Exits in multistable systems excited by coin-toss square-wave dichotomous noise: A chaotic dynamics approach

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We consider a wide class of multistable systems perturbed by a dissipative term and coin-toss square-wave dichotomous noise. These systems behave like their harmonically or quasiperiodically driven counterparts: depending upon the system parameters, the steady-state motion is confined to one well for all time or experiences exits from the wells. This similarity suggests the application to the stochastic systems of a Melnikov approach originally developed for the deterministic case. The noise induces a Melnikov process that may be used to obtain a simple condition guaranteeing the nonoccurrence of exits from a well. For systems whose unperturbed counterparts have phase space dimension 2, if that condition is not satisfied, weak lower bounds can be obtained for (a) the mean time of exit from a well and (b) the probability that exits will not occur during a specified time interval.

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I. INTRODUCTION

Numerous studies have been devoted, especially in the past decade, to dynamical systems driven by dichotomous noise, which is characterized primarily by whether it is "on" or "off" or whether it is "up" or "down" [1,2]. One example are systems where the excitation exceeds or does not exceed a specified threshold, situations described as on and off, respectively.

The purpose of this paper is to present a Melnikov-based procedure, applicable to a wide class of nonlinear multistable systems, which yields a necessary condition for the occurrence of exits from a well. The systems we consider are perturbed by a dissipative term and dichotomous noise excitation and have unperturbed Hamiltonian counterparts. In addition, for systems whose unperturbed counterparts have phase space dimension 2 we show that, if the necessary condition for the occurrence of exits is satisfied, our procedure can be used to obtain weak lower bounds for (a) the mean exit time from a well and (b) the probability that exits will not occur during a specified time interval. Our approach yields information on system behavior in a class of problems for which alternative approaches (e.g., the Fokker-Planck equation, otherwise a much more powerful approach) are impractical or inapplicable.

For specificity we consider the Duffing-Holmes equation perturbed by a linearly viscous dissipative term and a stochastic excitation. We assume that the latter consists of dichotomous noise of the coin-toss square-wave type [2]. However, we show that our approach can accommodate other types of noise.

Section II describes the class of systems to which our approach is applicable. Section III briefly reviews basic chaotic dynamics results pertaining to the exit problem for multistable systems with periodic or quasiperiodic excitation and with stochastic excitation. Section IV describes the Melnikov process induced by dichotomous noise and discusses the corresponding Melnikov-based criterion guaranteeing the nonoccurrence of exits. Section V discusses lower bounds for the mean exit time and the probability of no exits during a specified time interval, as well as the method we use to obtain mean upcrossing time estimates for the Melnikov process. Section VI presents our conclusions.

II. DYNAMICAL SYSTEMS

We consider second-order dynamical systems described by the equation

$$\ddot{z} = -V'(z) + \epsilon[\gamma G(t) - \beta \dot{z}] , \tag{2.1a}$$

where $\epsilon \ll 1$ and $V(z)$ is a potential function. The unperturbed counterpart of Eq. (2.1b) is the Hamiltonian system

$$\ddot{z} = -V'(z) . \tag{2.1b}$$

We assume that Eq. (2.1b) has a hyperbolic fixed point [3] connected to itself by a homoclinic orbit. However, all the results of this paper also apply to systems with two hyperbolic fixed points connected by a heteroclinic orbit. As an example, we consider in this paper the Duffing-Holmes equation, which has a double-well potential

$$V(z)=z^4/4-z^2/2\ , \tag{2.2}$$

shown in Fig. 1(a).

Equation (2.1b), with the potential (2.2), has the homoclinic orbits shown in Fig. 1(b). The homoclinic orbits constitute a separatrix, that is, a curve separating motions in (2.1b) that evolve around the centers C or C' and can never cross the potential barrier from motions that evolve around the hyperbolic fixed point O and cross the potential barrier periodically (Fig. 1). For the potential (2.2), integration of Eq. (2.1b) [which may be rewritten as $\dot{z}d\dot{z}=V'(z)dz$] with initial conditions $z=0$, $\dot{z}=0$ yields the expressions for the homoclinic orbits

$$z_0(t)=\pm(2)^{1/2}\mathrm{sech}(t)\ , \tag{2.3a}$$

$$\dot{z}_0(t)=\mp(2)^{1/2}\mathrm{sech}(t)\tanh(t)\ . \tag{2.3b}$$

For later use, we note that for the Duffing-Holmes equation the modulus of the Fourier transform of the function $h(t)\equiv\dot{z}_0(-t)$ is

$$S(\omega)=(2)^{1/2}\pi\omega\,\mathrm{sech}(\pi\omega/2) \tag{2.4}$$

and

$$c\equiv\int_{-\infty}^{\infty}\dot{z}_0^2(\tau)d\tau=\tfrac{4}{3}\ . \tag{2.5}$$

The approach presented in this paper is also applicable to higher-order systems if their unperturbed counterparts are completely integrable Hamiltonian systems or parametrized families of completely integrable Hamiltonian systems, with a degenerate homoclinic or heteroclinic structure. The perturbations are subject to conditions defined in [4].

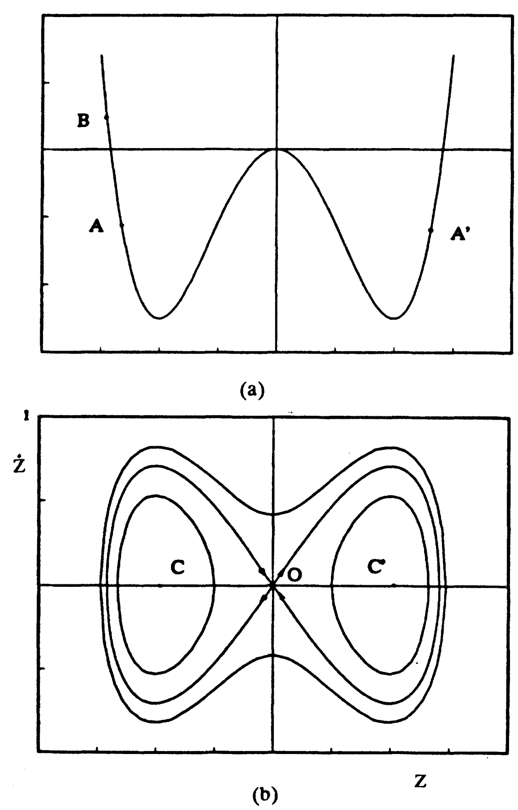

FIG. 1. (a) Potential wells for the bistable system and (b) phase plane diagram for the unperturbed Hamiltonian system.

III. NECESSARY CONDITION FOR THE OCCURRENCE OF EXITS

In this section we review briefly basic chaotic dynamics results pertaining to the exit problem for multistable systems with periodic or quasiperiodic excitation and with stochastic excitation. Those results were originally obtained for periodically excited systems [5]. They were subsequently extended to quasiperiodically excited systems in [6] and to stochastic systems in [7].

A. Periodic or quasiperiodic excitation

Let us consider the phase space $z,\dot{z},\theta$, where $\theta=\omega t$, ω is a constant, the coordinate axis $O\theta$, denoted by Γ, is normal to the plane $z,\dot{z}$, and O is a hyperbolic fixed point of Eq. (2.1b) [Fig. 2(a)]. The stable manifold $W^s(\Gamma)$ of the hyperbolic orbit Γ is defined as the set of points $(z_0^s(\theta),\dot{z}_0^s(\theta),\theta\}$ such that the orbits passing through those points approach Γ as $t\to\infty$. The unstable manifold $W^u(\Gamma)$ is defined as the set of points $\{z_0^u(\theta),\dot{z}_0^u(\theta),\theta\}$ such that the orbits passing through those points approach Γ as $t\to-\infty$. The cross section of the stable and unstable manifolds with any given plane $\theta=$const is a curve defined by the coordinates $z_0(t),\dot{z}_0(t)$ of the homoclinic orbit (2.3). For a planar system with a homoclinic orbit it is clear that the stable and unstable manifolds coincide.

The *persistence theorem* states that, for quasiperiodic $G(t)$ and sufficiently small ϵ, the perturbed system has a hyperbolic orbit Γ_ϵ with coordinates dependent on θ; Γ_ϵ

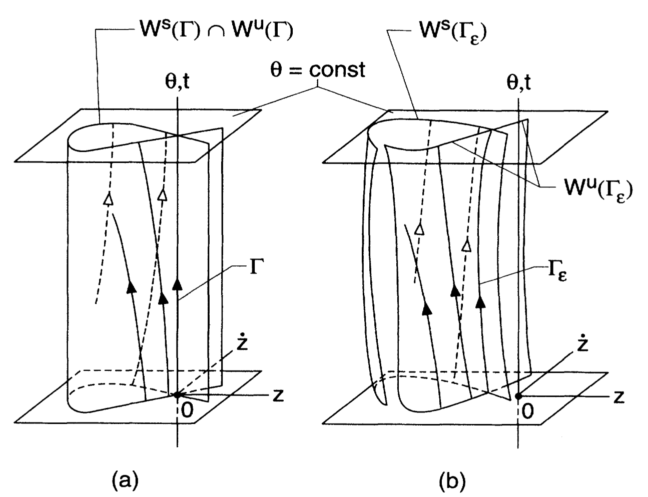

FIG. 2. Stable and unstable manifolds for (a) the unperturbed system and (b) the perturbed system (after [8]).

is contained in a close neighborhood of Γ and approaches Γ as $\epsilon \to 0$ [8]. The stable manifold $W^s(\Gamma_\epsilon)$ is defined as the set of points $\{z_0^s(\epsilon,\theta), \dot{z}_0^s(\epsilon,\theta), \theta\}$ such that the orbits passing through those points approach Γ_ϵ as $t \to \infty$. A similar definition holds, in reverse time, for the unstable manifold $W^s(\Gamma_\epsilon)$.

The stable and unstable manifolds of the perturbed system no longer coincide, as they do for $\epsilon=0$ [Fig. 2(b)]. The separation distance between $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ along a direction normal to the unperturbed manifolds, known as the *Melnikov distance,* is a function of θ and t. For any given cross section of the stable and unstable manifolds by a plane $\theta=$const (such a cross section is termed in [6] a *phase space slice;* in the particular case of harmonic excitation it is known as a *Poincaré section* [5]), the Melnikov distance is a function of t only. To first order, the Melnikov distance is proportional to the *generalized Melnikov function* (GMF) [9], which can be shown to have the expression

$$M(t) = -\beta \int_{-\infty}^{\infty} \dot{z}_0^2(\tau)\,d\tau + \gamma \int_{-\infty}^{\infty} h(\tau) G(\tau - t)\,d\tau \;, \qquad (3.1)$$

where the filter in the convolution integral of Eq. (3.1) is $h(t) = \dot{z}_0(-t)$ [5,6]. For sufficiently small ϵ, if $M(t)$ has simple zeros, $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ intersect transversely; if $M(t)$ is bounded away from zero, $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ do not intersect [5,6,8]. From the *Smale-Birkhoff theorem* it can be inferred that, for sufficiently small ϵ, the necessary condition for chaotic behavior (i.e., for the largest Lyapounov exponent to be positive or, equivalently, for the system to be sensitive to initial conditions) is that $M(t)$ have simple zeros [5,6,8].

Since motion starting on a manifold never leaves that manifold, the existence of a transverse intersection point in a phase space slice implies the existence of an infinity of intersection points. Areas in a phase space slice that are bounded by segments of stable and unstable manifolds between two successive intersection points are termed lobes. A set of lobe segments forming a shape roughly similar to the shape of the homoclinic orbit of the unperturbed counterpart of the system is termed a pseudoseparatrix [10] (Fig. 3). Unlike the homoclinic orbit (i.e., the separatrix) of Fig. 1(b), the pseudoseparatrix is permeable, that is, it can allow motions occurring within a well to exit from that well. The transport of phase space across the pseudoseparatrix is effected by detraining and entraining lobes. [Detraining (entraining) lobes are lobes that will cross or have crossed into the exterior (interior) region bounded by the pseudoseparatrix [6]]. No such transport can occur in the absence of lobes. It follows that the necessary condition for the occurrence of exits in quasiperiodically excited multistable systems (1.1) with sufficiently small perturbation is that the GMF have simple zeros.

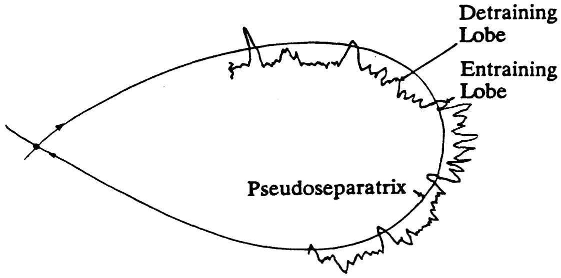

FIG. 3. Time slice showing a homoclinic tangle.

B. Stochastic excitation

The results just summarized are applicable for systems with quasiperiodic excitation. They can be applied to systems with stochastic excitation provided that the excitation can be approximated as closely as desired by sums of N harmonic terms with random parameters, where N is a finite, albeit large number. For Gaussian excitation, colored or white, such approximations are discussed in [7].

In this paper we consider excitation by dichotomous coin-toss square-wave noise, which has the expression

$$G(t) = a_n \quad [\alpha + (n-1)]t_0 < t \le (\alpha + n)t_0 \;, \qquad (3.2)$$

where $n = \ldots, -2, -1, 0, 1, 2, \ldots$ is the set of integers, α is a random variable uniformly distributed between 0 and 1, a_n are independent random variables that take on the values -1 and 1 with probabilities $\frac{1}{2}$ and $\frac{1}{2}$, respectively, and t_0 is a parameter of the process $G(t)$. A rectangular pulse wave of amplitude a_n and length t_0 centered at the coordinates $t_n = (\alpha + n - \frac{1}{2})t_1$ has a Fourier transform

$$F_n(\omega) = a_n \,|(2/\omega)\sin(\omega t_0/2)\exp(-j\omega t_n)|$$

[11]. The pulse itself can therefore be expressed as a sum of harmonic terms approximating as closely as desired the inverse Fourier transform of $F_n(\omega)$. Each realization of the coin-toss dichotomous square wave can be approximated arbitrarily closely by a superposition of such sums, which is itself a sum of harmonics, that is, a quasiperiodic function with parameters a_n.

Each realization of the noise, determined as it is by a set of parameters a_n, induces a GMF characterized by that set. The stochastic process $G(t)$ is an ensemble of realizations of the noise and induces an ensemble of realizations of the GMF. This ensemble is referred to as the *Melnikov process* induced by $G(t)$. The Melnikov process $M(t)$ can be obtained by using Eq. (3.1) in which $G(t)$ is given by Eq. (3.2), since each quasiperiodic realization of the approximating process can be assumed to be arbitrarily close to the corresponding realization of the process $G(t)$.

IV. MELNIKOV PROCESS AND CRITERION GUARANTEEING THE NONOCCURRENCE OF EXITS

From Eqs. (2.4), (2.5), and (3.1),

$$M(t) = -4\beta/3 + (2)^{1/2}\gamma F(t) \;, \qquad (4.1)$$

$$F(t) \approx \sum_{n=-l}^{l} a_n\{-\mathrm{sech}[(n+\alpha)t_0-t] + \mathrm{sech}[(n+\alpha-1)t_0-t]\} , \quad (4.2)$$

where l is sufficiently large for the error due to the assumption that l is finite to be as small as desired. The area under the curve $h(t) \equiv \dot{z}_0(-t)$ [Eq. (2.3b)] in a half plane is $(2)^{1/2}$. It follows immediately from the definition of $F(t)$ that $-2 < F(t) < 2$ [the second integral on the right-hand side of Eq. (3.1) yields $F(t)=2$ if $\alpha=0$ and $a_n=1$ for all n such that $t>0$ and $a_n=-1$ for all n such that $t<0$]. Since the necessary condition for chaos (i.e., for exits) is that $M(t)$ have simple zeros, it follows from Eq. (4.1) that chaos cannot occur if $F(t)$ does not reach the zero line $(4\beta/3)/[(2)^{1/2}\gamma]$ or

$$F(t) < 0.9428\beta/\gamma . \quad (4.3)$$

Since $|F(t)|<2$, chaos cannot occur if

$$\gamma/\beta < 0.471 . \quad (4.4)$$

The simplicity of Eq. (4.4) is noteworthy. Time histories of the function $F(t)$ for $t_0=1.0$, 0.35, and 0.1 are shown in Fig. 4. It is seen that, as a criterion guaranteeing the nonoccurrence of exits, Eq. (4.4) is increasingly weak as t_0 becomes smaller. We remark that Eq. (4.4) can also be applied, with no modification, for coin-toss dichotomous noise with random arrival times. More generally, criteria similar to Eq. (4.4) can be derived for other reasonable tail-limited random excitations.

We show in Figs. 5(a) and 5(b) time history realizations corresponding to the dichotomous noise of Eq. (3.2), $t_0=1.0$, parameters $\epsilon=0.1$, $\beta=1.5$, and, respectively, $\gamma/\beta=0.469<0.471$ and $\gamma/\beta=1.887$. The motion of Fig. 5(a) is confined to one well. Its irregularity is due to the stochastic nature of the excitation. The chaotic motion of Fig. 5(b) is similar to chaotic motions induced in the Duffing-Holmes oscillator by harmonic or quasiperiodic excitation. Its irregularity is due to both the chaotic nature of the motion and the stochastic nature of the excitation. Figure 5(b) shows that, as is the case for equations with harmonic forcing [12], the necessary condition for the occurrence of chaos is helpful in the search for chaotic regions of parameter space even for relatively large ϵ. Sensitivity to initial conditions (i.e., the positivity of the largest Lyapounov exponent) was verified numerically for the motion of Fig. 5(b).

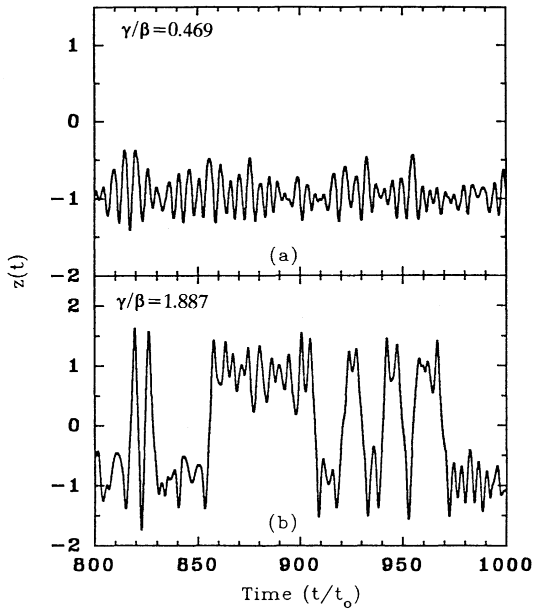

FIG. 5. Time histories of $z(t)$: (a) nonchaotic motion and (b) chaotic motion.

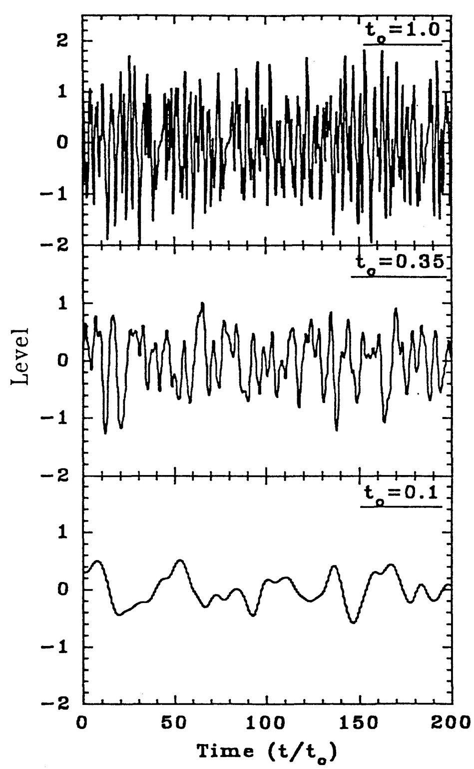

FIG. 4. Function $F(t)$ for $t_0=1.0$, 0.35, and 0.1.

We have so far assumed that the noise $G(t)$ is additive [see Eq. (2.1)]. If in Eq. (2.1) we consider instead multiplicative noise $r(z,\dot{z})G(t)$, then the filter $h(\tau)=\dot{z}_0(-\tau)$ in Eq. (3.1) is simply replaced by the filter

$$h_m(\tau)=\dot{z}_0(-\tau)r[z_0(-\tau),\dot{z}_0(-\tau)] . \quad (4.5)$$

V. MEAN EXIT TIME AND PROBABILITY OF NO EXITS DURING A SPECIFIED TIME INTERVAL

In this section we consider only systems whose unperturbed counterparts have phase space dimension 2.

A. Lower bound for the mean exit time

We refer to Fig. 4 and note that a line of constant ordinate $0.9428\beta/\gamma$ would represent the zero line for the

Melnikov process. The areas between the zero line and the positive ordinates of the GMF are the counterparts in Fig. 4 of entraining lobes such as those shown in Fig. 3. Similarity, the areas between the zero line and the negative ordinates of the GMF would represent the counterparts in Fig. 4 of detraining lobes.

For sufficiently high ratios β/γ, the zero up crossings of the process $M(t)$ are rare events. We denote the mean time between these up crossings by τ_u. It is seen from Fig. 3 that, on average, to within an approximation of order one, no transport across the pseudoseparatrix can occur during a time interval less than the mean zero upcrossing time τ_u of the Melnikov process, that is, τ_u is smaller than (a lower bound for) the mean exit time from a well τ_e. The type of Melnikov-based lower bound just described is applicable regardless of the nature of the excitation. In the case of excitation by white noise it has been shown analytically that this lower bound is weak [13]. Numerical simulations show that this is the case for other types of noise as well.

B. Mean zero up-crossing rates estimated by discrete probability function

We consider again Fig. 4, which shows typical realizations of the process $F(t)$ [Eq. (4.2)] for values $t_0=1.0,0.35,0.1$. Figure 4 shows that large excursions of $F(t)$ are more likely for large values of t_0. This is so because for large t_0 the number of dominant terms in the expression of $F(t)$ is small and the probability that successive dominant terms will have the same sign is therefore relatively large. For relatively small t_0 Monte Carlo estimates of the probability of occurrence of large excursions are poor owing to the rarity of such large excursions. For this reason, to calculate the probability density function of the stationary process $F(t)$ we use the discrete probability function (DPF) approach [14].

We define the sum

$$S_i=\sum_{k=-l}^{-l+i} a_k f_k(t) , \tag{5.1}$$

$i=0,\ldots,2l,$ and $f_k(t)=-\operatorname{sech}[(k+\alpha)t_0-t]+\operatorname{sech}[(k+1+\alpha)t_0-t]$. For any value of t, S_0 can take on only two values $\pm f_{-l}$ with probability 0.5 for each. Similarly, S_1 can take on four values $\pm f_{-l},\pm f_{-l+1}$ with probability 0.25 each. This can be carried forward to calculate the corresponding 2^{i+1} terms of S_i, with the corresponding probability of 0.5^{i+1} each. However, we can calculate the probabilities associated with S_i only for relatively small values of i owing to limited computer resources. This limitation can be circumvented by using the DPF method to construct histograms for S_i.

Let $[S_{i,n},P_{Si,n},\ n=1,2,\ldots,N]$ denote the histogram associated with S_i, where $[S_{i,n},S_{i,n+1})$ denotes the nth bin,

$$P_{Si,n}=\int_{S_{i,n}}^{S_{i,n+1}}\mathcal{P}(S_i)dS_i \tag{5.2}$$

denotes the probability that the variant S_i is contained in the nth bin, and $\mathcal{P}(S_i)$ denotes the probability density function (PDF) of S_i. The DPF of S_i, $P[S_i]$, is the set $P(S_i)$ consisting of $S_{i,n}$ and $P_{Si,n}$

$$P[S_i]=[S_{i,n}P_{Si,n}\ ,\ \ n=1,2,\ldots,N]\ , \tag{5.3}$$

where the set $P[S_i]$ is determined recursively from $P[S_{i-1}]$ [13].

The bins are determined recursively as follows. Start with $S_{0,n}$, $n=1,\ldots,N$, by setting $S_{0,1}=-|f_l|$, $S_{0,N}=+|f_{-l}|$, and $S_{0,n}=S_{0,1}+n\Delta_0$, where $\Delta_0=2|f_{-l}|/N$. Similarly, $S_{1,N}=S_{1,0}+n\Delta_1$, where $S_{1,1}=-|f_{-l}|-|f_{-l+1}|$, $S_{1,N}=+|f_{-l}|+|f_{-l+1}|$, and $\Delta_1=(S_{1,N}-S_{1,1})/N$.

The bin probabilities are determined by means of the recurrence formula

$$P_{Si+1,n}=0.5\sum_{k=1}^{N}\delta(n)P_{Si,k}\ , \tag{5.4}$$

where $\delta(n)=1$ if $S_{i+1,n}<S_{i,k}+a_{i+1}f_{i+1}<S_{i+1,n+1}$ and 0 otherwise. The simulation is started with $P[S_0]=[S_{0,n}P_{S0,n},\ n=1,\ldots,N]$ with $P_{S0,i}=0.5$ for $i=1,N$ and 0 for all other i. This approach dramatically reduces storage and computational requirements.

The advantage of explicit DPF calculations over Monte Carlo (MC) simulations is illustrated in Fig. 6. The Monte Carlo simulations represented in Fig. 6 were one order of magnitude more computationally intensive than the DPF calculations; the probability density function (PDF) of $F(t)$, shown in Fig. 6 by the dotted line, was calculated from 10^6 realizations of $F(t)$. Nevertheless, an absolute value of $F(t)$ exceeding unity was a sufficiently rare occurrence that the extremes of the PDF obtained by Monte Carlo simulations contained a high degree of statistical noise. For this reason, for rare exits, Monte Carlo simulations yield poor estimates or no estimates at all. The DPF method does not suffer from this limitation. In our calculations we used 100 bins ranging

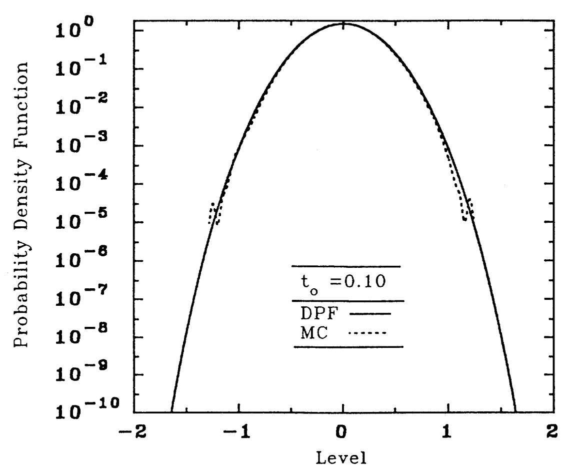

FIG. 6. Probability density function of $F(t)$ for $t_0=0.1$, estimated by the DPF approach (solid line) and by Monte Carlo simulation (dotted line).

from the lowest to the highest possible value of S_i.

Similarly, we can use the DPF method to calculate explicitly the conditional PDF of $dF(t)/dt$ given the value of the function F. The conditional PDF's based on Monte Carlo simulations and the DPF method are shown in Fig. 7 for $t_0=1.0$ and three levels of F. The conditional PDF's calculated by the DPF method are symmetric, as expected. The estimates obtained by the Monte Carlo method are affected by large statistical noise. Owing to the rare occurrences at the higher levels, the conditional PDF obtained by the Monte Carlo method is also skewed. It is noted that this skewness is due to sampling errors and could be reduced if the number of realizations used in the Monte Carlo procedure were larger. The PDF's obtained by the DPF method allow the calculation of the expected number of upcrossings by the function $F(t)$ of any desired levels [15].

Figure 8 shows the crossing rate as a function of upcrossing level for $t_0=0.1$, 0.35, and 1.0. For example, for $\beta/\gamma=0.53$ [as in Fig. 5(b)], the threshold for the function $F(t)$ is $0.9428\times 0.53=0.5$ [Eq. (4.3)] and, from Fig. 6, for $t_0=1.00$ the mean up-crossing time is $\tau_u\approx 3$. From a counterpart of Fig. 5(b), over a time interval $200<t<1000$ the estimated mean exit time was $\tau_e\approx 32$, which, as expected, is longer than the mean zero up-crossing time $\tau_u\approx 3$ of the Melnikov process.

C. Lower bound for the probability of no exit during a specified time interval

If zero up-crossings of the Melnikov process are rare events, the probability that no up-crossing occurs during a specified time interval T can be written as

$$p_T\approx\exp(-T/\tau_u)\ . \tag{5.5}$$

Since $\tau_u<\tau_e$, p_T is an approximate lower bound for the probability that exits from a well will not occur during the time interval T. For example, for $t_0=0.1$, $\gamma/\beta=1.6$, and $T=10^5$, $p_T=0.9999$. Even though the probability that no exit from a well can occur is higher, in such a case the Melnikov-based lower bound may be useful in some applications.

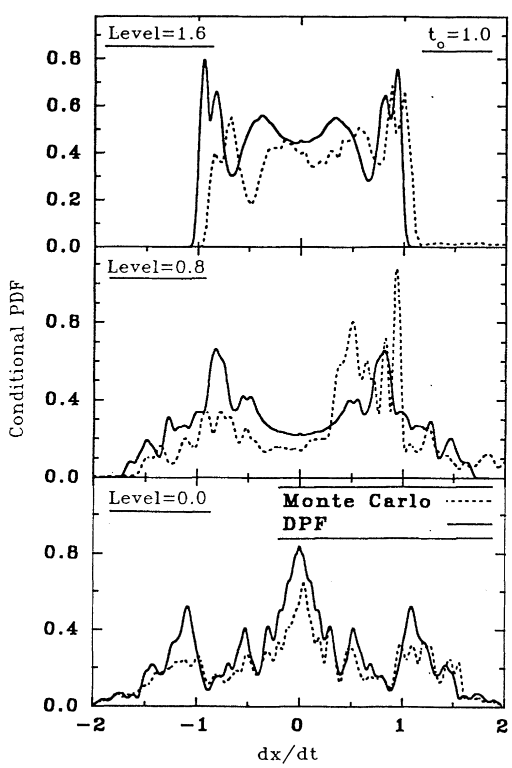

FIG. 7. Conditional probability density function of $dF(t)/dt$ given $F(t)=0$, $F(t)=0.8$, and $F(t)=1.6$, for $t_0=1.0$.

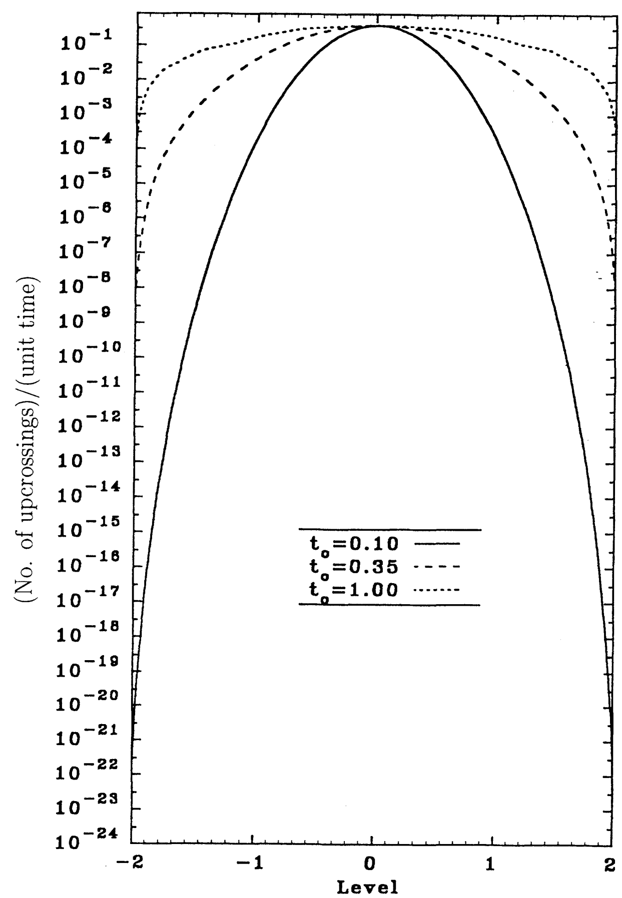

FIG. 8. Crossing rate of function $F(t)$ for $t_0=0.1$, 0.35, and 1.0 as a function of the up-crossing level.

VI. CONCLUSIONS

We showed that, for a wide class of nonlinear differential equations, forcing by dichotomous noise induces behavior that has useful similarities to behavior induced by harmonic or quasiperiodic forcing. For certain regions of parameter space, both the stochastic system driven by noise and the deterministic system driven harmonically experience behavior that may be chaotic or nonchaotic. Nonchaotic behavior precludes the occurrence of exits from the potential wells. However, if the behavior is chaotic, exits from the wells become possible. A necessary condition for the occurrence of chaos in the deterministic and stochastic systems is the existence of simple zeros in, respectively, the Melnikov function (which is a deterministic function) and the Melnikov process. This parallelism suggested extending, to our stochastic differential equations, an approach based on the theory of chaotic dynamics and originally developed for deterministic systems. This approach accommodates both additive and multiplicative noise and yields a remarkably simple criterion guaranteeing the nonoccurrence of exits. For second-order differential equations we obtained weak lower bounds for (a) the mean exit time from a well and (b) the probability of nonoccurrence of exits during a specified time interval.

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