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Noise-induced stabilization of the Lorenz system

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The flip-flop process of the Lorenz system is described as a symbolic dynamical system, studied in the presence of noise, and quantified with a specific complexity measure. For small noise amplitudes the perturbation results in a generation of structure within the space of symbol sequences (discretized trajectories), corresponding to a stabilization of the flip-flop process. This noise-induced stabilization effect is investigated for different noise amplitudes, particularly in the case where noise is added locally.

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INTRODUCTION

The behavior of nonlinear dynamical systems in the presence of small perturbations and noise has been the object of numerous and extensive studies since it has become clear that nonlinear dynamical systems no longer exhibit a simple relation between "cause and effect." In particular, the dynamics of chaotic systems depends sensitively on tiny perturbations of the initial values. Small perturbations can result in a variety of system behavior, which can be more but also less ordered than the unperturbed one. Perturbations can be added in a controlled (control theory) as well as uncontrolled way as is the case in the presence of noise (noise-induced chaos, stochastic resonance).

The Ott-Grebogi-Yorke (OGY) control mechanism [1,2] uses the fact that chaotic systems entail an infinite number of unstable periodic orbits, which can be used to stabilize the system. A control law specifies the time dependence of a system parameter required to stabilize an unstable periodic orbit in a chaotic system. In a similar way the sensitivity of chaotic systems to small perturbations can be used to direct chaotic trajectories rapidly to a desired state. For the Lorenz system the trajectory comes close to the origin (stationary state) in approximately one in 10^{10} orbits around each of the lobes of the attractor. By adding small perturbations to the model equations, this can be changed to the order of one in 10 orbits [3]. If a neighborhood of the stable manifold of the target state is reached, then the flow will automatically go to the desired target region along the stable manifold.

Not only control but also noise can change the system behavior in different ways. For systems possessing a periodic attractor and being close to a saddle-node bifurcation, a transition to intermittent chaos can be induced by small amounts of external noise [4]. Noise can transiently drive the system away from its stable periodic orbit towards neighboring unstable orbits, which it then follows briefly before returning again. The size of the largest Liapunov exponent as a function of noise intensity demonstrates that the system exhibits *noise-induced chaos.* It is found for several dynamical systems (logistic map [4], Lorenz model [5]). For the Lorenz model it has also been demonstrated [6] that intrinsic fluctuations can be chaotically amplified.

Stochastic resonance is a further remarkable nonlinear phenomenon, in which the signal-to-noise ratio of a periodically modulated, usually bistable system can be amplified by the addition of external noise [7]. Nonlinear cooperative effects between periodic and random perturbations can imply that incoherent noise leads to a coherent output signal (e.g., in ring lasers [8]).

A *suppression* of the sensitive dependence on initial states can result, if Gaussian white noise is added to a system (e.g., the logistic map [9]). Strange but nonchaotic attractors have been found for some ranges of amplitude of the external perturbation [9].

In general, the effects of perturbations can be quite difficult to predict; nevertheless, dramatic changes in the dynamics of chaotic systems have been found. For example, periodic and nearly periodic behavior can sometimes be produced in originally chaotic dynamical systems. In the present paper the influence of noise on the flip-flop process associated with the Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$), which is described as a symbolic dynamical system, is studied and characterized by a specific complexity measure (fluctuation complexity). For small noise levels the perturbation results in a stabilization of the flip-flop process in the sense that the number of successive rotations of a trajectory on the same lobe is significantly increased. It turns out that this stabilization effect is dominated by the influence of noise on those parts of the attractor where the maximum divergence rate is positive.

THE NOISY LORENZ SYSTEM

The Lorenz system [10] is a three-dimensional continuous dynamical system $[\mathbb{R}^3 \to \mathbb{R}^3, \mathbf{x} \mapsto \mathbf{F}(\mathbf{x})]$, which exhibits chaotic dynamics for certain parameter values:

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2 \ , \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3 \ , \\ \dot{x}_3 &= -b x_3 + x_1 x_2 \ . \end{aligned} \tag{1}$$

For the parameter set $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$, which is considered in this paper, the corresponding dynamics is characterized by a strange attractor with two unstable fixed points in addition to the origin [10,11]. A typical trajectory circulates away from the unstable fixed point on one lobe and switches to the other lobe after an apparently random number of rotations. The process corresponding to the change of lobes is called a *flip-flop process* [12].

This flip-flop process can be described as a *symbolic dynamical system.* The three-dimensional state space is partitioned according to the two lobes, which are represented by the symbols 0 and 1. Then each trajectory can be mapped to a unique symbol sequence $S = \{s_i\}_{i=0}^{\infty}$, such that a symbol 0 (1) is assigned to the symbol sequence if the trajectory circulates around the unstable fixed point $x_1 < 0$ ($x_1 > 0$). In other words, if a local maximum in the x_3 component corresponds to $x_1 < 0$, then $s_i = 0$, otherwise $s_i = 1$.

The corresponding symbolic dynamical system F is defined as [13]

$$\Sigma_F \to \Sigma_F, \quad S \mapsto \hat{\sigma}_F(S) = S' \ , \tag{2}$$

such that each symbol in the sequence $S = \{s_i\}_{i=0}^{\infty}$ satisfies the condition $s_{i+1} = s_i' = \hat{\sigma}_F(s_i)$. Σ_F is the space of all admissible symbol sequences. Admissible sequences are sequences that are induced by the dynamics of the system F (Lorenz system) for all initial states $\mathbf{x}_0$ on the attractor at time step $i = 0$. The operator $\hat{\sigma}_F$ is called the *shift operator on* Σ_F, and describes the dynamics generated by F in the space of symbol sequences Σ_F. The length L of a symbol sequence S is defined by $S = \{s_i\}_{i=0}^{L-1}$. In principle, the theory of symbolic dynamical systems deals with sequences of infinite length ($L = \infty$). For practical purposes, however, L is often regarded as a finite number.

The space of admissible symbol sequences Σ_F of the symbolic dynamical system is partitioned. The so-called *n-cylinder-induced partition* $P_n = \{A_{i,n}\}_{i=1}^{N}$ consists of all (N) words of length n, which appear in the sequence S. A word $A_{i,n}$ of length n is a subsequence of S, such that $A_{i,n} \in \{s_k s_{k+1} \cdots s_{k+n-1} | k = 0, 1, \ldots, L-n\}$. The set of admissible words P_n represents the set of all trajectories or sequences which coincide in the first n successive symbols. For the Lorenz system and the considered word length $n = 1, 2, \ldots, 9$ all words of length n which can be generated from a binary alphabet appear in the corresponding symbol sequences. All possible words are admissible; the number of words N increases with n as 2^n. A word $A_{i,n}$ is also called a state, and n is called the order of refinement of the partition P_n.

Because the system dynamics is characterized by a complexity measure related to the concept of information, a number of probabilistic quantities used to characterize the given states $A_{i,n}$ are required. The *state probability* p_i is the probability that a given word $A_{i,n}$ of length n appears in the symbol sequence S. The joint probability p_{ij} describes the probability that the system goes from $A_{i,n}$ to $A_{j,n}$ in two successive time steps. Using the joint probability p_{ij} the transition probabilities $p_{i\to j}$ are defined as conditional probabilities for the transition from a given state $A_{i,n}$ to a successive state $A_{j,n}$: $p_{i\to j} = p_{ij}/p_i$.

The dynamics of the flip-flop process is quantified by the fluctuation in net information gain (fluctuation complexity) σ_Γ^2, which has been introduced as a complexity measure by Bates and Shepard [14]. In [15,16] it has been found that σ_Γ^2 is a sensitive measure to particular types of dynamical behavior.

The net information gain Γ_{ij} of the system state A_i is defined as the difference of information gain $g_{ij} = -\log_2 p_{i\to j}$ and information loss $l_{ij} = -\log_2 p_{i\leftarrow j}$, whereby $p_{i\leftarrow j} = p_{ij}/p_j$. Information gain represents the information required to select a state A_j, if its preceding state A_i is given. Information loss determines the information a system has lost about a preceding state A_i after it has entered the successive state A_j. During the time evolution of a dynamical system the net information gain vanishes on average, $\sum_{ij} \Gamma_{ij} = 0$. However, the system can locally generate or store information. Therefore the net information gain may fluctuate about its mean value and yield a nonvanishing mean square deviation σ_Γ^2.

$$\sigma_\Gamma^2 = \langle \Gamma^2 \rangle - \langle \Gamma \rangle^2 = \sum_{i,j} p_{ij} \left(\log_2 \frac{p_i}{p_j} \right)^2 \ . \tag{3}$$

The quantity is denoted as the fluctuation complexity. It vanishes for purely regular behavior as well as for purely stochastic behavior, because in both cases the distribution of state probabilities is homogeneous. σ_Γ^2 is positive in between.

External noise

The Lorenz equations ($\sigma = 10$, $b = \frac{8}{3}$, $r = 28$) are integrated for 19 630 time units. This corresponds to about 26 300 rotations of the trajectory around the two lobes, or to 26 300 local maxima in the x_3 time series. Thus a typical trajectory is assigned to a symbol sequence of length $L = 26\,300$, which is based on the binary alphabet described above.

In the following the fluctuation complexity σ_Γ^2 of this symbol sequence is compared to the fluctuation complexity σ_Γ^2 of symbol sequences which are derived from the noisy Lorenz system. The Lorenz equations (1) are perturbed by an amplitude-dependent noise term ξ, which is added in the first component, $x_1 \to x_1 + \xi$, where ξ is defined in the following way:

$$\xi = \sqrt{|x_1|}\xi_z \ . \tag{4}$$

ξ_z is a random number, equally distributed in the interval $\xi_z \in [-\xi_0, \xi_0]$, and ξ_0 is called the noise amplitude.

In order to relate the magnitude of the noise term ξ to the amplitude of the unperturbed signal, the signal-to-noise ratio (SNR) is defined as

$$\mathcal{R}_{\mathrm{SNR}} = \frac{\langle |x_1| \rangle_{\xi_0=0}}{\langle |\xi| \rangle_{\xi_0}} = \frac{\langle |x_1| \rangle_{\xi_0=0}}{0.5\xi_0 \langle \sqrt{|x_1|} \rangle_{\xi_0=0}} \ , \tag{5}$$

where $\langle|x_1|\rangle_{\xi_0}$ is the average of the absolute value of the signal x_1 for a given noise amplitude ξ_0. Large (small) values of (SNR) correspond to a small (large) perturbation of the signal. A vanishing perturbation leads to an infinite (SNR). For the considered parameter set one gets $\langle|x_1|\rangle_{\xi_0=0} = 6.4$ and $\langle\sqrt{|x_1|}\rangle_{\xi_0=0} = 2.3$, such that $\mathcal{R}_{\rm SNR} = \frac{5.6}{\xi_0}$.

Throughout this paper the Lorenz system is considered for different noise amplitudes $\xi_0 \in \{0, 0.5, 1, 2, 3, 5, 10\}$, e.g., different signal-to-noise ratios $\mathcal{R}_{\rm SNR} \in \{\infty, 11.2, 5.6, 2.8, 1.9, 0.6\}$. Figures 1(a)–1(d) show how the structure of the unperturbed attractor gets more and more smeared out with increasing noise amplitude, corresponding to a decreasing signal-to-noise ratio. For $\xi_0 = 10$, respectively, $\mathcal{R}_{\rm SNR} = 0.6$ [Fig. 1(d)], the averaged noise signal is already larger than the unperturbed signal and dominates the structure of the corresponding attractor.

Figure 2 shows the influence of noise on the Lorenz system, which is reflected in the corresponding symbol sequences as quantified by fluctuation complexity (σ_Γ^2). In Fig. 2(a) fluctuation complexity is plotted versus the word length n for different noise amplitudes. (Concerning the convergence properties of σ_Γ^2, see [16].) The graphs for the unperturbed and the perturbed system differ drastically.

(i) For $n = 1$ all graphs coincide independently of the noise amplitude ξ_0, and $\sigma_\Gamma^2 = 0$. This reflects that both symbols 0 and 1 appear with equal probability in the symbol sequence. This is caused by the symmetry of the attractor (Lorenz equations), and does not indicate a purely stochastic flip-flop process.

(ii) σ_Γ^2 increases with increasing n for all considered ξ_0 except $\xi_0 = 10$. This reflects the inhomogeneity of the distribution of state probabilities for $n > 1$ and indicates that the flip-flop process is not a totally stochastic independent process.

(iii) For a large noise amplitude $\xi_0 = 10$ (small signal-to-noise ratio) the values of σ_Γ^2 are nearly vanishing independent of n, because in this case the dynamics is dominated by noise.

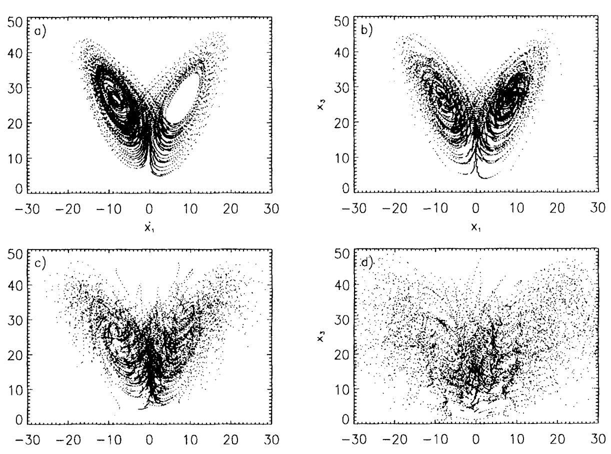

FIG. 1. The attractor of the Lorenz system for different noise amplitudes; the trajectory is plotted in time steps of 0.1. (a) $\xi_0 = 0.5$, (b) $\xi_0 = 1$, (c) $\xi_0 = 3$, and (d) $\xi_0 = 10$.

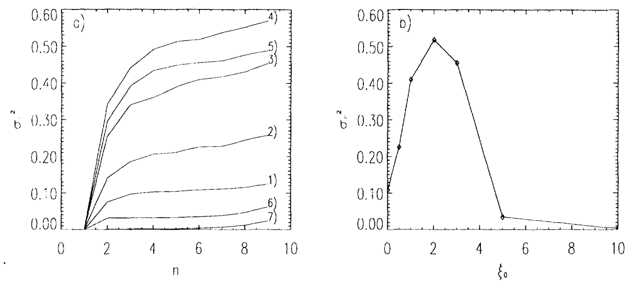

FIG. 2. (a) Fluctuation complexity σ_Γ^2 as a function of word length n for different noise amplitudes ξ_0. The indices $1, 2, \ldots, 7$ correspond to the values $\xi_0 = 0, 0.5, 1, 2, 3, 5, 10$. (b) σ_Γ^2 as a function of noise level ξ_0 for $n = 6$.

(iv) For a given n, $n > 1$, the complexity σ_Γ^2 first increases with increasing noise amplitude until it decreases for $\xi_0 > 2$. For $n = 6$ this behavior is plotted in Fig. 2(b). The reason is that for $\xi_0 \leq 2$ the distribution of state probabilities becomes more inhomogeneous with increasing noise level. Otherwise for $\xi_0 > 2$ the distribution of state probabilities becomes more homogeneous again. In this sense, the degree of inhomogeneity indicates the generation of structure in the Lorenz system by noise.

In order to investigate whether the increasing inhomogeneity in the distribution of state probabilities is caused by an enhancement or by a reduction of the switching of a trajectory between the two lobes, the distribution of state probabilities for different n has to be studied. For this purpose each state $A_{i,n}$ (word of length n) of the partition P_n is assigned to a number ν according to the binary number representation

$$s_k s_{k+1} \cdots s_{k+n-1} \mapsto \nu = s_k \times 2^{n-1} + s_{k+1} \times 2^{n-2} + \cdots + s_{k+n-1} \times 2^0 . \quad (6)$$

ν is normalized by $\nu \mapsto \frac{\nu}{2^n - 1}$ on the unit interval in order to compare words of different length n with regard to their state probability. Then, for instance, the word 111 as well as 1111 is assigned to $\nu = 1$. In Fig. 3 the frequency $N_{i,n}$ of each word $A_{i,n}$ is plotted versus ν for a given n, $n = 1, 5, 9$ for different noise amplitudes (a) $\xi_0 = 0$ and (b) $\xi_0 = 1$.

(i) In both cases (a) and (b) the distribution of state probabilities gets more inhomogeneous with increasing word length n, and the number of words increases with n as 2^n. This reflects the rules (the grammar), which are inherent in the Lorenz system and the corresponding symbol sequences, respectively.

(ii) The symmetry of all curves relative to the axis $\nu = 0.5$ exhibits the symmetry of the flip-flop process relative to the two lobes as mentioned above.

(iii) The most interesting feature with respect to the generation of structure is that the curves for $n > 1$ show maximum values for $\nu = 0, 1, 0.50, 0.25, \ldots$. For $n = 5$ these maxima correspond to the words $00000, 11111, 10000, 01000, \ldots$, e.g., to words with many

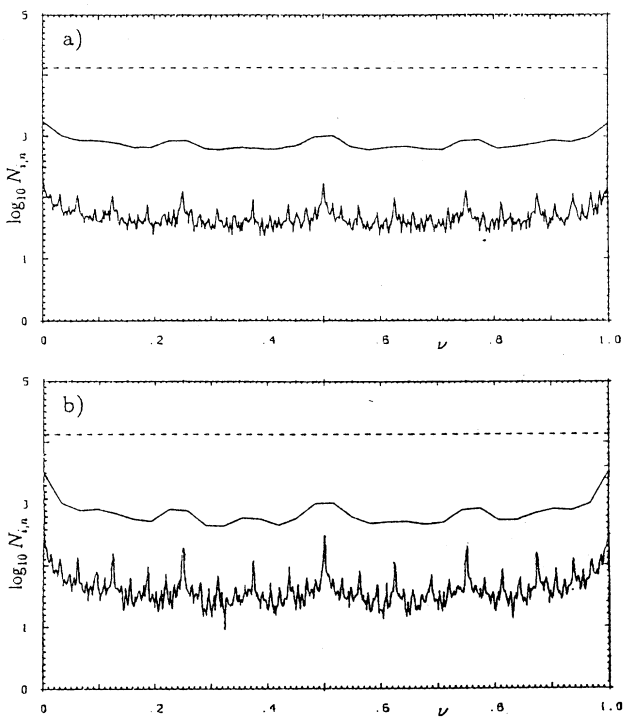

FIG. 3. Frequency distribution $N_{i,n}$ of states $A_{i,n}$ for different refinements n ($n = 1, 5, 9$ from top to bottom), where the states $A_{i,n}$ are characterized by their normalized binary representation ν: (a) $\xi_0 = 0$ and (b) $\xi_0 = 1$.

equal successive symbols. For $\xi_0 = 1$ [Fig. 3(b)] the maximum values are large compared to the unperturbed case $\xi_0 = 0$ [Fig. 3(a)], which again reflects the generation of structure by the influence of noise. Simultaneously this demonstrates that the generated structure corresponds to a stabilization of the flip-flop process, because a typical trajectory stays longer on the same lobe of the attractor. The switching between the two lobes under the influence of noise happens less frequently. This is indicated in Fig. 3(b) by a minimum value for $n = 9$ at $\nu = 0.33$, e.g., for the word 010101010.

Local external noise

In addition to the previous section, where noise is added homogeneously after each integration step on the entire attractor, the influence of *local noise* ($\xi_0 = 1$) on the Lorenz system has been studied. For this purpose the phase space is divided into two regions, Δ_1 and Δ_2 [Fig. 4(a)]. Δ_1 contains that part of the phase space from which the transitions between the two lobes take place. It is limited by the lines $g : x_3 = -10x_1/3 + 10$ and $h : x_3 = 10x_1/3 + 10$, and defined by the set $\Delta_1 = \{x_3 | x_3 > g \wedge x_3 > h\}$. Δ_2 is the complement of Δ_1 relative to the entire phase space.

Δ_2 contains that part of the attractor for which the maximum local divergence rate is positive [17], e.g., for

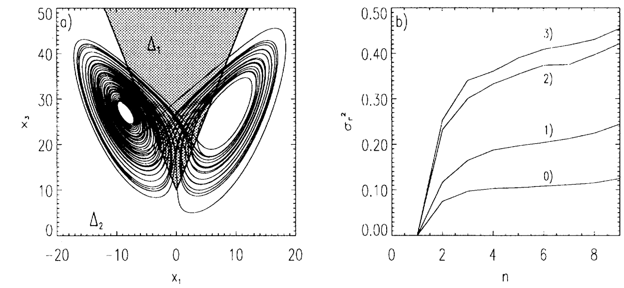

FIG. 4. (a) The two regions Δ_1 und Δ_2 of the phase space where the influence of local noise on the flip-flop process is investigated. (b) Fluctuation complexity σ_Γ^2 as a function of word length n, where the Lorenz system is perturbed by local external noise with amplitude $\xi_0 = 1$. The four cases indicated by 0, 1, 2, 3 are described in the text.

which nearby trajectories diverge locally. The maximum local divergence rate is negative for the complementary subset (Δ_1) of the attractor.

The following four cases can be distinguished: (0) Lorenz system without noise, (1) Lorenz system perturbed by local noise ($\xi_0 = 1$) in the region Δ_1, (2) Lorenz system perturbed by local noise ($\xi_0 = 1$) in the region Δ_2, and (3) Lorenz system perturbed globally. For each case the fluctuation complexity σ_Γ^2 is calculated as a function of word length n. The result is plotted in Fig. 4(b).

The graphs of σ_Γ^2 for which noise is added only on a partial region on the attractor (cases 1 and 2), are "between" the graphs of the unperturbed system (case 0) and the globally perturbed system (case 3). The curves corresponding to cases 2 and 3 are fairly close to each other indicating that the stabilizing effect of noise on the dynamics of the flip-flop process is dominated by the behavior in region Δ_2.

The number of constant words of length $n = 8$ (00000000 and 11111111) of the corresponding symbol sequences relative to the unperturbed case is enlarged by a factor of 1.6 for case 1, 3.6 for case 2, and 3.8 for case 3. This again shows the dominating effect of region Δ_2 for noise-induced stabilization.

Words with an alternating sequence of symbols ($n = 8$, 01010101 and 10101010) appear comparatively rarely in all considered symbol sequences. Their frequencies relative to the unperturbed case are given by a factor of 0.8 for case 1, 0.4 for case 2, and 0.6 for case 3. This means that noise leads to a reduction of the switching frequency between the two lobes in all cases. In case 1, where the perturbations take place in Δ_1, the switching between the two lobes happens more often than in cases 2 and 3.

In Δ_2 the maximum local divergence rate is positive [17], which implies that the forecasting of the system dynamics is restricted. The stabilizing effect of noise can be interpreted such that by the perturbation ξ the trajectory is mapped onto an orbit, which moves again around the same unstable fixed point.

The maximum local divergence rate is negative in Δ_1, which implies that small perturbations of the system are damped out. Switching processes are observed less frequently in cases 2 and 3 than in case 1.

In this context it is remarkable that for nonlinear ordinary differential equations the difference between numerically and algebraically (approximated) calculated limit cycles is reduced, where the maximum local divergence rate of the system is positive [18].

SUMMARY

In conclusion it is demonstrated that the influence of noise on the flip-flop process of the Lorenz system, which is characterized by two unstable fixed points, leads to a more inhomogeneous distribution of words (discretized orbits of finite length) for small noise amplitudes. As a (somewhat counterintuitive) consequence noise generates structure and stabilizes the flip-flop process, because the number of successive rotations around the same unstable fixed point is enlarged. Furthermore there is some indication that the described stabilization effect of noise on the flip-flop process is dominated by the influence of noise on those subsets of the attractor where the maximum local divergence rate is positive.

In contrast to OGY stabilization, noise-induced stabilization does not lead to a definite final target state, but influences a trajectory qualitatively by an increasing number of successive circulations on the same lobe. Besides stochastic resonance [7] and other noise-driven phenomena [4,9], noise-induced stabilization is a further example for the variety of dynamical behavior inherent in perturbed complex systems.

It might be interesting to look for experimental verifications of noise-induced stabilization. Good candidates are single-mode lasers, whose instabilities can be described by the Lorenz model. Another possibility is the instability between two alternative perceptual states in continuous observation of ambiguous patterns like the Necker cube [12], one of the original motivations to study flip-flop processes of the Lorenz system.

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