Topological characterization of a system with high-order symmetries

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Topological characterization of the Lorenz attractor taking into account the two-order equivariance of the vector field has been recently proposed by introducing a fundamental domain (typically a wing) and one copy of it. The present paper generalizes this approach to n-order equivariant systems. The general procedure is illustrated by taking a specific example, namely the so-called proto-Lorenz system which is derived from the Lorenz one. It is shown that symmetric attractors are tiled by n representations of a fundamental domain and that fundamental linking numbers therefore have to be introduced to conveniently validate the template.

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I. INTRODUCTION

In the past few years several works discussed the topological description of chaotic attractors. In particular, the idea has arisen that an attractor can be described by the population of periodic orbits, their related symbolic dynamics, and their linking numbers [1]. An attractor is then characterized by a template that constitutes a schematic view of the attractor. The case of asymmetric systems is now well documented and many templates of experimental attractors have been given [2–9].

Nevertheless, it could appear that an experimental system is described by an equivariant set of ordinary differentiable equations. Such an equivariant system is well exemplified by the Lorenz system whose equivariance defines an axial symmetry. In agreement with Cvitanovic and Eckhardt [10], we have pointed out that a special procedure is required to extract the template induced by a symmetric attractor and we have proposed an equivariant template for the common Lorenz attractor [11]. The equivariant template relies on the fact that the Lorenz attractor is tiled by a fundamental domain (roughly speaking, a wing) and one copy of it. The template may be validated by comparison between linking numbers predicted from the template and fundamental linking numbers that act on the fundamental domain, computed on a plane projection [12]. If our procedure was successfully applied to the Lorenz attractor whose equivariance is of order 2 (the fundamental domain appears twice on the Lorenz attractor, once for each wing), no description has been given for systems with an higher-order equivariance.

Consequently, this paper is devoted to the topological characterization of systems with higher-order equivariances, the procedure being illustrated by taking the example of the proto-Lorenz system. This system is discussed by Miranda and Stone [13] who have proposed a formula to obtain a vector field derived from the Lorenz system and for which the equivariance order can be arbitrarily chosen.

The paper is organized as follows. Section II provides a brief review concerning the general procedure of topological characterization. Section III presents the proto-Lorenz system introduced by Miranda and Stone and its induced template is given. Section IV deals with the topological characterization of the nth cover of the proto-Lorenz system whose equivariance is of order n. An explicit example with the quartic cover is discussed in Sec. V. Section VI is a conclusion.

II. TOPOLOGICAL CHARACTERIZATION

In three-dimensional phase spaces, periodic orbits may be viewed as knots [14] and, consequently, they are robust with respect to smooth parameter changes and allow the definition of topological invariants under isotopy.

The topological approach is based on the organization of periodic orbits whose linking properties severely constrain the topology of the strange attractor. Also, as recently advocated [8], [15], [16], the quantitative topological characterization of low-dimensional chaotic sets requires the assignment of a good symbolic encoding of trajectories. Consequently, the first step consists in defining a Poincaré section. The partition of the attractor is then given with respect to the critical points of the first-return map from the Poincaré section to itself. We note that such a partition can easily be performed only in the case of very dissipative systems. Otherwise, many problems arise as in defining critical points for encoding the symbolic dynamics. For more convenience, we label monotonic branches of the first-return map by integers related to the local torsion (expressed in terms of π) of the corresponding stripe, and thereafter introduce a symbolic dynamics. The population of unstable periodic orbits is then extracted by a close return method [17] and encoded by symbolic sequences.

In the next stage, ribbon subsets, here called stripes, are identified in a three-dimensional (3D) representation; i.e., the local torsion $L(i,i)$ of each stripe is determined

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and the linking numbers $L(i,j)$ of the ribbon graph for the ith and jth stripes are extracted. A linking matrix with diagonal matrix elements $L(i,i)$ and off-diagonal matrix elements $2L(i,j) = 2L(j,i)$ is proposed. The off-diagonal elements are equal to the sum of crossings between the ith and jth stripes of the ribbon graph with standard insertion [18].

To check the template, we then extract the linking number $L(N_1, N_2)$ of a pair of orbits N_1 and N_2 in a plane projection. For this process one only needs to count the signed crossings between orbits N_1 and N_2 in a *regular plane projection* of the pair (a drawing of it so that no more than two lines cross at any point). After assigning an orientation to the periodic orbits with respect to the flow and defining a number $\epsilon_{12}(p) = \pm 1$ for right-handed (+1) and left-handed (-1) crossings p between N_1 and N_2 [18], the linking number is given by

$$L(N_1, N_2) = \frac{1}{2} \sum_p \epsilon_{12}(p) \ , \qquad (1)$$

which is a topological invariant. Then, orbits N_1 and N_2 are constructed on the template (the procedure is completely reported in [14]). Finally, linking numbers are compared: the template is validated if the linking number obtained from the regular plane projection is equal to the one obtained from the corresponding orbits constructed on the template.

As the template that carries the periodic orbits is identified, the organization of the orbits is known. For a complete discussion about equivalence between periodic orbits embedded within a strange attractor and orbits of the template, see [2].

III. THE PROTO-LORENZ SYSTEM

A. The vector field

In order to eliminate the symmetry properties of the Lorenz system, Miranda and Stone [13] introduced a map $\pi : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ where $\boldsymbol{Y} \in I\!R^3$, with coordinates (u, v, z), written as

$$\pi = \begin{pmatrix} u = x^2 - y^2 \\ v = 2xy \\ z = z \end{pmatrix} \ . \qquad (2)$$

π is a local diffeomorphism since its Jacobian is nonsingular at any (x, y, z) with $(x, y) \neq (0, 0)$.

The vector field $\mathcal{C}_1$ on the orbit space $\boldsymbol{Y}$ is found to read as follows [13]:

$$\begin{pmatrix} \dot{u} = (-\sigma - 1)u + (\sigma - R)v + (1 - \sigma)N + vz \\ \dot{v} = (R - \sigma)u - (\sigma + 1)v + (R + \sigma)N - uz - Nz \\ \dot{z} = \frac{1}{2}v - bz \end{pmatrix} , \qquad (3)$$

where $N = \sqrt{u^2 + v^2}$.

Miranda and Stone [13] called this system the proto-Lorenz system on $\boldsymbol{Y} = I\!R^3$. The vector field $\mathcal{C}_1$ is not equivariant, i.e., no symmetry properties can be found on the attractor (displayed in Fig. 1).

The vector field $\mathcal{C}_1$ and the Lorenz one are related by the map π. Under this transformation, fixed points are mapped to fixed points, periodic orbits to periodic orbits but the symmetrical pair of fixed points of the Lorenz system are mapped to one fixed point of the proto-Lorenz system given by

$$\begin{pmatrix} u = 0 \\ v = 2b(R - 1) \\ z = R - 1 \end{pmatrix} \ . \qquad (4)$$

B. The partition

In order to define the partition of the attractor, we built a first-return map to a Poincaré section P, which is defined as follows:

$$P = \left\{ (u, v) \in I\!R^2 \mid z = R - 1, \ \dot{z} > 0 \right\} \ . \qquad (5)$$

The first-return map, displayed in Fig. 2, is constituted by two monotonic branches as for the Lorenz map [19]. The first-return map allows us to encode periodic orbits by using

$$K(v) = \begin{cases} 0 & \text{if} \quad v < v_c \ , \\ 1 & \text{if} \quad v > v_c \ . \end{cases} \qquad (6)$$

The population of periodic orbits may be easily obtained from the population of the Lorenz attractor by applying the map π on the coordinates of the orbits reported in [11].

C. The induced template

After many investigations, we found that the template induced by the attractor of the proto-Lorenz system (Fig. 3) is defined by a linking matrix written as

$$M_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & +1 \end{pmatrix} \ . \qquad (7)$$

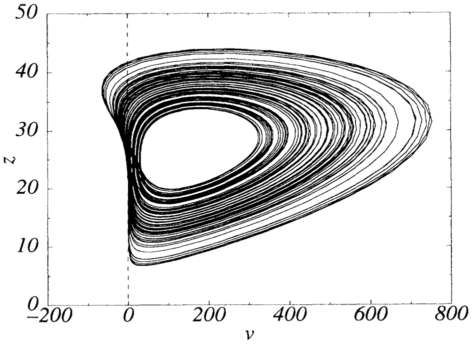

FIG. 1. The proto-Lorenz system at $(R, \sigma, b) = (28, 10, 8/3)$.

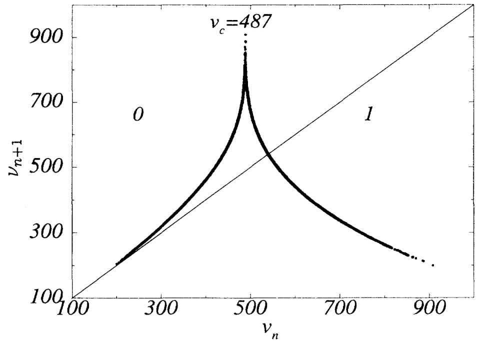

FIG. 2. First-return map to the Poincaré section built with the v coordinate.

We have validated this template by counting linking numbers from plane projections of pairs of orbits and checking that they are equal to the ones predicted by the template. For instance, from the plane projection of the couple (101,10) (Fig. 4), we found $L(101,10) = \frac{1}{2}(+4) = +2$, which is equal to the one predicted by the template.

In a previous paper [11], we have showed that the Lorenz system is characterized by such a template (for $R = 28$). Consequently, the topology is invariant under the action of the map π.

IV. THE Nth COVERS $\mathcal{C}_N$ OF THE PROTO-LORENZ SYSTEM

We are now interested in possessing a system whose attractor presents an n order of symmetry. Therefore, the vector field must be equivariant under a γ_n matrix such as $\gamma_n^n = I$. A procedure to obtain such a system is given by Miranda and Stone [13]. Vector fields $\mathcal{C}_n$ will present an equivariance defined by a γ_n matrix, which is written as

$$\gamma_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n & 0 \\ \sin\theta_n & \cos\theta_n & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (8)$$

where $\theta_n = \pm\frac{2\pi}{n}$ (the sign of the rotation is irrelevant). This γ_n matrix defines a rotation of θ_n with respect to

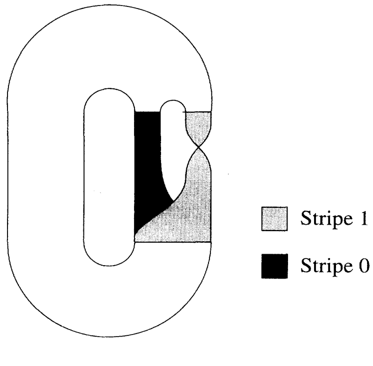

FIG. 3. Template of the proto-Lorenz system.

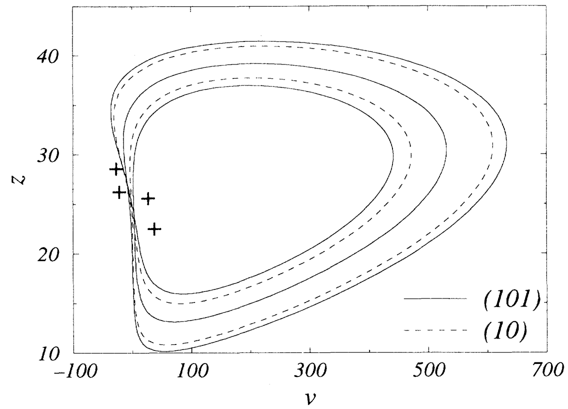

FIG. 4. Plane projection of the couple (101,10): $L(101,10) = \frac{1}{2}(+4) = +2$.

the z axis.

They call these vector fields *covers* of the proto-Lorenz system $\mathcal{C}_1$. Due to the relation between the Lorenz and the proto-Lorenz systems, the double cover $\mathcal{C}_2$ of the proto-Lorenz system identifies with the Lorenz system. The structure of the covers is conveniently understood by viewing the subspace (u, v) of $\mathbb{R}^2$ as the space $\mathbb{C}$ of complex numbers $w = u + iv$.

A. First-return map

Let $\mathcal{C}_n$ be a vector field that presents $n+1$ fixed points that are the origin F_0 $(0,0,0)$ and n fixed points F_j related by the following recurrence relation:

$$F_{j+1} = \gamma_n F_j \ . \quad (9)$$

This relation is valid for any system whose n-order equivariance defines a rotation by $\pm\frac{2\pi}{n}$.

In order to define the partition, we introduce a Poincaré set P constituted by n Poincaré sections P_j given by

$$P_j = \left\{ (w_n, z) \in \mathbb{C} \times \mathbb{R} \mid \theta = \theta_F - (j-1)\frac{2\pi}{n},\ \dot{\theta} < 0 \right\} , \quad (10)$$

where θ_F is the argument of the complex number defining the fixed point F_1, which is chosen among the n fixed points F_j for its computational advantages. A first-return map is thereafter built by using the invariant variable ρ, which is the modulus of w_n. The population of periodic orbits may then be encoded following the partition given by the first-return map.

B. Population of periodic orbits

In an n-order equivariant system, as the fundamental domain appears n times in the state space, all symbolic sequences of periodic orbits are described n times within the complete attractor in a way that depends on the symmetry properties of the orbits.

These symmetry properties of orbits may be directly known from the symbolic sequences. Let us assume that the considered n-equivariant vector field generates an attractor whose first-return map presents l monotonic branches labeled on the set $\Sigma = \{0, 1, ..., l-1\}$ of symbols. Each orbit of period p is encoded by a symbolic sequence (W) on Σ. Depending on the topological properties associated with each branch of the first-return map, a code may correspond to a transition between two successive representations of the fundamental domain, or not. In the case of the nth cover of the proto-Lorenz system, the symbol 0 is associated with the absence of transition while code 1 is associated with the existence of a transition from a copy of the fundamental domain to the next one under the action of γ_n.

Transition from one copy to another is the relevant feature to determine if an orbit is symmetric or not. Consequently, let us introduce a transition operator that maps a code K_j (the symbol associated with the jth intersection between the trajectory and the Poincaré set) pertaining to Σ to a transition number $\mathcal{T}(K_j)$ as follows:

$$\mathcal{T}(K_j) = \begin{cases} 1 & \text{if } K_j \text{ is associated with a transition from} \\ & \text{a copy of the fundamental domain to the} \\ & \text{next one generated by the action of } \gamma_n. \\ 0 & \text{otherwise .} \end{cases} \tag{11}$$

We define the transition number $\mathcal{T}_W$ of a period-p orbit encoded by the symbolic sequence (W) as the sum of the p transition numbers $\mathcal{T}(K_j)$ associated with the p codes K_j, which constitute the symbolic sequence (W). From this transition number $\mathcal{T}_W$, we can distinguish different kinds of periodic orbits as follows.

Depending on the transition number $\mathcal{T}_W$, an orbit must describe its symbolic sequence m times before returning to its initial conditions where m is the smallest integer such as:

$$\text{mod } (m\mathcal{T}_W, n) = 0 \ , \tag{12}$$

where mod $(m\mathcal{T}_W, n)$ is the rest of the integer division of $m\mathcal{T}_W$ by n. For instance, in the $\mathcal{C}_3$ cover ($n = 3$), the orbits encoded by (1) and (1011) have a transition number $\mathcal{T}_{(1)}$ and $\mathcal{T}_{(1011)}$ equal to 1 and 3, respectively. The orbits, displayed in Fig. 5, are found to return to their initial conditions after describing their symbolic sequence three and one times, respectively. This is easily checked from the relation (12) since the smallest integer m ($\neq 0$) such as $\text{mod}(m\mathcal{T}_{(1)}, 3) = 0$ are $m = 3$ and $m = 1$ for orbits (1) and (1011), respectively. In a similar way, in the $\mathcal{C}_4$ cover, orbits encoded by (10), (101), and (10111) (displayed in Figs. 6, 7, and 8), describe their symbolic sequence four times, twice, and once before returning to their initial conditions, respectively.

Let us consider an asymmetric orbit ξ of period p whose transition number $\mathcal{T}_W$ is equal to n ($m = 1$). For instance, such an orbit may be viewed as the orbit (1011) of the $\mathcal{C}_3$ cover or the orbit (10111) of the $\mathcal{C}_4$ cover. Orbit ξ describes its symbolic sequence once before returning to its initial conditions; i.e., it is degenerated n times. In particular, each one of the p periodic points is visited once in the Poincaré set. These p periodic points, which are actually dispatched on Poincaré sections (we recall that the Poincaré set is constituted by n Poincaré sections), may then be divided in n subsets, each subset

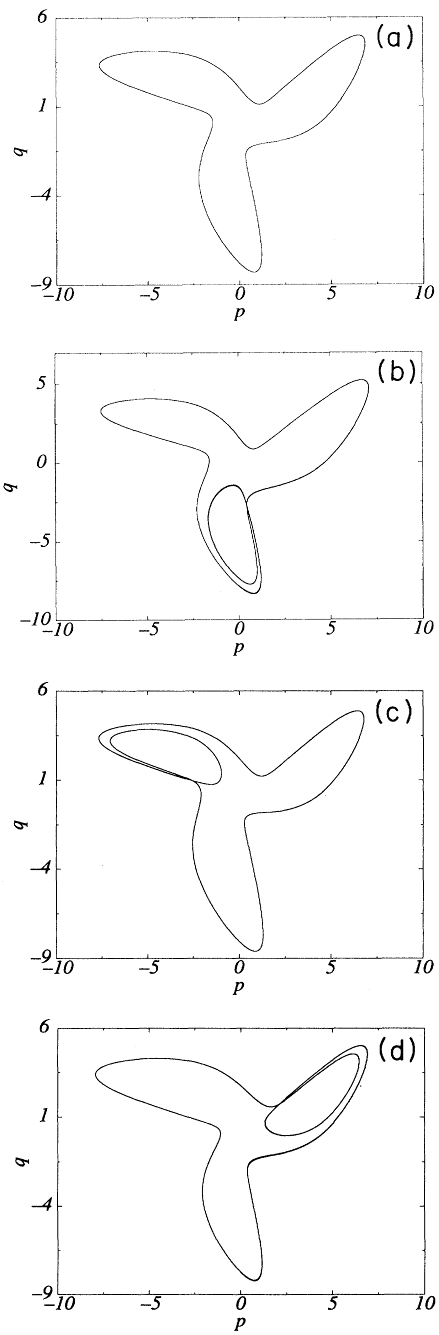

FIG. 5. Examples of a symmetric orbit and of a triplet of asymmetric orbits extracted from the triple cover $\mathcal{C}_3$ of the proto-Lorenz system. (a) Symmetric orbit encoded (1), (b) asymmetric orbit encoded (1011), and its symmetric configurations, (c) and (d).

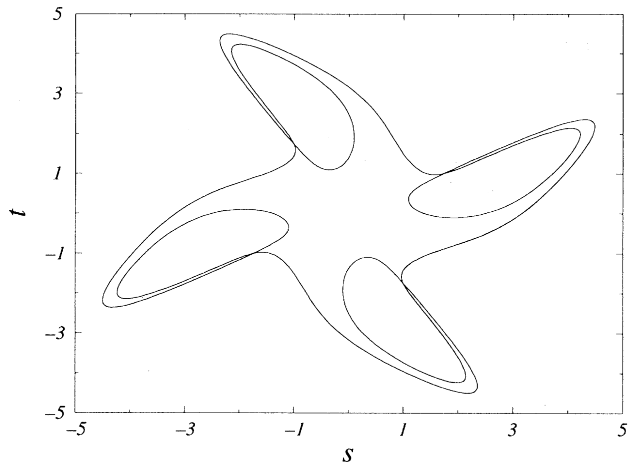

FIG. 6. Symmetric orbit encoded by (10) extracted from the quartic cover $\mathcal{C}_4$.

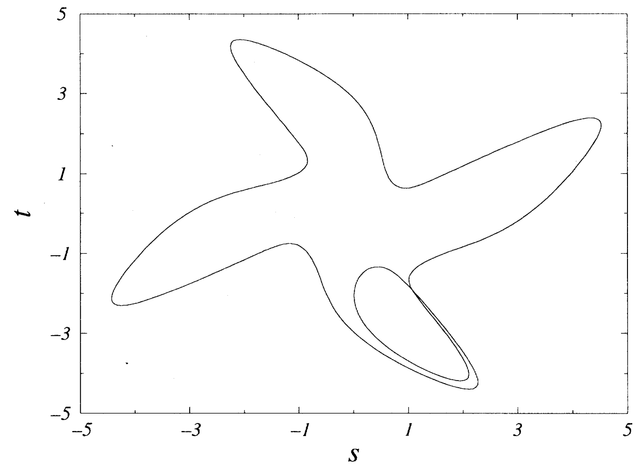

FIG. 8. Asymmetric orbit encoded by (10111) whose degeneracy is equal to 4 (from $\mathcal{C}_4$).

corresponding to the periodic points visited in the jth Poincaré section. Periodic points that are not visited in a Poincaré section P_j are visited in different symmetric configurations $\gamma^j\xi$ of the orbit ξ. Also, each symmetric configuration of orbit ξ visits a different subset of periodic points in the Poincaré sections; i.e., each copy of ξ may be associated one-to-one with each subset. Such an orbit ξ of time period T_ξ satisfies

$$\{\boldsymbol{Y}_{n\xi}(t)\}_{t=0}^{T_\xi} \cap \{\gamma_n^j \boldsymbol{Y}_{n\xi}(t)\}_{t=0}^{T_\xi} = \emptyset \qquad \forall j \in [1, n-1] ,$$

where $\boldsymbol{Y}_{n\xi}(t)$ are the coordinate vectors at time t of the periodic orbit ξ in the state space.

Let us now consider an orbit ζ of period p with a transition number $\mathcal{T}_W$ such as there exists an integer $m \neq 1$, which is a divisor of n less than n. The orbit describes therefore m times its symbolic sequence before returning to its initial conditions. Such an orbit is asymmetric since it visits the fundamental domain and its copies in a different way. Following the same arguments as in the previous case, $\frac{n}{m}$ subsets of periodic points may be distinguished in the Poincaré set. The orbit ζ is then degenerated $\frac{n}{m}$ times. For instance, the orbit (101) of the quartic cover ($n = 4$) has a transition number $\mathcal{T}_{(101)}$ which is equal to 2. m is therefore equal to 2 and the orbit is degenerated twice; i.e., two symmetric configurations appear in the state space. Such an orbit satisfies

$$\{\boldsymbol{Y}_{n\zeta}(t)\}_{t=0}^{T_\zeta} \cap \{\gamma_n^j \boldsymbol{Y}_{n\zeta}(t)\}_{t=0}^{T_\zeta} = \emptyset \qquad \forall j \in [1, m-1]$$

and

$$\{\boldsymbol{Y}_{n\zeta}(t)\}_{t=0}^{T_\zeta} = \{\gamma_n^m \boldsymbol{Y}_{n\zeta}(t)\}_{t=0}^{T_\zeta} .$$

Consequently, there exists one kind of asymmetric orbit with its specific degeneracy for each divisor of n.

Lastly, let us consider an orbit η of period p whose transition number $\mathcal{T}_W$ satisfies $\mathrm{mod}(m\mathcal{T}_W, n) = 0$ with $m = n$. $\mathrm{mod}(\mathcal{T}_W, n)$ is therefore not a divisor of n. The orbit η describes n times its symbolic sequence before returning to its initial conditions. Each periodic point is therefore visited by the orbit η within each Poincaré section before returning to its initial conditions. The orbit η satisfies

$$\{\boldsymbol{Y}_{n\eta}(t)\}_{t=0}^{T_\eta} = \{\gamma_n^j \boldsymbol{Y}_{n\eta}(t)\}_{t=0}^{T_\eta} \qquad \forall j \in [1, n-1]$$

and is consequently a symmetric one.

In summary, there exists one kind of periodic orbit associated with each divisor of n. The product $m \times k$ where k is the degeneracy is always equal to n. Consequently, the symmetry properties of an orbit may be easily found from its symbolic sequence.

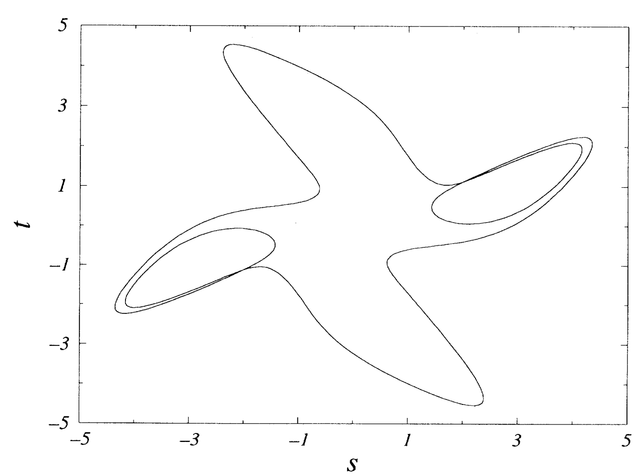

FIG. 7. Asymmetric orbit encoded by (101) whose degeneracy is equal to 2 (from $\mathcal{C}_4$).

C. Fundamental linking numbers

For any system whose equivariance defines a rotation around an axis by $\frac{\pm 2\pi}{n}$, the precise definition of the fundamental domain and of its copies is not required to evaluate fundamental linking numbers. Indeed, fundamental linking numbers $\mathcal{L}_n(N_i, N_j)$ between orbits N_i and N_j

are simply defined as follows:

$$\mathcal{L}_n(N_i,N_j) = \frac{1}{n}L(N_i,N_j) = \frac{1}{2n}\sum_p \epsilon_{ij}(p) \qquad (i \neq j) \ , \tag{13}$$

where $\epsilon_{ij}(p)$ are the numbers associated with oriented crossings between N_i and N_j on a regular plane projection. In this projection, symbolic sequences of orbit must appear n times; i.e., complete symmetric orbits or the k configurations of asymmetric orbits degenerated k times must be projected.

Such fundamental linking numbers are then equal to the linking numbers predicted by the template induced by the fundamental domain.

D. Template

The symmetry induces specific considerations such as defining a fundamental domain $\mathcal{D}$, which tasselates the complete state space [10]. Indeed, when a dynamical system is invariant under the action of γ_n, the state space can be tiled by a fundamental domain $\mathcal{D}$ and its copy $\gamma_n^i\mathcal{D}$ (note that $\gamma_n^n\mathcal{D} = \mathcal{D}$). For instance, in the Lorenz attractor, the fundamental domain may be viewed as a wing [11]. It has been shown that the dynamical analysis of such a system must be achieved by working in the fundamental domain in which the trajectory is projected [10,11].

Then, let us recommend building a mask of the attractor projected in the plane perpendicular to the rotation axis. In this plane, copies of the fundamental domain can be easily defined by identifying the angle $\frac{2\pi}{n}$ of the rotation induced by γ_n. The proposed template must be checked by comparison between linking numbers predicted by the template induced by the fundamental domain and the fundamental linking numbers computed from projection of orbit pairs in the plane perpendicular to the rotation axis.

V. THE QUARTIC COVER $\mathcal{C}_4$ OF THE PROTO-LORENZ SYSTEM

We now exemplify the application of the general procedure by characterizing a 4-order equivariant system, i.e., on the quartic cover $\mathcal{C}_4$ of the proto-Lorenz system.

The vector field of the quartic cover $\mathcal{C}_4$ of the proto-Lorenz system is written as

$$\left(\begin{array}{l} \dot{r} = \dfrac{\left[-\sigma r^3 + (2\sigma + R - z)r^2 s + (\sigma - 2)rs^2 - (R-z)s^3\right]}{2\left(r^2+s^2\right)} \\ \dot{s} = \dfrac{\left[R - z)r^3 + (\sigma - 2)r^2 s + (-2\sigma - R + z)rs^2 - \sigma s^3\right]}{2\left(r^2+s^2\right)} \\ \dot{z} = 2r^3 s - 2rs^3 - bz \end{array} \right) . \tag{14}$$

This vector field presents five fixed points that are the origin $F_0(0,0,0)$ and four fixed points F_j defined as

$$F_1 = \begin{pmatrix} r_F \\ s_F \\ z_F \end{pmatrix}, \quad F_2 = \begin{pmatrix} r_F \\ -s_F \\ z_F \end{pmatrix},$$

$$F_3 = \begin{pmatrix} -r_F \\ -s_F \\ z_F \end{pmatrix} \quad F_4 = \begin{pmatrix} -r_F \\ s_F \\ z_F \end{pmatrix},$$

where

$$r_F = 3.200\,412\,582,$$
$$s_F = 1.325\,642\,97,$$
$$z_F = 27.0$$

for $(R,\sigma,b) = (28, 10, 8/3)$. One may remark that these four fixed points F_j have a z coordinate equal to the z coordinate of the fixed points $F_\pm$ of the Lorenz attractor, i.e., $z_F = R - 1$. Moreover, if we use the complex coordinate $w_4 = r + is$, we obtain that fixed points F_j are defined by

$$F_j = \begin{pmatrix} \rho e^{i\left[\theta_F - (j-1)\frac{\pi}{2}\right]} \\ z_F \end{pmatrix} , \tag{15}$$

where $\rho = \sqrt{r_F^2 + s_F^2}$ and $\theta_F = \tan^{-1}(\frac{s_F}{r_F})$. Consequently, we may easily check that fixed points F_j are connected by the recurrence rule (9) for $n = 4$ where

$$\gamma_4 \equiv \begin{pmatrix} \cos\theta_4 & -\sin\theta_4 & 0 \\ \sin\theta_4 & \cos\theta_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{16}$$

with $\theta_4 = -\frac{\pi}{2}$. One may also check that $\gamma_4^4 = I$. The

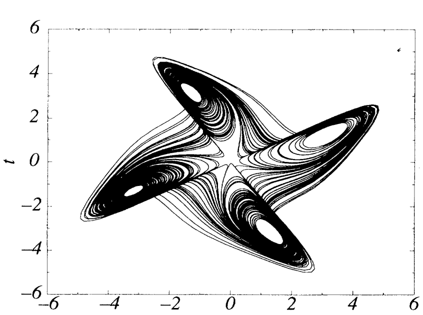

FIG. 9. The quartic cover $\mathcal{C}_4$ of the proto-Lorenz system.

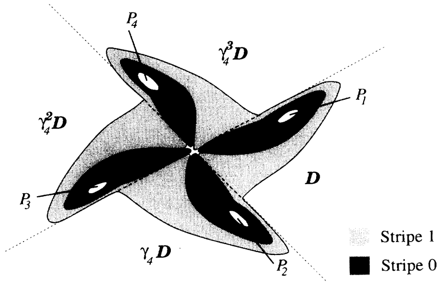

FIG. 10. Mask of the quartic cover $\mathcal{C}_4$ of the proto-Lorenz system.

vector field $\mathcal{C}_4$ is therefore equivariant under the action of the γ_4 matrix and the equivariance order is 4.

By integrating numerically the vector field $\mathcal{C}_4$, we obtain the attractor displayed in Fig. 9. As this attractor is globally invariant under the action of the γ_4 matrix, it may be tasselated by a fundamental domain $\mathcal{D}$ and three of its copies. A mask is then built with a schematic view of the fundamental domain and of its copies (Fig. 10).

We use the Poincaré set P constituted by four Poincaré sections P_j defined as

$$P_j = \left\{ (w_4, z) \in \mathbb{C} \times \mathbb{R} \mid \theta = \theta_F - (j-1)\frac{\pi}{2}, \dot{\theta} < 0 \right\} , \tag{17}$$

where $w_4 = r + is$ is a complex number whose θ is the argument. A first-return map to this Poincaré set is built with the invariant variable ρ. This map is similar to the map of the proto-Lorenz system displayed in Fig. 2. Two monotonic branches are exhibited and are associated with the two stripes of the mask displayed in Fig. 10.

The periodic orbits are then extracted and encoded by the symbolic dynamics defined as

$$K(\rho_n) = \begin{cases} 0 & \text{if} \quad \rho < \rho_c, \\ 1 & \text{if} \quad \rho > \rho_c, \end{cases} \tag{18}$$

where $\rho_c = 4.22$. The population of periodic orbits is found to be the same as on the proto-Lorenz system or as on the Lorenz system [11].

As an example, we check the proposed template with the pair of orbits encoded as (1,10). Eight positive crossings are counted on the plane projection of the couple (1,10) in Fig. 11. The fundamental linking number $\mathcal{L}_4(1,10)$ is therefore equal to $\frac{1}{8}(+8) = +1$, which is indeed equal to the linking number $L(1,10)$ predicted by the template induced by the proto-Lorenz system.

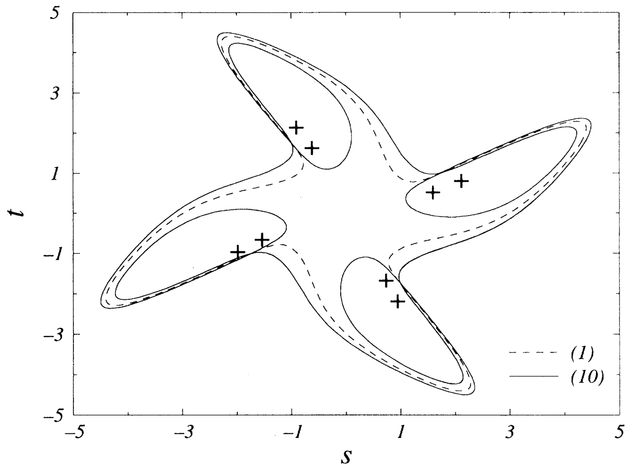

FIG. 11. Plane projection of the couple (1,10): $\mathcal{L}_4 = \frac{1}{8}(+8) = +1$.

VI. CONCLUSION

Topological characterization of n-equivariant vector fields is solved in the case where the equivariance defines an invariance under a rotation by $\frac{2\pi}{n}$, in modulus. It is shown that a fundamental domain is conveniently defined. The fundamental domain and its $(n-1)$ copies completely tasselate the attractor. In order to determine the partition of the attractor, we showed that a Poincaré set constituted by n Poincaré sections (one per copy of the fundamental domain) is required. The first-return map to this Poincaré set is thereafter built with an invariant variable (given by the modulus of the complex coordinate defined on the plane perpendicular to the rotation axis). A fundamental linking number is also introduced to check the linking numbers predicted by the template induced by the fundamental domain.

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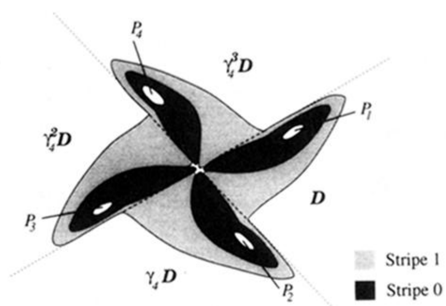

FIG. 10. Mask of the quartic cover $\mathcal{C}_4$ of the proto-Lorenz system.

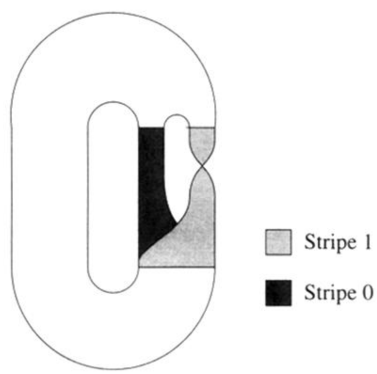

FIG. 3. Template of the proto-Lorenz system.