(2-\gamma)^3}{\gamma^3d^5}, \quad \text{at } d\to\infty \quad (2.34)$$

$$C_0^{(3)} \to \frac{2(2-\gamma)^3}{\gamma^3d^5}, \quad \text{at } d\to\infty \quad (2.35)$$

$$C_0^{(1)} \to -\frac{(5d-2-d^2)(2-\gamma)^3}{2(d-2)(d-1)^3d^3}, \quad \text{at } \gamma\to 2^- \quad (2.36)$$

$$C_0^{(2)} \to -\frac{(2+d^2-4d)(2-\gamma)^3}{4(d-2)(d-1)^3d^3}, \quad \text{at } \gamma\to 2^- \quad (2.37)$$

$$C_0^{(3)} \to +\frac{(2-\gamma)^3}{4d^2(d-1)(d-2)}, \quad \text{at } \gamma\to 2^- \ . \quad (2.38)$$

It is natural that our $\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1$ expansion gives a small correction in the limit $d\to\infty$ while we should admit that the same behavior in the limit $\gamma\to 2$ is surprising for us. That means that the $\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1$ expansion might be meaningful also when d is arbitrary while $2-\gamma$ is small. To establish that, one should check whether nonlogarithmic terms in $\Gamma^{(1)}$ are proportional to $(2-\gamma)^3$ as well, which is beyond the scope of the present approach. A nontrivial task to develop a consistent $(2-\gamma)$ expansion will be the subject of further publications. As the $\gamma\to 2$ velocity spectrum diverges at small scales, one should introduce an ultraviolet cutoff r_η in (1.9) so that $r^{2-\gamma}/(2-\gamma) \to \ln(r/r_\eta)$. The scalar correlation functions contain logarithms already at $\gamma = 2$ (in the paper [10], the limit $\gamma\to 2, D/(2-\gamma)\to$const has been considered where such logarithms are absent).

On the contrary, the perturbation theory in $1/d$ which is the main subject of this section is regular and uniform with respect to distances r_{ij}. Note that we assume $1/d$ to be the smallest parameter in the problem, in particular, $1/d \ll \gamma$, so that our results do not describe the limit of small γ which will be the subject of further publications.

We have found all the logarithmic terms entering the fourth-order correlator on the first step of the iteration procedure:

$$F_{12,34} = F^{(0)}_{12,34} + \gamma\Big\{C_0^{(1)}\sum r_{ij}^{\gamma}r_{kl}^{\gamma}\ln[L/r] + C_0^{(2)}\sum r_{ij}^{\gamma}r_{ik}^{\gamma}\ln[L/r] + C_0^{(3)}\sum r_{ij}^{2\gamma}\ln[L/r]\Big\} + L\text{-independent or nonlogarithmic terms} + \cdots, \quad (2.39)$$

where the summations are performed over all the possible combinations supposing all the indexes i, j, k, l to be different. The expression in front of the logarithms in (2.39) is a sum of the well-known zero modes ($\sim r^{2\gamma}$), which is guaranteed by the identity that could be directly checked,

$$C_0^{(3)} + \frac{d}{2(d+\gamma)}\left(C_0^{(1)} + 4C_0^{(2)}\right) = 0. \quad (2.40)$$

It agrees with the absence of any logarithms among zero modes of $\hat{\mathcal{L}}_0$.

We have thus found the first (in $1/d$) term of the expansion of the zero-mode part of the solution of the full operator,

$$\mathcal{Z} = \frac{(2-\gamma)^2}{\gamma^2 d^4}\sum Z_{ij,kl} + \gamma\Big(C_0^{(1)}\sum \mathcal{Z}_{ij,kl}\ln[L/r] + C_0^{(2)}\sum \mathcal{Z}_{ij,ik}\ln[L/r]\Big) + \cdots. \quad (2.41)$$

Note that there are two types ($Z_{ij,kl}$ and $Z_{ij,ik}$) of zero modes of $\hat{\mathcal{L}}_0$ possessing scaling 2γ; only one of them ($Z_{ij,kl}$) is present on the zero step of the iteration procedure. It is clear that (2.41) cannot be described in terms of a single anomalous exponent $\Delta(d)$ because the expression in the last set of brackets does not coincide with $\sum Z_{ij,kl}$. We thus come to the conclusion that $\mathcal{Z}$ should consist of at least two terms with different scaling. In other words it means that one gets a degenerate case. Indeed, the full operator $\hat{\mathcal{L}}$, the bare operator $\hat{\mathcal{L}}_0$, and the perturbative one $\hat{\mathcal{L}}_1$ are scale invariant. The bare zero modes scale similarly, and it is very essential that the first perturbation step should dismiss the degeneracy. On the first step of the iteration procedure there should appear two zero modes,

$$\mathcal{Z}_1 = \sum\Big[\left(\alpha Z_{ij,kl} + \beta Z_{ij,ik}\right)\left(1 - \Delta_1\ln[r/L]\right)\Big], \quad (2.42)$$

$$\mathcal{Z}_2 = \sum\Big[\left[(1-\alpha)Z_{ij,kl} - \beta Z_{ij,ik}\right]\left(1 - \Delta_2\ln[r/L]\right)\Big], \quad (2.43)$$

such that

$$\mathcal{Z} = \frac{(2-\gamma)^2}{\gamma^2 d^4}(\mathcal{Z}_1 + \mathcal{Z}_2), \quad (2.44)$$

where $\mathcal{Z}$ was found above in (2.41).

To extract additional information that will fix all the numbers α, β, Δ_i, we should calculate the first-order logarithmic correction $-\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1\sum \mathcal{Z}_{in,jn}$ to the second bare zero mode $\sum \mathcal{Z}_{in,jn}$. First of all one finds

$$-\hat{\mathcal{L}}_1\sum \mathcal{Z}_{in,jn} = \frac{(d-1)\gamma^2}{2(2-\gamma)}\sum\left(r_{ij}^2 - r_{in}^2 - r_{jn}^2\right)r_{jn}^{\gamma-2} + \frac{\gamma^2}{4}\sum\frac{r_{in}^{\gamma-2}r_{jn}^{\gamma-2}}{r_{ij}^{\gamma}}\Big[\frac{d+1-\gamma}{2-\gamma}r_{ij}^2\left(r_{in}^2 + r_{jn}^2 - r_{ij}^2\right) + \frac{1}{2}\left(r_{ij}^2 + r_{in}^2 - r_{jn}^2\right)\left(r_{ij}^2 + r_{jn}^2 - r_{in}^2\right)\Big]. \quad (2.45)$$

Performing calculations analogous to what was done at the calculations of the coefficients C_0^i one finds that all the logarithmic terms that appeared are proportional to the bare zero mode $\sum \mathcal{Z}_{in,jn}$ only. It is a manifestation of the fact that the functions on a triangle constitute an invariant subspace of the full operator $\hat{\mathcal{L}}$: $\hat{\mathcal{L}}$ acting on an arbitrary function of three distances from a triangle r_{in}, r_{ij}, r_{jn} produces a function of the same three distances again. Thus the cancellation of a coefficient before the logarithm proportional to another four-point zero mode will occur on all the higher steps of the iteration of $\sum \mathcal{Z}_{in,jn}$ too. Thus, the resulting zero mode (say $\mathcal{Z}_1$ with $\alpha = 0$) is scale invariant with the following asymptotics of the anomalous exponent at $d \to \infty$:

$$\Delta_1 \to -\frac{(2-\gamma)(2+\gamma)}{2d}. \quad (2.46)$$

The asymptotics of all the remaining coefficients defining the second (mixed) scale-invariant zero mode $\mathcal{Z}_2$ are restored from (2.41)–(2.44) and (2.46),

$$\Delta_2 \to \frac{4(2-\gamma)}{d} \quad \beta \to \frac{4}{10+\gamma}. \quad (2.47)$$

Note that the above $\hat{\mathcal{L}}_1^{-1}\hat{\mathcal{L}}_0$ iteration procedure was not, strictly speaking, a direct $1/d$ expansion: not only major terms $\sim d^2$ but also subleading ones $\sim d$ have been included into $\hat{\mathcal{L}}_0$. That made it possible to have both unperturbed operator and perturbation of the same (second) order. To reinforce the results and obtain the explicit solution for Γ_1 (not only its logarithmic part as above), let us briefly describe the direct $1/d$ procedure where the main part of $\hat{\mathcal{L}}$ in the convective interval is the differential operator of the first order,

$$\hat{\mathcal{L}}_0' = d^2\frac{D}{2-\gamma}\sum_{i>j} r_{ij}^{1-\gamma}\partial_{r_{ij}}, \quad (2.48)$$

yet we will directly account for the necessary boundary conditions while integrating over characteristics below. The zero modes of (2.48) which we are going to iterate are

$d\to\infty$ limits of $\sum \mathcal{Z}_{ij,kl}$ and $\sum \mathcal{Z}_{ij,ik}$: $\mathcal{Z}_0'=\sum[2r_{ij}^{\gamma}r_{kl}^{\gamma}-(r_{ij}^{2\gamma}+r_{kl}^{2\gamma})]$ and $\tilde{\mathcal{Z}}_0'=\sum[2r_{ij}^{\gamma}r_{ik}^{\gamma}-(r_{ij}^{2\gamma}+r_{ik}^{2\gamma})]$. To find the first-order corrections to the bare zero modes one has to solve the differential equation

$$\hat{\mathcal{L}}_0'\mathcal{Z}_1'=-\hat{\mathcal{L}}_1'\mathcal{Z}_0',\quad \hat{\mathcal{L}}_0'\tilde{\mathcal{Z}}_1'=-\hat{\mathcal{L}}_1'\tilde{\mathcal{Z}}_0', \tag{2.49}$$

$$\hat{\mathcal{L}}_1'=d\frac{D}{2-\gamma}\left\{\sum_{i>j}(r_{ij}^{2-\gamma}\partial^2_{r_{ij}}-\gamma r_{ij}^{1-\gamma}\partial_{r_{ij}})\right.$$
$$-\frac{1}{2}\sum(r_{in}^2-r_{ij}^2-r_{jn}^2)\frac{r_{ij}^{1-\gamma}}{r_{jn}}\frac{\partial^2}{\partial r_{ij}\partial r_{jn}}$$
$$-\frac{1}{4}\sum\frac{r_{ij}^{2-\gamma}}{r_{im}r_{jn}}(r_{in}^2+r_{jm}^2-r_{ij}^2-r_{mn}^2)$$
$$\left.\times\frac{\partial^2}{\partial r_{im}\partial r_{jn}}\right\}, \tag{2.50}$$

where $\hat{\mathcal{L}}_1'$ is the part of the full operator (in the convective interval) proportional to d; note that it stems both from $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_1$. Equations (2.49) are integrated by characteristics, for example,

$$\mathcal{Z}_1'=-\int dt\hat{\mathcal{L}}_1'\mathcal{Z}_0'\left[r_{ij}^{\gamma}\to\frac{d^2\gamma D}{2-\gamma}t+B_{ij}\right], \tag{2.51}$$

where B_{ij} should be considered as constants, and $\mathcal{Z}_1'$ itself is defined up to zero modes of the bare operator $\hat{\mathcal{L}}_0'$. The result of integration (2.51) at an arbitrary γ is bulky. We present here the explicit expression in the simplest possible case of $\gamma=1$,

$$\mathcal{Z}_1'(\gamma=1)=\frac{4}{d}\sum_{i>j}r_{ij}^2-\frac{1}{2d}\sum\left(\frac{(r_{im}-r_{ij})[(r_{im}-r_{in})^2+(r_{im}-r_{jm})^2-(r_{im}-r_{ij})^2-(r_{im}-r_{mn})^2]}{r_{im}-r_{jn}}\ln[L/r_{im}]\right.$$
$$\left.+\frac{(r_{jn}-r_{ij})[-(r_{jn}-r_{in})^2-(r_{jn}-r_{jm})^2+(r_{jn}-r_{ij})^2+(r_{jn}-r_{mn})^2]}{r_{im}-r_{jn}}\ln[L/r_{jn}]\right),$$
$$\tilde{\mathcal{Z}}_1'(\gamma=1)=-\frac{8}{d}\sum_{i>j}r_{ij}^2-\frac{1}{d}\sum\frac{(r_{ij}-r_{ik})(r_{ij}-r_{kj})}{r_{ik}-r_{kj}}\Big((r_{ij}+r_{kj}-2r_{ik})\ln[L/r_{ik}]-(r_{ij}+r_{ik}-2r_{kj})\ln[L/r_{kj}]\Big).$$

One can directly check that those are the solutions of (2.6) at the first order in $1/d$. Being interested only in finding the overall scaling we can just set all the separations under the logarithm to be the same. Generally, one can extract the logarithmic parts of $\mathcal{Z}_1',\tilde{\mathcal{Z}}_1'$ directly from the right-hand side of (2.51) setting formally the upper limit of the integral to be very large and looking for the logarithmically divergent terms

$$\mathcal{Z}_1'=\frac{2(2-\gamma)}{d}(2\mathcal{Z}_0'-\tilde{\mathcal{Z}}_0')\ln\left(\frac{L}{r}\right)+(\text{nonlog. terms}), \tag{2.52}$$

$$\tilde{\mathcal{Z}}_1'=\frac{\gamma^2-4}{d}\tilde{\mathcal{Z}}_0'\ln\left(\frac{L}{r}\right)+(\text{nonlog. terms}). \tag{2.53}$$

This is the direct analog of (2.41). The formulas (2.52) and (2.53) immediately give the leading $d\to\infty$ asymptotics (2.46) and (2.47) obtained above from the general iteration procedure.

The structure of the resolvent (2.16a) at small r determined by (2.17) shows that for $\Gamma^{(1)}$ there exists the limit $\kappa\to 0$. This property being reproduced at any step of the iteration procedure enables one to construct a solution $\Gamma^{(co)}$ which is the limit of Γ at $\kappa\to 0$. The function $\Gamma^{(co)}$ is close to Γ in the convective interval and remains finite at any $r_{ij}\to 0$. The behavior of $\Gamma^{(co)}$ is determined by the estimation (2.54), where r is the maximal value among r_{ij}.

We thus come to the conclusion that for separations r from the convective interval $r_d\ll r\ll L$,

$$\Gamma(r)-\Gamma(0)\sim P_2^2r^{2\gamma}(L/r)^{\Delta}/D^2,\quad \Gamma(0)\sim P_2^2L^{2\gamma}/D^2 \tag{2.54}$$

up to dimensionless constants depending on d and γ and the anomalous dimension Δ is given by the largest exponent (Δ_2), which is asymptotically

$$\Delta=\Delta_2=4\frac{2-\gamma}{d}+O(1/d^2)\ . \tag{2.55}$$

Finally, we treat the case where all separations are in the diffusion interval. The contribution to $\Gamma^{(1)}$ associated with $r'\gg r_d$ gives the constant $\sim L^{2\gamma}(L/r_d)^{\Delta}P_2^2/D^2$. The contribution associated with scales $r'\lesssim r_d$ has to be analyzed carefully since for all separations from the diffusion interval the typical integration time t in (2.13) is also characteristic of the diffusion region. We use (2.16a) for $\mathcal{R}^{(p)}$ and (1.20a) for f. The estimate for the r-independent contribution gives $\sim r_d^{2\gamma}P_2^2/D^2$. To analyze an r-dependent contribution associated with $r'\lesssim r_d$ it is convenient to return to the initial form (2.6) of the equation for Γ. At the analysis we will not divide $\hat{\mathcal{L}}$ into $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_1$; the consequent conclusions are consequently nonperturbative. In the diffusion region, $\hat{\mathcal{L}}$ is reduced to the sum of Laplacians. The right-hand side of (2.6b) is proportional to $r^{4-\gamma}$, which generates the forced part of the solution proportional to $r^{6-\gamma}$. To find the principal r-dependent term one has to compare the forced solution with zero modes of the sum of Laplacians. All the modes are well known: they are constructed from even powers of r_{ij}. There are no modes with second powers of r_{ij} in Γ

since they are not invariant under permutations of points $\mathbf{r}_i$. The zero mode of $\hat{\mathcal{L}}$ of the fourth power possessing the permutation symmetry,

$$\sum_{\{ij\}\neq\{nm\}}\left(r_{ij}^2 r_{nm}^2-\frac{d}{2(d+2)}(r_{ij}^4+r_{nm}^4)\right), \qquad (2.56)$$

turns out to produce a larger contribution than the forced term. Thus, we conclude that the principal term in Γ is of the fourth power in r. The parametric dependence of the coefficient at the power can be easily established from a matching at $r\sim r_d$:

$$\Gamma-(C_1L^{2\gamma}+C_1' r_d^{2\gamma})P_2^2/D^2$$

$$\sim r^4 r_d^{2\gamma-4}(L/r_d)^{\Delta}P_2^2/D^2, \quad \text{at} \quad r\ll r_d. \qquad (2.57)$$

The consideration of this section shows that the degree of non-Gaussianity increases as the distances decrease in the convective interval and eventually comes to a constant when the distances are in the diffusion interval. For example, the flatness factor is proportional to $(L/r)^{\Delta}$ in the convective interval $L\gg r\gg r_d$ and it is an r-independent constant of the order of $(L/r_d)^{\Delta}$ in the diffusive interval $r_d\gg r$.

III. FOUR-POINT OBJECTS WITH STRONGLY DIFFERENT SEPARATIONS

In Sec. II we established the overall scaling behavior of the four-point irreducible correlation function Γ_{1234}. In particular, one may check that at any step of the iteration procedure (2.11) the contributions to Γ_{1234} associated with possible small values of the separation ratios are negligible in comparison with the leading term. To show that the summation over n could not convert subleading terms into leading ones, here we check the self-consistency of our approach nonperturbatively in $1/d$ using the full operator $\hat{\mathcal{L}}$. In the next subsection, we show that if any separation ρ is much less than the other ones, the ρ-dependent contribution to Γ_{1234} is much less than the leading term. This property enables one to find such two-point objects as $\langle(\theta_1-\theta_2)^4\rangle$, which can be extracted from Γ_{1234} by fusing some points. Besides, to establish the properties of the correlation functions $\langle\langle\epsilon_1\theta_2^2\rangle\rangle$ and $\langle\langle\epsilon_1\epsilon_2\rangle\rangle$ one should know not only the limit of Γ_{1234} for coinciding points but also the dependence of Γ_{1234} on distances between nearby points in groups strongly separated from each other. It is the aim of this section to obtain the dependence. Here, we shall use perturbation theory with respect to the ratio between small and large distances. Calculating the corresponding contributions to Γ_{1234} one can obtain the two-point correlation functions of products of different spatial derivatives. We shall demonstrate how an absence of an r_d-related anomalous scaling in Γ_{1234} prescribes it for $\langle\langle\epsilon_1\theta_2^2\rangle\rangle$ and $\langle\langle\epsilon_1\epsilon_2\rangle\rangle$.

A. Four-point correlator with a small separation

Suppose that among separations r_{ij} there is a separation ρ, say r_{34}, which is much smaller than other separations. Then the main term in the operator $\hat{\mathcal{L}}$ (2.4) is associated with derivatives over the vector $\boldsymbol{\rho}=\mathbf{r}_3-\mathbf{r}_4$. This term can be written as

$$\hat{\mathcal{L}}_{\boldsymbol{\rho}}\equiv\mathcal{K}^{\alpha\beta}(\boldsymbol{\rho})\nabla_\rho^\alpha\nabla_\rho^\beta+2\kappa\triangle_\rho\,, \qquad (3.1)$$

where $\nabla_\rho^\alpha\equiv\partial/\partial\rho_\alpha$. To solve Eq. (2.6a) for Γ_{1234} in this case, we can formulate an iteration procedure where $\hat{\mathcal{L}}_{\boldsymbol{\rho}}$ is treated as the main contribution to $\hat{\mathcal{L}}$. Namely, a solution of (2.6a) can be represented as

$$\Gamma_{1234}=\Gamma_0(\mathbf{r}_{13},\mathbf{r}_{23})+\sum_{n=1}^{\infty}\Gamma_n(\mathbf{r}_{13},\mathbf{r}_{23},\boldsymbol{\rho})\,, \qquad (3.2a)$$

$$-\hat{\mathcal{L}}_{\boldsymbol{\rho}}\Gamma_n=\Phi_n(\mathbf{r}_{13},\mathbf{r}_{23},\boldsymbol{\rho})\,, \qquad (3.2b)$$

$$\Phi_1=\hat{\mathcal{L}}\Gamma_0(\mathbf{r}_1,\mathbf{r}_2)+\Phi_{12;34}+\Phi_{13;24}+\Phi_{14;23}, \qquad (3.2c)$$

$$\Phi_n=(\hat{\mathcal{L}}-\hat{\mathcal{L}}_{\boldsymbol{\rho}})\Gamma_{(n-1)}, \quad n>1\,, \qquad (3.2d)$$

where $\mathbf{r}_{13}$ and $\mathbf{r}_{23}$ are separations much larger than $\boldsymbol{\rho}$ and Γ_0 is a ρ-independent part of Γ. The procedure introduced by (3.2) can be considered as a series over the small parameters ρ/r_{13}, ρ/r_{23}. We impose two boundary conditions on Γ_n: to be of order of Γ_0 at $\rho\sim r_{13},r_{23}$ and to remain finite at $\rho=0$.

The right-hand side of (3.2b) can be assumed to be known. A solution of (3.2b) satisfying the imposed boundary conditions has the following form:

$$\Gamma_n(\mathbf{r}_1,\mathbf{r}_2,\boldsymbol{\rho})=\int_0^\infty dt\int d\mathbf{r}\mathcal{R}(t;\boldsymbol{\rho},\mathbf{r})\Phi_n(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r})\,, \qquad (3.3)$$

where $\mathcal{R}$ is the kernel of the resolvent of the operator $\hat{\mathcal{L}}_{\boldsymbol{\rho}}$:

$$(\partial_t-\hat{\mathcal{L}}_{\boldsymbol{\rho}})\mathcal{R}(t;\boldsymbol{\rho},\mathbf{r})=\delta(t)\delta(\mathbf{r}-\boldsymbol{\rho})\,. \qquad (3.4)$$

Since the operator $\hat{\mathcal{L}}_{\boldsymbol{\rho}}$ explicitly depends on $\boldsymbol{\rho}$ the resolvent $\mathcal{R}$ is not a function of the difference $|\,\mathbf{r}-\boldsymbol{\rho}\,|$ only as it would be in a homogeneous case. The symmetry enables treating $\mathcal{R}$ as a function of ρ,r and $x=\boldsymbol{\rho}\mathbf{r}/\rho r$. It is convenient to expand the function in the series over Jacobi polynomials [23] which are the eigenfunctions of the x-dependent part of $\hat{\mathcal{L}}_{\boldsymbol{\rho}}$:

$$\mathcal{R}(\rho,r,x)=\sum_{n=0}^{\infty}\mathcal{R}^{(2n)}(\rho,r)P_{2n}^{(\nu,\nu)}(x)\,, \qquad (3.5)$$

where $\nu=(d-3)/2$ and only the polynomials with even numbers participate since $\mathcal{R}$ is an even function of x (at $d=3$, Jacobi polynomials turn into Legendre ones). One can obtain from (3.4) separate equations for $\mathcal{R}^{(2n)}$:

$$\left[\partial_t - \hat{\mathcal{L}}_\rho^{(p)} + \left(\frac{d+1-\gamma}{2-\gamma}D\rho^{2-\gamma} + 2\kappa\right)\frac{2n(2n+d-2)}{\rho^2}\right]\mathcal{R}^{(2n)}(\rho, r)$$

$$= \delta(\rho - r)\rho^{-(d-1)}\frac{(d-2+4n)\Gamma(d-2+2n)}{2^{d-1}\pi^{(d-1)/2}\Gamma[(d-1)/2+2n]}, \quad (3.6)$$

where $\hat{\mathcal{L}}_\rho^{(p)}$ was introduced in (1.14b).

Solving (3.6) one gets the behavior of $\mathcal{R}^{(2n)}$ in the diffusive and convective intervals explicitly:

$$\mathcal{R}^{(2n)} = \frac{(d-2+4n)\Gamma(d-2+2n)}{2^{d+1}\pi^{(d-1)/2}\Gamma(2n-1+d/2)} \times \exp\left(-\frac{r^2+\rho^2}{8t\kappa}\right)I_{d/2-1+2n}\left(\frac{r\rho}{4t\kappa}\right)(r\rho)^{1-d/2}, \quad r, \rho, \sqrt{t\kappa} \ll r_d \quad (3.7a)$$

$$\mathcal{R}^{(2n)} = \frac{(2-\gamma)(d-2+4n)\Gamma(d-2+2n)}{\gamma(d-1)2^{d-1}\pi^{(d-1)/2}\Gamma(2n-1+d/2)} \times \exp\left(-\frac{(r^\gamma+\rho^\gamma)(2-\gamma)}{\gamma^2(d-1)tD}\right)I_{\eta_n}\left(\frac{2(2-\gamma)(r\rho)^{\gamma/2}}{\gamma^2(d-1)tD}\right), \quad r, \rho, (Dt)^{1/\gamma} \gg r_d \quad (3.7b)$$

where

$$\eta_n = \frac{1}{\gamma}\sqrt{(d-\gamma)^2 + \frac{8(d+1-\gamma)n(2n+d-2)}{d-1}}, \quad (3.8)$$

and I_η designate modified Bessel functions. Note that $\mathcal{R}^{(0)} = \mathcal{R}^{(p)}\Gamma(d/2)r^{1-d}/(2\pi^{d/2})$, where $\mathcal{R}^{(p)}$ was introduced by (2.15) and (2.16). It is possible to formulate for $\mathcal{R}^{(2n)}$ interpolation formulas of the (2.17) type.

To establish the behavior of Γ_n determined by (3.2) we should analyze the integral (3.3) using (3.7). The main part of the integral is determined by the region $r \sim r_{13}, r_{23}$. Thus the ρ dependence is associated with the limit of (3.7) at $\rho \ll r$. We see that the resolvents in this case tend to zero with decreasing ρ. We can also assert based on (3.8) that the higher the number n of the angular harmonic is, the faster is the decay of the corresponding resolvent. Since the procedure (3.2) produces a convergent series over a small parameter, the above statements can be extended to the sum of the series. Thus we proved that the ρ-dependent part of Γ is parametrically smaller than its ρ-independent part at $\rho \ll r_1, r_2$, which was the purpose of this subsection.

B. Four-point correlator with two separated pairs

In the rest of this section, we consider Γ_{1234} in the case of the special geometry with two separations between points being much smaller than all the other ones, namely $R \gg \rho_{1,2}$, where

$$\mathbf{r}_1 - \mathbf{r}_2 = \boldsymbol{\rho}_1, \qquad \mathbf{r}_3 - \mathbf{r}_4 = \boldsymbol{\rho}_2, \qquad \mathbf{R} = (\mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_1 - \mathbf{r}_2)/2. \quad (3.9)$$

Here, the mutual orientations of the vectors and the ratio ρ_1/ρ_2 are supposed to be arbitrary. We denote by $\Gamma_0(R)$ the main contribution to Γ_{1234}, which is equal to Γ_{1234} at $\rho_1 = \rho_2 = 0$. The estimate for Γ_0 can be extracted from the results that have been obtained in Sec. II and justified in the preceding subsection. Indeed, the statements (2.54) and (2.57) are valid for r being the largest length R of the tetrahedron. One thus gets for Γ_0

$$\Gamma_0 \approx \begin{cases} (C_1L^{2\gamma} - C_2R^{2\gamma}(L/R)^{\Delta})P_2^2/D^2, & r_d \ll R \ll L \\ (C_1L^{2\gamma} - C_3R^4(L/R)^{\Delta}r_d^{2\gamma-4})P_2^2/D^2, & R \ll r_d \end{cases} \quad (3.10a)$$

where C_1, C_2, C_3 are dimensionless constants depending on d and γ.

Generally, we do not assume here and below in this section that Δ is small (or d is large), yet the condition $\Delta < 2\gamma$ is assumed. At the limit $d \to \infty$, the numerical coefficients C_1, C_2, C_3 could be found from the results of Sec. II. In particular, we can get the following asymptotic expressions:

$$\langle(\theta_1 - \theta_2)^4\rangle \approx \frac{24(2-\gamma)^2}{\gamma^2 d^4}r^{2\gamma}\left[(1-2\beta)\left(\frac{L}{r_{12}}\right)^{\Delta_2} + 2\beta\left(\frac{L}{r_{12}}\right)^{\Delta_1}\right] + \cdots, \quad (3.11)$$

at $r_{12} \ll r_{13} \approx r_{23}$,

$$\langle(\theta_1 - \theta_2)^2\theta_3^2\rangle - \langle(\theta_1 - \theta_2)^2\rangle\langle\theta_3^2\rangle \approx \frac{4(2-\gamma)^2\beta r_{12}^\gamma r_{13}^\gamma}{\gamma^2 d^4}\left[-\left(\frac{L}{r_{13}}\right)^{\Delta_2} + \left(\frac{L}{r_{13}}\right)^{\Delta_1}\right] + \cdots, \quad (3.12)$$

at $r_{12}, r_{34} \ll r_{13} \approx r_{23} \approx r_{24} \approx r_{14}$,

$$\approx \langle(\theta_1 - \theta_2)^2(\theta_3 - \theta_4)^2\rangle - \langle(\theta_1 - \theta_2)^2\rangle\langle(\theta_3 - \theta_4)^2\rangle \approx \frac{4(2-\gamma)^2}{\gamma^2(d-1)^2d^2}r_{12}^\gamma r_{34}^\gamma\left(\frac{L}{r_{13}}\right)^{\Delta_2} + \cdots, \quad (3.13)$$

where dots designate the terms with a normal scaling $(r^{2\gamma})$. Note that if one expands (3.11) in small Δ's, then the remarkable cancellation of the terms linear in logarithms happens: $(1-2\beta)\Delta_2 + 2\beta\Delta_1 = 0$.

Now, we aim at finding leading corrections to $\Gamma_0(R)$ at small values of ρ_1/R and ρ_2/R. For this we should solve (2.6a) at the conditions $\rho_1, \rho_2 \ll R$. In this limit the principal part of $\hat{\mathcal{L}}$ is $\hat{\mathcal{L}}_{\rho_1} + \hat{\mathcal{L}}_{\rho_2}$, where $\hat{\mathcal{L}}_\rho$ is determined by (3.1). It is the reason which caused us to formulate an iteration procedure, where $\hat{\mathcal{L}}_{\rho_1} + \hat{\mathcal{L}}_{\rho_2}$ is treated as a main contribution to $\hat{\mathcal{L}}$. Namely, a solution of Eq. (2.6a) can be represented as

$$\Gamma_{1234} = \Gamma_0(R) + \sum_{n=1}^{\infty} \Gamma_n(\mathbf{R}; \boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\,, \tag{3.14a}$$

$$-(\hat{\mathcal{L}}_{\rho_1} + \hat{\mathcal{L}}_{\rho_2})\Gamma_n = \Phi_n(\mathbf{R}; \boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\,, \tag{3.14b}$$

$$\Phi_1 = \hat{\mathcal{L}}\Gamma_0 + \Phi_{12;34} + \Phi_{13;24} + \Phi_{14;23}, \tag{3.14c}$$

$$\Phi_n = (\hat{\mathcal{L}} - \hat{\mathcal{L}}_{\rho_1} - \hat{\mathcal{L}}_{\rho_2})\Gamma_{(n-1)}, \quad n > 1. \tag{3.14d}$$

The procedure introduced by (3.14) can be considered as a series over the small parameters ρ_1/R, ρ_2/R. We impose two conditions on Γ_n: to be of order of Γ_0 at $\rho_1, \rho_2 \sim R$ and to turn into zero at $\rho_{1,2} = 0$.

Before consistent derivation, let us notice that the r_d dependence could be readily found from (3.14b) and the additional anomalous scaling of the subleading (ρ-dependent) terms naturally appears already at the first step as a consequence of the given scaling of Γ_0. Let us consider $\rho \ll r_d$. The solution of (3.14b) for Γ_1 will be $\rho_1^2 + \rho_2^2$ multiplied by the R-dependent terms Φ and $\hat{\mathcal{L}}\Gamma_0$, which are both proportional to $R^\gamma (L/R)^\Delta$ — see (3.21) below. Assuming that no special cancellations happen, one may conclude, in particular, that $\langle\langle \theta^2(0)\epsilon(R)\rangle\rangle \propto r_d^{\gamma-2} R^\gamma (L/R)^\Delta$ instead of $R^{2\gamma-2}(L/R)^\Delta$, which would be given by a naive counting of powers.

Let us start now a regular examination of (3.14b). A solution of that equation has the following form:

$$\Gamma_n(\mathbf{R}; \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int_0^\infty dt \int\!\!\int d\mathbf{r}' d\mathbf{r}'' \mathcal{R}(t; \boldsymbol{\rho}_1, \mathbf{r}') \times \mathcal{R}(t; \boldsymbol{\rho}_2, \mathbf{r}'')\Phi_n(\mathbf{R}; \mathbf{r}', \mathbf{r}'')\,, \tag{3.15}$$

where $\mathcal{R}$ is the kernel of the resolvent of the operator $\hat{\mathcal{L}}_\rho$ introduced by (3.4), (3.5), and (3.7). The main contribution to the integral (3.15) is determined by the region $r' \sim R$, $r'' \sim R$. The higher the number of the angular harmonic n in (3.5) and (3.7) is, the smaller is the respective contribution into Γ due to additional powers of the small parameter ρ/R in $\mathcal{R}^{(2n)}$. Therefore, besides the first harmonic, we consider further only the second one to show how a nontrivial anomalous exponent appears due to an angular dependence.

C. Zero modes

The asymptotic formulas (3.7) for the resolvent enable one to extract a scaling behavior of the general expression (3.15) for a solution of (3.14b). We consider the case where the largest scale R of the tetrahedron is much smaller than the pumping scale L. That means that the only external scales for the integrations in (3.15) can be R or r_d but not L. The interest in the contribution to Γ_n stemmed from large enough scales $r', r'' \sim R$ and time $t \sim \tau_R$ (where one defines τ_r as r^2/κ at $r \ll r_d$ but as r^γ/D at $r \gg r_d$), and one can expand the resolvents in (3.15) in a series over the small parameters τ_r/t and analyze the convergence of the temporal integral for different terms of the expansion. Of course, the integrals depend on the concrete form of Φ. However, in any case only a finite number of the first terms of the expansion are determined by the time $t \sim \tau_R$. The respective contributions into Γ are the zero modes of the operator $\hat{\mathcal{L}}_{\rho_1} + \hat{\mathcal{L}}_{\rho_2}$ that grow with ρ and are symmetric under the permutation $\rho_1 \leftrightarrow \rho_2$. All the remaining terms (stemmed from $r', r'' \ll R$) are summed up into the term estimated as $\tau_\rho \Phi$. We will name this part of the solution the forced one, Γ^f.

The r dependence of the forced part of the term Γ_n depends on a concrete form of the function Φ_n in (3.14b). On the contrary, the form of the zero modes is universal: they depend on the source term Φ_n only via coefficients. We thus start by determining the zero modes that give the main contributions at small scales. To find the zero modes that contribute to the solution, we will follow an indirect (but, probably, the simplest) way. Since the zero modes are formed at the large times $t \sim \tau_R$ in the integral (3.15) (which means that they appear due to matching at $\rho \sim R$), we can extract their scaling behavior expanding the expressions (3.7) for the resolvents in the series over τ_ρ/t (3.7) and keeping the first terms of the expansion. Thus we can extract the exponents of the first zero modes and their angular dependence but not coefficients at different terms of the same order. To find the coefficients we can construct a combination of a given order with arbitrary coefficients and then find relations between the coefficients demanding that the combination is a zero mode of $\hat{\mathcal{L}}_{\rho_1} + \hat{\mathcal{L}}_{\rho_2}$, which can be established by directly applying the operator.

Let us realize the scheme using the expressions (3.7). We see that the ρ expansion of the expressions produces powers of ρ^γ and ρ^δ, where as follows from (3.8)

$$\delta = \frac{1}{2}\left(\gamma - d + \sqrt{(d-\gamma)^2 + \frac{8(d+1-\gamma)d}{d-1}}\right). \tag{3.16}$$

The angular structure of the zero modes also can be established if we take into account the explicit form of two first Jacobi polynomials:

$$P_0^{(\nu,\nu)}(x) = 1\,, \quad P_2^{(\nu,\nu)}(x) = \frac{d+1}{8}(dx^2 - 1)\,.$$

Then one can directly check that the zero modes of the first and of the second orders are as follows:

$$\mathcal{Z}_2^{(0,0)} \sim \frac{P_2^2}{D^2}\left(\frac{L}{R}\right)^\Delta \times \begin{cases} \rho_1^\gamma \rho_2^\gamma - \frac{d}{2(d+\gamma)}(\rho_1^{2\gamma} + \rho_2^{2\gamma}), & \rho \gg r_d, \\ \left(\rho_1^2\rho_2^2 - \frac{d}{2(d+2)}(\rho_1^4 + \rho_2^4)\right) r_d^{2\gamma-4}, & \rho \ll r_d; \end{cases} \tag{3.17a}$$

$$\mathcal{Z}_1^{(0,2)} \sim \frac{P_2^2}{D^2} R^{2\gamma-\delta}\left(\frac{L}{R}\right)^{\Delta} \times \begin{cases} \left(\{\rho_1^{\delta-2}[d(\mathbf{R}\boldsymbol{\rho}_1)^2 - R^2\rho_1^2]\} + \{\boldsymbol{\rho}_1 \leftrightarrow \boldsymbol{\rho}_2\}\right)R^{-2}, & \rho \gg r_d, \\ [\{d(\mathbf{R}\boldsymbol{\rho}_1)^2 - R^2\rho_1^2\} + \{\boldsymbol{\rho}_1 \leftrightarrow \boldsymbol{\rho}_2\}]r_d^{\delta-2}R^{-2}, & \rho \ll r_d; \end{cases} \tag{3.17b}$$

$$\mathcal{Z}_2^{(0,2)} \sim \frac{P_2^2}{D^2} R^{\gamma-\delta}\left(\frac{L}{R}\right)^{\Delta} \times \begin{cases} \{(\rho_2^\gamma - b_1\rho_1^\gamma)\rho_1^{\delta-2}[d(\mathbf{R}\boldsymbol{\rho}_1)^2 - R^2\rho_1^2]\}R^{-2} + \{\boldsymbol{\rho}_1 \leftrightarrow \boldsymbol{\rho}_2\}, & \rho \gg r_d, \\ \left(\{(\rho_2^2 - b_2\rho_1^2)[d(\mathbf{R}\boldsymbol{\rho}_1)^2 - R^2\rho_1^2]\}R^{-2} + \{\boldsymbol{\rho}_1 \leftrightarrow \boldsymbol{\rho}_2\}\right)r_d^{\gamma+\delta-4}, & \rho \ll r_d; \end{cases} \tag{3.17c}$$

$$\mathcal{Z}_2^{(2,2)} \sim \frac{P_2^2}{D^2} R^{2(\gamma-\delta)}\left(\frac{L}{R}\right)^{\Delta} \times \begin{cases} \left(\{\rho_1^{\delta-2}[d(\mathbf{R}\boldsymbol{\rho}_1)^2 - R^2\rho_1^2]\}\{\boldsymbol{\rho}_1 \leftrightarrow \boldsymbol{\rho}_2\}\right)R^{-4} + c_1(\rho_1\rho_2)^{\delta-2}[d(\boldsymbol{\rho}_1\boldsymbol{\rho}_2)^2 - \rho_1^2\rho_2^2], & \rho \gg r_d, \\ \left([\{d(\mathbf{R}\boldsymbol{\rho}_1)^2 - R^2\rho_1^2\}\{\boldsymbol{\rho}_1 \leftrightarrow \boldsymbol{\rho}_2\}]R^{-4} + c_2[d(\boldsymbol{\rho}_1\boldsymbol{\rho}_2)^2 - \rho_1^2\rho_2^2]\right)r_d^{2\delta-4}, & \rho \ll r_d; \end{cases} \tag{3.17d}$$

where $c_{1,2}$ are dimensionless constants and

$$b_1 = \frac{(\delta+\gamma)(d+2\gamma-2) - 2d}{d\gamma}, \quad b_2 = \frac{d+4}{d}. \tag{3.18}$$

The upper indices of $\mathcal{Z}$, introduced in (3.17), characterize the type of a zero mode with respect to the angular structure. The lower index denotes the original order of the resolvent's expansion over τ_ρ/t producing the concrete term. We keep the terms of the second order (3.17a), (3.17c), and (3.17d) besides the first-order term (3.17b) since they possess qualitatively different ρ dependence: the term (3.17b) is an additive one while the terms (3.17a), (3.17c), and (3.17d) contain cross contributions.

Note that higher-order angular harmonics will be characterized by the exponents δ_n, which can be extracted from the asymptotic behavior of the resolvents (3.7) characterized by (3.8). Let us write the explicit expressions for the exponents:

$$\delta_n = \frac{1}{2}\left(\gamma - d + \sqrt{(d-\gamma)^2 + \frac{8n(d+1-\gamma)(2n+d-2)}{d-1}}\right) \tag{3.19}$$

which at $n=1$ gives (3.16). The exponents δ_n will figure in the expressions of the (3.17) type for higher harmonics. It is worthwhile to emphasize here that the exponents δ_n are nontrivial: to obtain them by the $\hat{\mathcal{L}}_0^{-1}\hat{\mathcal{L}}_1$ expansion that we used in Sec. II, one would get terms like $\rho^\gamma \ln(\rho/R)$, which should be summed up into ρ^δ.

We introduced in (3.17) the dimensional factors (in front of the braces) with which the zero modes appear at Γ. The factors in front of the braces are taken from the analysis of the integral (3.15); since the contributions stem from the region $\rho \sim R$ (where the resolvent is not precisely known), the dimensionless coefficients (including c_1 and c_2) could not be found within our approach with one exception, which will be described below in Sec. III E. The dependence of the zero modes on r_d can be established by matching the respective term from inertial and diffusive interval at r_d and using the r_d independence of Γ in the convective interval.

D. Forced terms

Let us begin the discussion of the forced terms with Γ_1^f, which appears at the first step of the iteration procedure (3.14c). The right-hand side of (3.14c) contains the terms of different order in ρ/R, different angular functions, and different types of the dependence on ρ_1 and ρ_2 (additive and multiplicative). We shall analyze the respective contributions order by order in ρ/R separately for the different angular functions. The terms with the cross dependence (like $\rho_1^2\rho_2^2$) are of special interest since they contribute to the $\langle\epsilon\epsilon\rangle$ correlation function.

The leading contribution in ρ/R on the right-hand side of (3.14c) is ρ independent (and consequently angular independent). The contribution originates (a) from $\hat{\mathcal{L}}\Gamma_0$,

$$\hat{\mathcal{L}}\Gamma_0 \to (\hat{\mathcal{L}}_R - \kappa\triangle_R)\Gamma_0(R); \tag{3.20a}$$

and (b) from $\Phi_{13;24} + \Phi_{14;23}$,

$$\Phi_{13;24} + \Phi_{14;23} \to \Phi_R \equiv 2\mathcal{K}_R^{\alpha\beta}\frac{R^\alpha R^\beta}{R^2}\left(\frac{df(R)}{dR}\right)^2. \tag{3.20b}$$

The expression for the respective forced solution is

$$\Gamma_{1,\mathrm{add}}^f \approx \left[-\frac{(2-\gamma)}{(d-1)dD}\Phi_R + (\hat{\mathcal{L}}_R - \kappa\triangle_{\mathbf{R}})\Gamma_0(R)\right] \times\left(\int_0^{\rho_1}\frac{r\,dr}{2r_d^{2-\gamma}+r^{2-\gamma}} + \int_0^{\rho_2}\frac{r\,dr}{2r_d^{2-\gamma}+r^{2-\gamma}}\right). \tag{3.21}$$

The subsequent additive terms are produced by taking into account the next ρ^2 terms on the right-hand side of (3.14c). They produce the additive forced contributions to Γ_1 with the scaling $\propto \rho^{2+\gamma}$ in the convective interval.

The leading cross term originates from $\Phi_{12;34}$; the main term of its expansion is

$$\Phi_{12;34} \approx \nabla_{\mathbf{R}}^\nu \nabla_{\mathbf{R}}^\mu \mathcal{K}_{\mathbf{R}}^{\alpha\beta}\frac{\rho_1^\nu\rho_1^\alpha\rho_2^\beta\rho_2^\mu}{\rho_1\rho_2}\frac{df(\rho_1)}{d\rho_1}\frac{df(\rho_2)}{d\rho_2}. \tag{3.22}$$

Apart from (3.22), we have to take into account the cross terms originating from other terms on the right-hand side of (3.14c). The terms have a character of a regular expansion in ρ/R and are consequently proportional to $\rho_1^2\rho_2^2$. Such terms are much less than (3.22) in the convective interval and are of the same order in the diffusive interval. Since we are interested not in the factors but only in the scaling behavior, we can restrict ourselves with the analysis of (3.22). The term is proportional to $\rho^{2\gamma}$ in the convective interval and to ρ^4 in the diffusive one, which gives for the principal behavior of the corresponding forced term $\rho^{3\gamma}$ and ρ^6, respectively.

The source (3.22) produces not only cross terms in Γ of scalar and tensor structures but also additive and generally mixed ones possessing all the variety of angular and $\rho_1 \leftrightarrow \rho_2$ symmetries used in the classification of the zero modes (3.17). All those terms scale with R as $R^{-\gamma}$ both in the convective and diffusive intervals by ρ. To match those forced terms from the convective and diffusive intervals one has to introduce the additional zero modes in the diffusive interval by ρ,

$$\mathcal{Z}_{2,d}^{(0,0)} \sim \mathcal{Z}_2^{(0,0)} (R/L)^{\Delta} (r_d/R)^{\gamma}, \tag{3.23a}$$

$$\mathcal{Z}_{1,d}^{(0,2)} \sim \mathcal{Z}_1^{(0,2)} (R/L)^{\Delta} (r_d/R)^{3\gamma-\delta}, \tag{3.23b}$$

$$\mathcal{Z}_{2,d}^{(0,2)} \sim \mathcal{Z}_2^{(0,2)} (R/L)^{\Delta} (r_d/R)^{2\gamma-\delta}, \tag{3.23c}$$

$$\mathcal{Z}_{2,d}^{(2,2)} \sim \mathcal{Z}_2^{(2,2)} (R/L)^{\Delta} (r_d/R)^{3\gamma-2\delta}. \tag{3.23d}$$

Let us now consider the forced term Γ_2^f arising at the second step of the iteration procedure. Analyzing the structure on the right-hand side of (3.14d), one concludes that it decreases downscales with ρ not slower than $[\Gamma_1^f$+(zero modes)]. Consequently, the forced term Γ_2^f decreases downscales with ρ not slower than $\tau_\rho[\Gamma_1^f$+(zero modes)], which is faster than Γ_1^f. Thus, one can drop Γ_2^f in comparison with Γ_1^f and all the set of zero modes. The same argument allows one to drop the higher forced terms Γ_n^f, $n > 2$.

E. Comparison between the zero modes and forced terms

To find the main contributions into Γ one should compare the zero modes and forced terms of the same angular structure. The leading term in the expansion in ρ/R is the isotropic additive forced term (3.21) inside of both the convective and diffusive intervals.

The zero modes are L dependent but the nonadditive forced terms are not. Thus keeping the parameter L/R large enough, one forces the zero modes to dominate. A possibility for the forced terms to prevail appears at the smallest ρ separations, when the dominance of the zero modes could be compensated by different ρ dependencies of the zero modes and the forced terms. The present subsection is devoted to such a comparison.

We compare (3.17a) with the forced terms generated by (3.22). Both in the convective and diffusion interval we get the dominance of the angular independent zero mode (3.17a) containing the cross contribution. To be sure that (3.17a) does give the main contribution into $\langle\langle\epsilon_1\epsilon_2\rangle\rangle$ (and thus provides for its anomalous scaling), we have to check whether or not the respective dimensionless coefficient in front of that mode can turn into zero. We consider that mode in a special case $R \gg L \gg \rho$ when the coefficient can be found unambiguously contrary to other cases where the matching at $\rho \sim R$ is necessary. If $R \gg L \gg \rho$, then one has to consider the integral (3.15) with (3.22) and the respective term from (3.7) of the expansion of $\mathcal{R}^{(0)}$ – see (2.16a) and (2.17). One can directly check that the main contribution into the integral stems from the scales L and time τ_L so that no matching at R is necessary. We do not write here the bulky expression for the integral; it turns into zero only at a single value $\gamma/d \approx 0.086$ found numerically. That numerical value depends on the shape of χ. For the remaining values of γ/d, the integral is nonzero and provides the contribution of the structure (3.17a).

Next we consider the second angular harmonic presented in (3.17b), (3.23b), (3.17c), and (3.23c) and the product of the second angular harmonics in (3.17d) and (3.23d). The scaling of the respective harmonics of the forced terms coincides with that of the isotropic contribution: $\rho^{3\gamma}$ and ρ^6 in the convective and diffusive intervals, respectively. The zero modes (3.17b), (3.17c), and (3.17d) scale as ρ^δ, $\rho^{\delta+\gamma}$, and $\rho^{2\delta}$ in the convective interval, and as ρ^2, ρ^2, and ρ^4 in the diffusive one, respectively. One concludes (a) in the convective interval, the zero mode $\mathcal{Z}_1^{(0,2)}$ ($\mathcal{Z}_2^{(0,2)}$ or $\mathcal{Z}_2^{(2,2)}$) is dominant if $3\gamma > \delta$ (respectively $2\gamma > \delta$ or $\gamma > 2\delta/3$), otherwise at the smallest ρ the forced terms prevail; (b) in the diffusive interval, at $3\gamma > \delta$ ($2\gamma > \delta$ or $\gamma > 2\delta/3$) the zero mode (3.17b) [respectively (3.17c) or (3.17d)] becomes dominant at large enough R; otherwise the zero mode (3.23b) [respectively (3.23c) or (3.23d)], matching with the forced solution originating from (3.22), prevails.

IV. POSSIBLE GENERALIZATIONS

We remind the reader that the above results on r_d dependence are general while those on L dependence are formally obtained only when the respective anomalous exponent Δ is much less than the normal exponent 2γ. The whole approach of Sec. II was developed for the fourth-order correlation function at $d > 2$. The generalization for the nth correlation function at $d > n-2$ is straightforward by means of the same representation of $\hat{\mathcal{L}}$ in terms of r_{ij}. Such a generalization leads to the new predictions for $n > 4$: the scaling exponent $2\gamma - \Delta$ should appear at all high-order functions. The direct check shows that the respective zero modes appear in the partially reducible contributions only. For example, the six-point object $F_{123456} \equiv \langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\theta(\mathbf{r}_3)\theta(\mathbf{r}_4)\theta(\mathbf{r}_5)\theta(\mathbf{r}_6)\rangle$ should necessarily contain terms like $f(r_{12})\Gamma_{3456}$, i.e., $r^{2\gamma-\Delta}L^{\gamma+\Delta}$. Such contributions should cancel out in structure functions like $\langle\langle(\theta_1-\theta_2)^n\rangle\rangle$. To find the scaling exponents of the nth structure functions in the limit $d \to \infty$, one should iterate the zero mode with the scaling exponent $n\gamma$, which will be published elsewhere.

Quite a different picture may appear for $d \le n-2$. In this case, $n(n-1)/2$ distances between points are not longer than independent variables. Additional constraints should be imposed on that set of (otherwise very convenient) variables, which may lead to the possibility of additional zero modes.

As a final remark, let us emphasize that the above results are valid for a Gaussian pumping only and discuss what might be the consequences of the pumping non-Gaussianity. Let us add to the fourth-order pumping correlation function (1.4) the irreducible part $\chi_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)\delta(t_1-t_2)\delta(t_1-t_3)\delta(t_1-t_4)$. One may model the function χ_4 by the step function: $\chi_4^m(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)=P_4$ if all $r_{ij}<L$ and zero if any $r_{ij}>L$. The production rate of θ^4 is $3P_2f(0)+P_4$ (remember that the advection preserves the integral of any power of the scalar field). The ratio $\tau_*=P_4/P_2^2$ having the dimensionality of time is a correlation time of the pumping, which is assumed to be the smallest time in the problem. As a result, the terms generated by χ_4 will be small, yet we keep them to show what qualitatively new terms may appear in the case of a non-Gaussian pumping with a finite correlation time. Equation (2.6a) acquires an additional term:

$$-\hat{\mathcal{L}}\Gamma_{1234}=\chi_4+\Phi_{12;34}+\Phi_{13;24}+\Phi_{14;23}.$$

The presence of the function χ_4 changes scaling in the convective interval; it causes the terms with anomalous scaling $\tau_* r_{ij}^\gamma$ to appear in Γ which corresponds to the scaling $\zeta_n=\zeta_2$ discussed in [17,6]. Also, the terms with r_{12}^0 appear in $\langle\langle\epsilon_1\theta_2^2\rangle\rangle$, while no additional terms appear in $\langle\langle\epsilon_1\epsilon_2\rangle\rangle$. Note that within our approach the τ_*-related terms are small corrections even at the diffusion scale despite the fact that they decrease slower than the main terms as distances decrease. Most probably, finite correlation time of the pumping will lead to a substantial contribution with another anomalous scaling, yet this is beyond the scope of the present analysis. The development of the theory for the finite correlation times of the velocity and pumping is necessary for a meaningful comparison between theory and experiment. This will be the subject of future publications. On the other hand, the more detailed experimental results are desirable, in particular, direct measurements of the correlation function of the dissipation field.

V. CONCLUSION

We have shown that the fourth-order correlation function F_{1234} has the scaling exponent $2\gamma-\Delta$ in the convective interval at $d>2$. The anomalous exponent Δ is analytically found at $d\to\infty$. Considering the behavior of F_{1234} in the convective interval where some separations ρ tend to zero, we established that the ρ-dependent contribution to F_{1234} also tends to zero. Moreover, nothing terrible happens when ρ passes through the diffusion scale r_d and we find that

$$\langle(\theta_1-\theta_2)^4\rangle\propto|\mathbf{r}_1-\mathbf{r}_2|^{2\gamma-\Delta}L^\Delta, \tag{5.1}$$

for r_{12} from the convective interval.

We analyzed also the case where two separations are much smaller than the distance R between the pairs and found the contribution, mixing both small distances, which determines the correlator

$$\langle(\theta_1-\theta_2)^2(\theta_3-\theta_4)^2\rangle-\langle(\theta_1-\theta_2)^2\rangle\langle(\theta_3-\theta_4)^2\rangle\,, \tag{5.2}$$

at $r_{12},r_{34}\ll R$. We established that the scaling behavior of that correlation function in the convective interval is characterized by the additional anomalous exponent δ defined by (3.16). The corresponding contributions to the irreducible part Γ_{1234} of the correlator F_{1234} are presented in (3.17). The isotropic contribution (3.17a) behaves $\propto r_{12}^\gamma r_{34}^\gamma(L/R)^\Delta$, while the anisotropic one (3.17d) behaves $\propto r_{12}^\delta r_{34}^\delta R^{(2\gamma-2\delta)}(L/R)^\Delta$. The cross term (3.17c) behaves $\propto(r_{12}^\delta r_{34}^\gamma+r_{34}^\delta r_{12}^\gamma)R^{\gamma-\delta}(L/R)^\Delta$. The analysis of the continuation of the correlation function (5.2) for r_{12}, r_{34} passing to the diffusive interval gives the following scaling laws:

$$\langle\langle\epsilon_1\epsilon_3\rangle\rangle=\kappa^2\langle\langle(\nabla\theta_1)^2(\nabla\theta_3)^2\rangle\rangle\propto r_d^0(L/r_{13})^\Delta\,, \tag{5.3}$$

$$\langle\langle[\nabla_\alpha\theta_1\nabla_\beta\theta_1-d^{-1}\delta_{\alpha\beta}(\nabla\theta_1)^2][\nabla_\mu\theta_3\nabla_\nu\theta_3-d^{-1}\delta_{\mu\nu}(\nabla\theta_3)^2]\rangle\rangle\propto r_d^{2\delta-4}r_{13}^{2\gamma-2\delta}(L/r_{13})^\Delta\,, \tag{5.4}$$

$$\langle\langle\epsilon_1[\nabla_\alpha\theta_3\nabla_\beta\theta_3-d^{-1}\delta_{\alpha\beta}(\nabla\theta_3)^2]\rangle\rangle\propto r_d^{\delta-2}r_{13}^{\gamma-\delta}(L/r_{13})^\Delta\,, \tag{5.5}$$

where the separation r_{13} lies in the convective interval. The double angular brackets designate irreducible correlation functions. The analogous analysis can be performed for the correlation function

$$\langle(\theta_1-\theta_2)^2\theta_3^2\rangle-\langle(\theta_1-\theta_2)^2\rangle\langle\theta_3^2\rangle \tag{5.6}$$

at $r_{12}\ll r_{13}$. The contribution (3.21) gives the law $\propto r_{13}^\gamma r_{12}^\gamma$ for the isotropic part of the correlation function (5.6) which leads to

$$\langle\langle\epsilon_1\theta_3^2\rangle\rangle\propto r_d^0 r_{13}^\gamma(L/r_{13})^\Delta\,. \tag{5.7}$$

The anisotropic part of the correlation function (5.6) is determined by the contribution (3.17b), which is proportional to $r_{13}^{2\gamma-\delta}r_{12}^\delta$. It leads to

$$\langle\langle\theta_1^2[\nabla_\alpha\theta_2\nabla_\beta\theta_2-d^{-1}\delta_{\alpha\beta}(\nabla\theta_2)^2]\rangle\rangle\propto r_d^{\delta-2}r_{13}^{2\gamma-\delta}(L/r_{13})^\Delta. \tag{5.8}$$

The appearance of an r_d dependence in the correlation functions (5.3)–(5.8) may correspond to the presence of the ultraviolet divergences found in [16,11] in the diagrams for powers of gradients both of the velocity and of the passive scalar. That means that, contrary to L-related scaling, r_d-related anomalous scaling of the scalar derivatives could be caught perturbatively [24,25]. Moreover, in the asymptotic limits of the normal overall scaling of Γ_{1234} (at $d\to\infty$) it is tempting to describe the structure of (5.1), (5.3), (5.4), (5.5), (5.7), and (5.8) using the language of the so-called operator algebra [26–28] (developed in the context of second-order phase transition theory), of which the validity for turbulence was argued in [16]. Namely, at $d \to \infty$ in accordance with those expressions, the passive scalar θ has dimensionality $-\gamma/2$, $(\nabla\theta)^2$ has dimensionality 0, and $[\nabla_\alpha\theta\nabla_\beta\theta - d^{-1}\delta_{\alpha\beta}(\nabla\theta)^2]$ has dimensionality $\delta - \gamma$. In the general case of arbitrary d and γ when the L-related anomalous scaling does exist, the underlying algebraic structure (if it exists at all) is unclear at the moment.

All said above concerns the terms associated with zero modes of the operator $\mathcal{L}$ in the convective interval. There exists also a forced solution in the convective interval originating from (3.22). The tails in the diffusion intervals created by this forced solution can be interpreted as zero modes (in the diffusive interval only), which are determined by the estimates (3.23). Those zero modes produce the contributions to the correlation functions (5.4), (5.5), and (5.8), which are $\propto r_{13}^{-\gamma}$ and can prevail if at large enough r_{13}, $\delta > 3\gamma/2$, $\delta > \gamma/2$, and $\delta > 3\gamma$ correspondingly. Those contributions are not governed by the operator algebra even in the asymptotic cases of $d \to \infty$ or $\gamma \to 2^-$. It is natural from the diagrammatic point of view since those contributions correspond to "one-bridge" diagrams, which do not contain series producing anomalous scaling. Thus the factor $\propto r_{13}^{-\gamma}$ is determined simply by the "bridge" factor $\nabla\nabla\mathcal{K} \propto R^{-\gamma}$. Of course analogous contributions exist in all above correlation functions, but only in (5.4), (5.5), and (5.8) can they prevail at some δ.

To conclude, the $1/d$ expansion tells us that the anomalous scaling is present already in the oversimplified model of the δ-function-correlated velocity field. If one formally uses the formula in the Introduction for the anomalous exponent in the Richardson-Kolmogorov case $d = 3, \gamma = 2/3$, it gives $\Delta = 16/9$, which is substantially larger than the experiments and numerics give ($\Delta \simeq 0.4 - 0.5$). The difference could be accounted for by the inaccuracy of both an asymptotic formula at finite d and the model with δ-function-correlated velocity. Note that our results are valid not only for the steady state but also for decaying turbulence of a passive scalar with a compact spectrum as an initial condition. In that case, the scalar cascade is accelerated as it goes towards small scales (with typical time $t \propto r^{(2-\gamma)/2}$) so that the small-scale part of the scalar distribution is quasisteady.

ACKNOWLEDGMENTS

We are indebted to E. Balkovsky, V. Belinicher, A. Fairhall, U. Frisch, O. Gat, V. L'vov, A. Migdal, M. Nelkin, I. Procaccia, and V. Yakhot for discussing the manuscript of the present paper. We are deeply grateful to R. Kraichnan for numerous illuminating explanations. This work was partly supported by the Clore Foundation (M.C.), by the Rashi Foundation (G.F.), and by the Minerva Center for Nonlinear Physics (I.K. and V.L.).

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