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Approximately conserved quantity in the Hénon-Heiles problem

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A possible relation between an approximately conserved quantity in the Hénon-Heiles problem and the integrals of the Toda lattice is investigated. A relation good to fifth order exactly and seventh order when averaged is proposed. A form of averaging appears to be significant in the successful construction of approximately conserved quantities.

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I. INTRODUCTION

The Hénon-Heiles problem [1] represents the third order truncation of all three-particle systems interacting with a next-neighbor central potential. The Toda lattice [2] is an n-particle, next-neighbor interacting system and is completely integrable [3]. It has been shown that starting from the third order truncation of the three-particle Toda lattice, the Hénon-Heiles problem, its truncations are not integrable [4]. However, the Hénon-Heiles problem (henceforth referred to as the HH problem) shows near-regular behavior for a relatively large range of values of the Hamiltonian, $0 \leq H \leq 0.11$, and hence admits a second approximately conserved quantity. Gustavson [5] developed a power series for such a quantity through normal form transformations. Recently, Finkler *et al.* [6] have constructed a power series for a quasiconserved quantity K in the HH problem. This series (hereafter referred to as the FJS series) shows approximate conservation for $H \leq 0.11$, but behaves as an asymptotic series and possesses possible convergence problems, as discussed in Ref. [6]. Contopoulos and Polymilis [7] have numerically studied the integrability of various truncations of the three-particle Toda lattice. This and the work of Yoshida *et al.* [4] imply that although the truncations are not integrable, a seemingly integrable behavior for a considerable range of the values of the Hamiltonian can be observed.

The aim of this work is to understand the mechanism by which an approximately conserved quantity can exist for the HH problem. For this reason, the Liapunov spectrum for both the integrable Toda lattice and the nonintegrable HH problem has been calculated. The results confirm the Poincaré surface of section analysis that for $0 \leq H \leq 0.11$ the system shows an orderly transition to chaotic behavior in accordance with the Kolmogorov-Arnold-Maser (KAM) theorem. The frequency spectrum of the HH system has been studied, and a two-period spectrum is observed for $0 \leq H \leq 0.11$.

Since the HH problem is the third order truncation of the three-particle Toda problem, it is worthwhile to investigate a possible connection between the approximately conserved infinite series proposed by Finkler *et al.* and truncations of the third integral of the Toda problem. We have observed that this third integral or its truncations do not exactly coincide with the FJS series. Truncations of the third Toda integral function as an HH integral for up to the fourth order in the dynamical variables of the problem. It can also be shown that linear combinations of the square of the third integral and the Toda Hamiltonian give the FJS series up to and including the fifth order. Higher order terms of the FJS series do not coincide with any combination of the two integrals of the Toda problem (the Hamiltonian and the third integral).

The fact that, in the range $0 \leq H \leq 0.11$, an approximately conserved quantity can be constructed makes it worthwhile to study the averaged version of the HH system over its larger frequency. Under these conditions, it is shown that the FJS series (with the exception of the fourth order term) average to zero. A combination of the Toda Hamiltonian and the Toda third integral agrees with the FJS series up to and including the seventh order when averaged.

In Sec. II, we present our numerical results concerning the Liapunov spectra and the periodicity. Section III is devoted to the details of the calculations involving the relations between the FJS series and Toda integrals. A discussion and interpretation of the results are presented in Sec. IV.

II. LIAPUNOV AND FREQUENCY SPECTRA OF THE HÉNON-HEILES SYSTEM

The three-particle Toda system has the following Hamiltonian:

$$H_T = \frac{p_1^2 + p_2^2 + p_3^2}{2} + e^{-(q_1 - q_2)} + e^{-(q_2 - q_3)} + e^{-(q_3 - q_1)}, \tag{2.1}$$

with the three integrals

$$I_1 = p_1 + p_2 + p_3, \tag{2.2}$$

$$I_2 = H_T, \tag{2.3}$$

$$I_3 = p_1 p_2 p_3 - p_1 e^{-(q_2 - q_3)} - p_2 e^{-(q_3 - q_1)} - p_3 e^{-(q_1 - q_2)}. \tag{2.4}$$

If one switches to a new coordinate system by letting

$$q_1 = \frac{2\sqrt{6}}{3}x_{cm} + \frac{2\sqrt{3}}{3}x + 2y, \tag{2.5}$$

$$q_2 = \frac{2\sqrt{6}}{3}x_{cm} - \frac{4\sqrt{3}}{3}x, \tag{2.6}$$

$$q_3 = \frac{2\sqrt{6}}{3}x_{cm} + \frac{2\sqrt{3}}{3}x - 2y, \tag{2.7}$$

I_1 is trivially satisfied. With the scaling $d\tau = \sqrt{3}dt$ and $H \rightarrow H/24$, one gets

$$H_T = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{24}(e^{(2y+2\sqrt{3}x)} + e^{(2y-2\sqrt{3}x)} + e^{-4y}) - \frac{1}{8}. \tag{2.8}$$

A third order truncation of H_T gives the HH Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - y^3/3. \tag{2.9}$$

It has been observed that the Wiesel [8] procedure gives the most reliable performance in the calculation of Liapunov exponents for the two systems (Toda and HH) that we have studied. The derivation of the equations from the dynamical system has been automated by the use of the GENTRAN and SCOPE [9] packages in REDUCE [10]. Numerical integrations have been carried out using a Bulirsch-Stoer type integrator.

We first consider the Liapunov spectrum of the Toda lattice. Since this is a Hamiltonian system, the Liapunov exponents should add up to zero. Furthermore, it is a completely integrable system, which tells us that it does not have any chaotic behavior; hence it does not have any positive Liapunov exponent. Thus we are faced with the fact that all four Liapunov exponents of the system are exactly zero. Bearing this in mind, we examined our result concerning the spectrum. The result is presented in Fig. 1 for different ranges of the value of the Hamiltonian, where we see the four Liapunov exponents versus the value of the Hamiltonian of the system.

It is apparent that, when such an iterative calculation is made using computer simulations, the definition of *zero* should be revised, basically because

(i) Liapunov exponents that are close to zero are difficult to calculate, as the local behavior of the system dominates the overall performance.

(ii) The floating point truncation error during the calculations adds numerical noise.

(iii) The numerical integration procedure involves an effective series truncation through both the discretization in the numerical integrator and the handling of the floating point by the computer. Since any truncation of the Toda lattice above the second order is nonintegrable, we have not been able to observe an appreciable improvement in the departure of the Liapunov exponents from zero, neither through increase of floating point precision, nor by the use of higher order integrals.

Looking at the results of Fig. 1, we can say that any λ value below 0.03 should be considered as zero. Keeping this in mind, we could say that the Liapunov spectrum of the HH system (Fig. 2) consists of zeros for $H < 0.11$. This spectrum is an indication that a nearly conserved quantity for $H < 0.11$, where all Liapunov exponents are compatible with zero as specified above, should exist.

Next, we consider the frequency spectrum of the HH system for various H values (Fig. 3). We can identify a two-periodicity for the system when the value of its Hamiltonian is less than 0.11, though as we approach this upper limit, deformation of this nature gradually begins; for $H > 0.11$, the spectrum becomes continuous, which is yet another indication that we should expect a nearly conserved quantity other than its Hamiltonian, for $H < 0.11$. We observe that the relative position of the larger frequency is stable, while that of the smaller frequency shifts slowly, and new frequencies emerge as the value of the Hamiltonian increases.

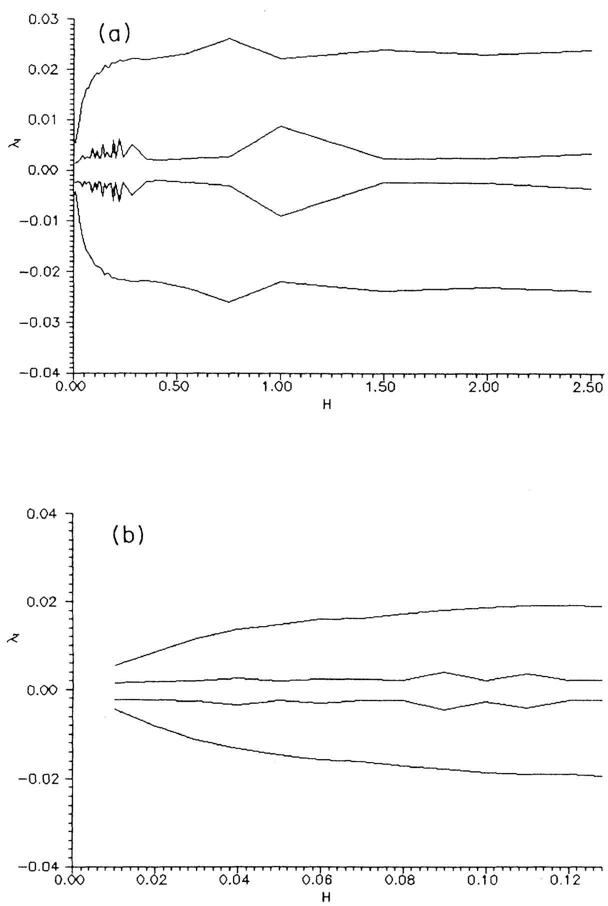

FIG. 1. (a) Liapunov spectrum of the Toda system vs the value of the Hamiltonian. Deviation of the numerical values from the theoretical values of zero reflects the numerical sensitivity of the computation. (b) A closer look at the interval [0, 0.12].

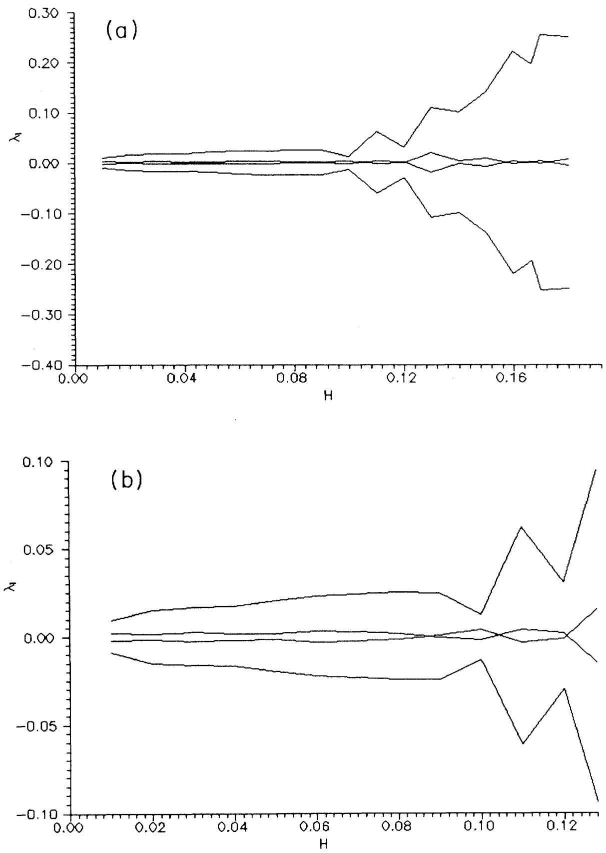

FIG. 2. (a) Liapunov spectrum of the HH system vs the value of the Hamiltonian. There is a clear passage from a regular characteristic to a chaotic one near the value $H \sim 0.11$. (b) A closer look at the interval $[0, 0.12]$. Comparing it with Fig. 1, one can say that the spectrum consists of zeros on the interval $[0, 0.11)$ vaguely. Units in this figure are in pixels. The pixel/mm ratio is indicated in the text.

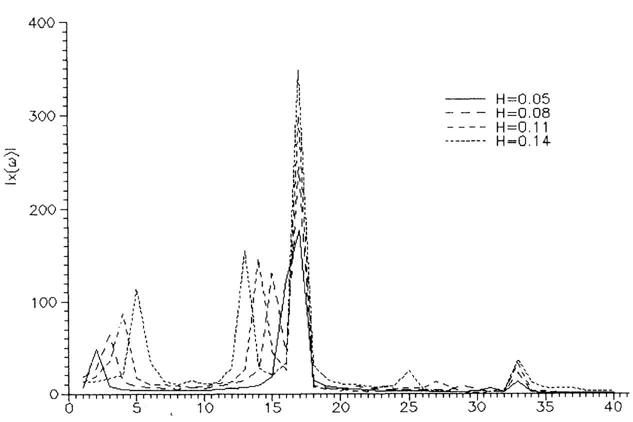

FIG. 3. Frequency spectrum of the HH dynamics for various values of the Hamiltonian. There is a fixed peak at about $w = 17$, and new harmonics are generated while the lower peak slides to higher frequencies as the value of the Hamiltonian increases.

III. FJS SERIES AND THE TRUNCATED INTEGRALS OF TODA LATTICE

Since the HH problem is the third order truncation of the exactly integrable three-particle Toda lattice, it is natural to expect a relation between the approximately conserved FJS series and the Toda integrals. The simplest ansatz of seeking a relation connecting the FJS series directly to the series expansion of the third integral

$$I_3 = -\frac{2}{3}p_x(p_x^2 - 3p_y^2) - \frac{1}{24}[(2p_x - 2\sqrt{3}p_y)e^{(2y+2\sqrt{3}x)} + (2p_x + 2\sqrt{3}p_y)e^{(2y-2\sqrt{3}x)} - 4p_x e^{-4y}] \tag{3.1}$$

is impossible, since the FJS series starts with fourth order terms in the dynamical variables, whereas it is apparent from (3.1) that the series expansion of I_3 starts with second order terms. It can be immediately verified that the FJS series, which has a power series expansion starting from fourth order terms, is not also the same as the square of the third integral. However, it is worth noting here that the time derivative of I_3 can be shown to start with fifth order terms, so that it is a good integral up to and including the fourth order, in spite of the fact that the HH problem is a third order truncation of the Toda problem.

Another candidate for an approximate integral is the Toda Hamiltonian (2.1), which also expands to a power series starting with second order terms, and possesses a fourth order time derivative as expected. Since the second order terms in the series expansions of H_T and I_3 are linearly independent, a linear combination of the two cannot be annihilated even to achieve a series starting from the third order. The fact that H_T and I_3 start with the second order terms is a motivation for trying a linear combination of the squares of these two in order to form an expression agreeing with the FJS series in its fourth order term. The combination

$$N_{45} = \alpha H_T^2 + \beta I_3^2 \tag{3.2}$$

with $\alpha = -5/12$ and $\beta = 7/12$, when expanded to fifth order, agrees with the FJS series not only in the fourth but also in the fifth order terms. Nevertheless, it is not possible to push this agreement to higher powers by adding higher powers of H_T and I_3, or cross terms involving the two to N_{45}.

The two-period property of the HH problem, as well as the fact that the higher angular frequency is nearly 1 for $H < 0.11$, suggest averaging over this frequency. Thus we introduce

$$x = u_x \cos t + v_x \sin t, \tag{3.3}$$

$$y = u_y \cos t + v_y \sin t, \tag{3.4}$$

and perform a first order averaging. One obtains, using the seventh order truncations, with $\langle\ \rangle$ denoting averaging,

$$\langle K \rangle = \langle K_4 \rangle, \tag{3.5}$$

$$\langle H_T \rangle^2 = (u_x^2 + u_y^2 + v_x^2 + v_y^2)^2, \tag{3.6}$$

$$\langle I_3\rangle^2 = (u_x v_y - u_y v_x)^2, \tag{3.7}$$

$$\langle N_{45}\rangle = \alpha\langle H_T\rangle^2 + \beta\langle I_3\rangle^2, \tag{3.8}$$

and agreement between the averaged forms of the FJS series and the linear combination of $\langle H_T\rangle^2$ and $\langle I_3\rangle^2$ is thus maintained up to and including the seventh order terms.

It is actually possible to show that the averages of all terms in the FJS series are zero, with the exception of K_4, i.e., (3.5) holds for the infinite series K.

To this end, we apply (3.3) and (3.4) to the generator operator $D_0^{-1}D_1$ of Finkler *et al.*, which is used to obtain K_{n+1} from K_n, where

$$D_0 = i\left(\beta\frac{\partial}{\partial\beta} + \beta^*\frac{\partial}{\partial\beta^*} + \bar\beta\frac{\partial}{\partial\bar\beta} + \bar\beta^*\frac{\partial}{\partial\bar\beta^*}\right), \tag{3.9}$$

$$D_1 = \frac{1}{4}\left((\beta^* + \bar\beta^*)^2\frac{\partial}{\partial\beta} + (\beta+\bar\beta)^2\frac{\partial}{\partial\bar\beta^*} - (\beta+\bar\beta)^2\frac{\partial}{\partial\beta^*} - (\beta^*+\bar\beta^*)^2\frac{\partial}{\partial\bar\beta}\right) \tag{3.10}$$

and

$$\beta = (x+iy) + i(\dot x + i\dot y), \tag{3.11}$$

$$\beta^* = (x-iy) + i(\dot x - i\dot y), \tag{3.12}$$

$$\bar\beta = (x+iy) - i(\dot x + i\dot y), \tag{3.13}$$

$$\bar\beta^* = (x-iy) - i(\dot x - i\dot y), \tag{3.14}$$

as described in [6]. Note that the operator D_0 does not change the order of the polynomial terms it is applied to.

Now, consider the series

$$S = \sum_{n>0}(a_n\cos nt + b_n\sin nt), \tag{3.15}$$

which is the most general form with $\langle S\rangle = 0$, where a_n and b_n are polynomials in u_x, u_y, v_x, and v_y, and apply the operator D_1. A direct calculation shows

$$D_1 S = \sum_{n>0}(\hat a_n\cos nt + \hat b_n\sin nt), \tag{3.16}$$

$\hat a_n$ and $\hat b_n$ again being polynomials in u_x, u_y, v_x, and v_y, and thus $\langle D_1 S\rangle = 0$.

Finally, noting that $\langle K_5\rangle = 0$ permits us to write it in the form given by S, we conclude that

$$\langle K_n\rangle = 0, \quad \forall n \geq 5. \tag{3.17}$$

IV. CONCLUSION

Our study of the Hénon-Heiles problem reveals that, although it is the third order truncation of the three-particle Toda problem, the third integral of Toda is a quasi-conserved quantity of the HH system up to the fifth order.

In the regime where the FJS series stays more or less constant, the HH system shows a two-period behavior. Thus, making an averaging over the higher frequency is expected to increase the agreement between the FJS series and N_{45}. Indeed the agreement is pushed up by two orders, from the fifth to the seventh.

The fourth and the fifth order terms of the FJS series can be written in terms of series truncations of Toda integrals. The fifth and higher order terms average to zero. These two observations make clear the reason why the FJS series works so well. In the argument presented by Yoshida *et al.* [4], the existence of a nearly conserved quantity can be explained by the approach of the Hénon-Heiles system to the integrable harmonic oscillator form in the low energy limit. This limit corresponds to averaging in a first approximation.

The averaged system possesses two derived quantities, $\langle H_T\rangle$ and $\langle I_3\rangle$ as given by Eqs. (3.6) and (3.7), respectively, in which we can describe both of the average integrals. This shows that averaging is an important consideration in the successful construction of an approximately conserved quantity in the Hénon-Heiles problem. However, the elementary averaging that has been employed here ignores the slow frequency shift in the lower frequency as well as higher harmonic generation as the value of the Hamiltonian increases. This is consistent with the slowly varying function assumption inherent in averaging. Further work on taking into account the possible effects of this assumption and the effects of the choice of integrator, step size, and reorthonormalization interval on the accuracy of the result is in progress.

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