Phase problem associated with the determination of the longitudinal shape of a charged particle bunch from its coherent far-ir spectrum

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The coherent far-infrared radiation induced from relativistic electron bunches of submillimeter length provides a way to characterize the bunch shape. Once the spectrum of the bunch form factor is measured, one can apply the Kramers-Kronig relations to the spectral form factor to find the minimal phase and then calculate the bunch shape from the complete Fourier transform. One potential problem is the uniqueness of the phase so determined since, in general, the phase shift associated with the Blaschke product must also be included. Here we examine a variety of possible asymmetric bunch shapes in order to identify any errors inherent in this approach.

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The production of relativistic electron bunches in submillimeter size and the proposals for even smaller bunches definitely tax the limits of streak camera measurement techniques [1]. In addition, the calculations that show that specifically shaped charge distributions can minimize the energy spread within a single bunch [2] have brought attention to the experimental problem of characterizing the shape of such objects on a subpicosecond time scale. A developing technique makes use of the spectral information in the coherent far-infrared synchrotron or transition radiation that is produced under appropriate conditions at long wavelengths when the emitted wavelength is comparable to the bunch length [3–11]. In all of these cases the bunch shape has been calculated from the spectrum with a cosine Fourier transform so that only a symmetric shape can be found. Recently it has been proposed that a logarithmic Hilbert transform of the spectrum be used to find the phase information so that a more complete description of the bunch shape can be determined from the data [12]. The coherent synchrotron or transition radiation spectrum emanating from millimeter long electron bunches produced at the Cornell University linear accelerator has been analyzed in this way and the bunches are found to be asymmetric [13]. In both of these works it is assumed that the minimal phase obtained from the Hilbert transform is the appropriate phase to use in calculating the bunch shape. In all such phase retrieval problems, it is the modulus of the complex function and its phase that are of interest and it has been shown, in general, that when the logarithmic Hilbert transform is used, the dispersion relation for the phase shift includes both the minimal phase shift and also the phase shift associated with the Blaschke product [14]. A great deal of effort has gone into characterizing this phase problem because of its potential use in optics [15]. In some cases, such as the Kramers-Kronig analysis of the reflectivity of solids, it is possible to show that because the imaginary part of the dielectric function is always positive for positive frequencies, the Blaschke product can be ruled out [16]. In addition, it has been demonstrated for a broad spectrum such as the blackbody one that the Blaschke product does not contribute so that only the minimal phase results [17]. On the other hand, it also has been shown for both Lorentzian and Gaussian line shapes that the Blaschke product does contribute to the reconstructed spectrum [18]. In this paper we examine in some detail the application of this spectrum transform technique to bunch shape determination to identify it strengths and weaknesses.

The total intensity of far-infrared radiation produced by an electron bunch can be obtained by summing the electric field over all the N electrons with the phase relations between different electrons taken into account. This leads to [19]

$$I_{\mathrm{tot}}(\omega)=I(\omega)[N+N(N-1)F(\omega)]\ , \tag{1}$$

where $I(\omega)$ is the intensity produced by a single electron and $F(\omega)$ is the form factor. For relativistic electrons the radiation appear in the forward direction with the maximum intensity at $\theta\sim 1/\gamma$. For this $\theta\to 0$ limit, the form factor can be simplified as

$$F(\omega)=\left|\int_0^\infty dz\, S(z)e^{i(\omega/c)z}\right|^2 . \tag{2}$$

Here $NS(z)dz$ is the number of electrons in the region between z and $z+dz$, where $S(z)$ is the normalized longitudinal distribution function of electrons in the bunch.

To obtain maximum information about the bunch shape from the measured form factor data, we first redefine the integral in Eq. (2). Let

$$\hat{S}(\omega)\equiv\int_0^\infty dz\, S(z)e^{i(\omega/c)z}\equiv\rho(\omega)e^{i\psi(\omega)} . \tag{3}$$

The complex function $\hat{S}(\omega)$ defined in Eq. (3) is a regular function in the upper half of the complex frequency plane. Note the formal similarity between Eq. (3) and the corresponding expression for the complex degree of coherence, which involves an integral of similar form over positive frequencies [20] or the input-output response analysis used in optics to obtain the complex reflectivity at an interface, which involves an integral of

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similar form over positive time [16]. In analogy with these earlier analyses, we write

$$\ln\hat{S}(\omega)=\ln \rho(\omega)+i\psi(\omega) \ . \tag{4}$$

The regularity of $\hat{S}(\hat{\omega})$ in the upper half plane imposes a severe restriction on the phase factor $\psi(\omega)$ associated with any given $\rho(\omega)$ and the dispersion relation for Eq. (4) has the same form as for the real and imaginary parts of $\ln[\hat{r}(\omega)]$ [16] so that [14,21]

$$\begin{aligned}\psi(\omega)&=\psi_m(\omega)+\psi_{\text{Blaschke}}(\omega)\\&=-\frac{2\omega}{\pi}\,\mathrm{P}\int_0^\infty dx\frac{\ln\rho(x)}{x^2-\omega^2}+\sum_j \arg\left[\frac{\omega-\hat{\omega}_j}{\omega-\hat{\omega}_j^*}\right],\end{aligned} \tag{5}$$

where the first term $\psi_m(\omega)$ is the minimal phase, and the second term $\psi_{\text{Blaschke}}(\omega)$ is the contribution from the $\hat{\omega}_j$'s, namely, the zeros of $\hat{S}(\hat{\omega})$ in the upper half of the complex frequency plane, and $\hat{\mathrm{P}}$ denotes the principal value.

Once $F(\omega)$ is experimentally determined over the entire frequency spectrum, then by Eq. (3) so is $\rho(\omega)$. With the aid of Eq. (5), the normalized distribution function can now be obtained from the inverse Fourier transform of Eq. (3), namely,

$$S(z)=\frac{1}{\pi c}\int_0^\infty d\omega\,\rho(\omega)\left[\psi(\omega)-\frac{\omega z}{c}\right] . \tag{6}$$

However, only the minimal phase can be uniquely determined from $\rho(\omega)$ since the Blaschke phase in general cannot be determined experimentally. In the reflectivity problem the possibility of a zero in $\hat{r}(\hat{\omega})$ is excluded since there is always some absorption process, while in the bunch shape problem we cannot arbitrarily exclude the possibility of zeros in $\hat{S}(\hat{\omega})$. If $\hat{S}(\hat{\omega})$ has no zeros in the upper half plane, the phase contribution from the zeros $\psi_{\text{Blaschke}}(\omega)=0$ and the minimal phase can be used to recover the bunch distribution. Although the analytic continuation of the form factor of many distribution functions has no zeros in the upper half of the complex frequency plane, it is not the general case.

Some insight into the second term in Eq. (5) is of value. First, zeros on the real axis give no contribution to the phase. It was shown by Titchmarsh that as $\mathrm{Re}(\omega)\to\infty$, the distant zeros tend to be located along the real axis with a spacing of $2\pi c/\sigma_z$, where σ_z is the full bunch length. Therefore, those zeros make little contribution to the phase and contain no information about the bunch shape [22]. Second, zeros of $\hat{S}(\hat{\omega})$ are always paired together, i.e., if $\hat{\omega}_j$ is a zero, so is $-\hat{\omega}_j^*$ since the distribution function $S(z)$ is necessarily real [18]. One can show that the contribution from such a pair of zeros is linear with ω and negligible provided $|\hat{\omega}_j|\gg\omega$. Therefore, only "nearby" zeros, that is, zeros near the region over which the spectrum is measured, have an effect on the bunch shape. It follows that the minimal phase evaluated from the first term in Eq. (5) is a good approximation to the actual phase in cases where the form factor has no nearby zeros in the upper half of the complex frequency plane.

Both measured coherent synchrotron and transition radiation spectra of electron bunches show strong interference patterns, an indication of a structured bunch. To describe a multiple peaked bunch shape we examine it from the point of view of a superposition of simple Gaussian shapes since it has been shown by Nussenveig [18] that the Fourier transform of a truncated Gaussian distribution has no nearby zeros in the upper half plane. This means that if one were to use the form factor of a Gaussian bunch to calculate the minimal phase, then an undistorted Gaussian bunch shape would be found. To test more complex shapes we model the bunch distribution function by a superposition of three truncated Gaussians to include both asymmetry and local concentration, i.e.,

$$S(z)=\sum_{j=1}^{3}\alpha_j G(z,z_j,\sigma_j)/N, \quad 0<z<\sigma_z \ , \tag{7}$$

where σ_z is the full bunch length over which $S(z)$ is nonzero, α_j/N is the fractional weight of the jth component, and $G(z,z_j,\sigma_j)$ is a normalized Gaussian distribution function centered at z_j with standard deviation σ_j.

The solid line in Fig. 1 shows the bunch distribution function given by Eq. (7) for a specific set of parameters. The bunch consists of an asymmetric main contribution and a satellite postbunch. The form factor of such a complex bunch shape plotted as a function of frequency in Fig. 2(a) contains an interference pattern. The procedure now is to use this frequency-dependent form factor data to calculate the minimal phase for this complex spectrum and then determine the bunch shape to see if a Blaschke phase contribution has been missed.

The solid curve in Fig. 2(b) shows the actual phase of $\hat{S}(\omega)$ and the dot-dashed curve is the minimal phases associated with the spectrum in Fig. 2(a) calculated from the first term of Eq. (5). The dot-dashed curve agrees with the solid curve up to $\omega/\omega_0=30$, above which small discrepancy between the minimal phase and actual phase appears. This indicates that $\hat{S}(\omega)$ has no nearby zeros but does have "distant" zeros. The corresponding bunch

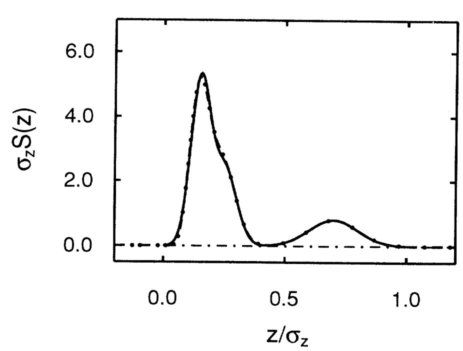

FIG. 1. Comparison of the exact longitudinal bunch distribution function with that calculated from the form factor. Solid curve, the exact longitudinal bunch distribution function given by Eq. (7) with a specific set of parameters $\alpha_1=0.5$, $z_1/\sigma_z=0.15$, $\sigma_1/\sigma_z=0.04$; $\alpha_2=0.3$, $z_2/\sigma_z=0.25$, $\sigma_2/\sigma_z=0.05$; $\alpha_3=0.2$, $z_3/\sigma_z=0.7$, $\sigma_3/\sigma_z=0.1$. Dot-dashed curve, the longitudinal bunch distribution function calculated from the form factor with the minimal phase approximation.

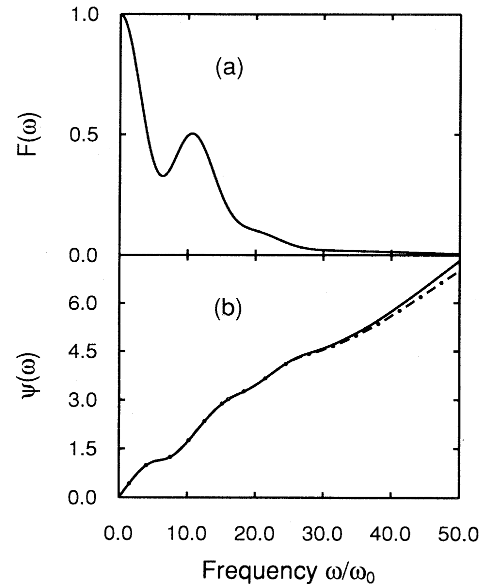

FIG. 2. Form factor and phases as functions of frequency for the superposition of Gaussians shown as the solid curve in Fig. 1. (a) Form factor versus frequency. (b) Phase versus frequency, calculated two different ways: solid curve, the actual phase; dot-dashed curve, the calculated minimal phase.

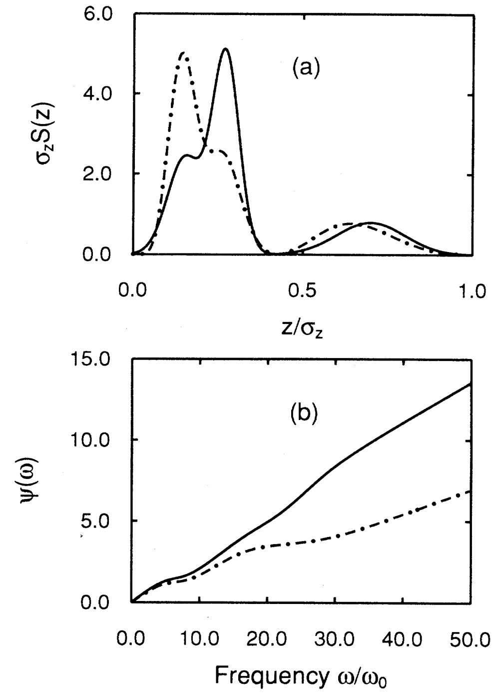

FIG. 3. Example of a bunch distribution in which the actual phase is very different from the minimal phase. (a) Solid curve, the longitudinal bunch distribution function given by Eq. (7) is such that a weak Gaussian precedes a strong Gaussian component. Parameters $\alpha_1=0.3$, $z_1/\sigma_z=0.15$, $\sigma_1/\sigma_z=0.05$; $\alpha_2=0.5$, $z_2/\sigma_z=0.27$, $\sigma_2/\sigma_z=0.04$; $\alpha_3=0.2$, $z_3/\sigma_z=0.7$, $\sigma_3/\sigma_z=0.1$. Dot-dashed curve, the longitudinal distribution function calculated from the form factor (not shown) with the minimal phase approximation. (b) Phases versus frequency: solid curve, the actual phase; dot-dashed curve, the minimal phase.

distribution calculated from Eq. (6) using the minimal phase is plotted in Fig. 1 as the dot-dashed curve. The good agreement between the solid and dot-dashed curves in Fig. 1 demonstrates that the minimal phase is indeed a good approximation to the actual phase in this case. This is also confirmed by our numerical calculations on bunch distribution functions with different sets of parameters as long as the first Gaussian is the strongest.

The minimal phase, however, may not be a good approximation to the actual phase when a weak Gaussian appears first. Plotted in Fig. 3(a) as the solid curve is another superposition of the same Gaussian used in Fig. 1 but with the centers changed so that the strongest one is in the middle. The calculated minimal phase is plotted in Fig. 3(b) as the dot-dashed curve. The significant discrepancy between the minimal phase and the actual phase (solid curve) indicates that the Blaschke phase cannot be ignored in this case. For comparison, the corresponding bunch distribution calculated from the minimal phase is plotted in Fig. 3(a) as the dot-dashed curve. It is interesting that the widths of both the main bunch and the satellite bunch are close to the actual ones, although the bunch shape resulting from the minimal phase is significantly different from the actual shape.

The condition for the minimal phase approximation—that the first Gaussian is the strongest—may be understood on the basis of the Rouche's theorem [23], which states that if two functions $\hat{S}_1(\hat{\omega})$ and $\hat{S}_2(\hat{\omega})$ are regular in a region C of the complex plane and if $|\hat{S}_1(\hat{\omega})|>\hat{S}_2(\hat{\omega})|$ at every point of the boundary of C, then this is a sufficient but not necessary condition for $\hat{S}_1(\hat{\omega})$ and $\hat{S}_1(\hat{\omega})+\hat{S}_2(\hat{\omega})$ to have the same number of zeros in C. It then follows from this theorem that the form factor of a distribution function with a strong component followed by some weaker components may have no zeros or nearby zeros in the upper half plane as long as $\hat{S}(\omega)$ of the strong component has no zeros or nearby zeros, such as a Gaussian.

In conclusion, the Kramers-Kronig transform technique can be used to determine the minimal phase from the bunch form factor if it is known over a sufficiently large frequency interval and as long as weak bunch components follow the strong one. When this minimal phase is then used together with the form factor amplitude the asymmetric bunch shape can be determined. In principle, the phase determined from the form factor could be larger than this minimal value and then the Kramers-Kronig transform approach would fail in the sense that the detailed shapes would be wrong, although the calculated widths would still be correct. We have examined a number of different bunch shapes to see how and when this problem would appear. Bunches with more than one peak were simulated by superimposing a number of Gaussian components. Our study of this case shows that the minimal phase is the appropriate phase to use to describe this complex structure as long as the largest Gaussian component comes first.

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