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Relaxation, noise-induced transitions, and stochastic resonance driven by non-Markovian dichotomic noise

A. Fuliński

Institute of Physics, Jagellonian University, Reymonta 4, PL-30-059 Kraków, Poland

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Under the influence of non-Markovian dichotomic noise, the linear process behaves nonlinearly: relaxation becomes nonmonotonic, and the process exhibits a series of transient noise-induced transitions. The latter give rise to stochastic resonance when an external oscillating field is applied.

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It is well known that the deterministic kinetics can be drastically changed by the influence of stochastic forces. In some cases the presence of noise may result in highly nontrivial effects. Of these, the best known are: noise-induced transitions [1,2] and stochastic resonance [3,4].

To the best of the author's knowledge, in all cases of noise-induced transitions and stochastic resonance discussed so far, only Markovian driving noises have been taken into account. Non-Markovian stochastic processes are more difficult to deal with than Markovian ones. Besides, very little is known about the behavior of stochastic flows driven by non-Markovian noises. Only very recently have a few papers been published which use non-Markovian driving, either explicitly [5] or implicitly [6]. A systematic theory of explicitly non-Markovian noises with exponential damping of memory has been recently proposed by the present author [7]. On the other hand, Markovianity seems to be but an approximation for natural fluctuations occurring in the real systems. Therefore it seems natural to ask how non-Markovianity of the driving noise changes the behavior of driven stochastic processes.

To avoid misunderstandings, it seems proper to mention at this point that, in fact, almost any stochastic flow $\dot{X}(t)$ driven by a colored Markovian noise, or by white noise and oscillating force, is a non-Markovian process by itself. In particular, almost all systems exhibiting the phenomenon of stochastic resonance are non-Markovian. In this sense there is a vast literature on non-Markovian stochastic processes, though this fact is mentioned explicitly very rarely. What is said above is that, to the best of author's knowledge, a discussion of the use of non-Markovian noises as driving processes is almost absent from the literature.

I propose in this paper to consider one of the simplest stochastic flows, viz., the linear process (relaxation):

$$\dot{x}(t) = -[a + \xi(t)]x(t) + b\,, \tag{1}$$

driven by multiplicative *non-Markovian* symmetric dichotomic noise $\xi(t)$ [7]:

$$\xi(t) \in \{-\Delta, +\Delta\}\,,\ \xi^2(t) = \Delta^2\,,\ \langle\xi(t)\rangle = 0, \tag{2}$$

with the probability distributions fulfilling the non-Markovian master equation, and with the correlation function

$$\langle\xi(t)\xi(t+\tau)\rangle \equiv C_2(\tau) = (\Delta^2/\Gamma)\left[(\theta_1 - \nu)e^{-\theta_1\tau} - (\theta_2 - \nu)e^{-\theta_2\tau}\right], \tag{3}$$

$$\theta_{1,2} = \tfrac{1}{2}(\nu + 2\gamma_0\lambda \pm \Gamma)\,,\quad \Gamma = \sqrt{(2\gamma_0\lambda - \nu)^2 - 8\gamma_1\lambda}\,,$$

where λ is the probability of switching between states $\pm\Delta$ per unit time, ν is the inverse memory time, and γ_0, γ_1 describe, respectively, Markovian and non-Markovian contributions to the master equation [cf. Eq. (8) below].

It is generally believed that the highly nontrivial effects of noise can be observed in nonlinear processes only. In particular, it is believed that the system in parabolic potential does not exhibit stochastic resonance [8], because the latter is related to the enhancement of the switching of the system between its two (or more) possible states. However, we will show that the non-Markovianity of the driving noise introduces a sequence of noise-induced locally stable transient states even in the linear process, and that the joint effect of non-Markovianity and of the external (additive) sinusoidal driving leads to a behavior identical with that commonly ascribed to the stochastic resonance.

Consider first the simplest case: the noisy relaxation: $a = a_0 \neq a(t)$, $b = 0$. The (exact) solution for the average value of $x(t)$ reads

$$\langle x(t)\rangle = e^{-a_0 t}\langle\Xi(t,0)\rangle x_0\,, \tag{4}$$

where $x_0 = x(0)$ and

$$\Xi(t,t_1) = \exp\left[-\int_{t_1}^{t} dt'\xi(t')\right]. \tag{5}$$

The average of Ξ can be calculated exactly for dichotomic noise (DN) (either Markovian or non-Markovian), but the derivation is lengthy (details are given in Ref. [7(b)]). The result is

$$\langle\Xi(t,t_1)\rangle = \sum_{j=1}^{3} s_j e^{z_j(t-t_1)},$$
$$s_1 = (z_1+\theta_1)(z_1+\theta_2)/(z_1-z_2)(z_1-z_3)\,, \tag{6}$$

etc., and z_j are the solutions of the cubic equation

$$z^3 + (\theta_1+\theta_2)z^2 + (\theta_1\theta_2 - \Delta^2)z - \nu\Delta^2 = 0.$$

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For the Markovian case ($\gamma_0 = 1$, $\gamma_1 = 0$) the noise-driven relaxation is monotonic and the process becomes divergent when $a_0 < \sqrt{\lambda^2 + \Delta^2} - \lambda$. Non-Markovian-driven relaxation is illustrated in Fig. 1. Such a process is highly nonmonotonic, which results, among others, in much longer effective relaxation times than in the noiseless case. Dispersion of $x(t)$, calculated in a similar fashion, exponentially tends to infinity in all cases.

The behavior of the average value of the process variable $x(t)$ is rather trivial. More interesting are the most probable values of $x(t)$, identified with the location of maxima of the probability distribution $P(x,t)$. The latter can be found exactly only for the Markovian case, for non-Markovian driving DN one has to resort to approximations. The best approximation found in [7(a)] leads to the following master equation describing kinetics (1):

$$\left[\frac{\partial}{\partial t} - (a_0 + \Delta)\frac{\partial}{\partial x}x\right]\left[\frac{\partial}{\partial t} - (a_0 - \Delta)\frac{\partial}{\partial x}x\right]P(x,t)$$

$$= -2\lambda \int_0^t Q(t-t')\left(\frac{\partial}{\partial t} - a_0\frac{\partial}{\partial x}x\right)P(x,t'), \quad (7)$$

$$Q(\tau) = \gamma_0\delta(\tau) + \gamma_1 C_2(\tau)e^{-\nu\tau}. \quad (8)$$

The numerical solution of Eq. (7), for an initial distribution centered around $x = 1$, is shown in Fig. 2. The most striking non-Markovian effects are (i) the presence of oscillations in the time evolution, visible along the time axis, and (ii) the appearance in the course of time of several additional peak splittings, visible along the x axis for constant time. Note that Markovian DN is able to force at most one such splitting. According to some interpretations [1], the appearance of additional peaks in $P(x)$ (at a given t) means the appearance of the noise-induced transitions between macroscopic states having no deterministic counterpart. Assuming this philosophy to be true, the non-Markovianity may lead to a multitude of such transitions: more and more new transient,

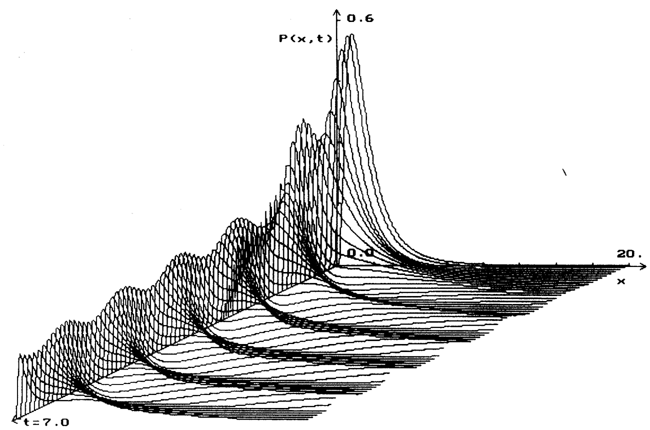

FIG. 2. Temporal evolution of probability density for relaxation driven by non-Markovian dichotomic noise. $a_0 = 1$, $\gamma_0 = 0$, $\gamma_1 = 1$, $\lambda = 2.5$, $\Delta^2 = 5$, $\nu = 0.05$.

locally stable states (local maxima of probability density) appear in the course of the relaxation process driven by non-Markovian DN. This point is illustrated in Fig. 3, where the traces of local maxima of $P(x)$ at subsequent times t are drawn on the (x,t) plane. Dashed lines correspond to unstable local states, full lines correspond to transient (meta) stable local states. Corresponding evolution under Markovian driving is denoted by crosses. In this case the Markovian driving leads to distribution with only one maximum (it coincides with the evolution of one of the most probable non-Markovian states). A logarithmic scale is used to make clearer the behavior near $x = 0$.

Figure 2 shows that there is nonzero probability density between these local states, therefore transitions between such states are possible. It is such noise-induced transitions between new noise-induced states that makes possible the occurrence of stochastic resonance effects when regular driving is added to the kinetics (1).

The stochastic resonance (SR) is the phenomenon of an increase of the response of the system to the deterministic forcing by an increase in the input noise, occurring

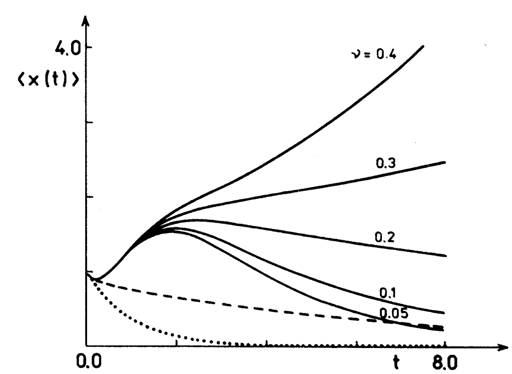

FIG. 1. Relaxation driven by non-Markovian dichotomic noise (full lines). $a_0 = 1$, $x_0 = 1$, $\gamma_0 = 0$, $\gamma_1 = 1$, $\lambda = 2.5$, $\Delta^2 = 5$. Dashed line shows the Markovian-driven case, $\gamma_0 = 1$, $\gamma_1 = 0$; dotted line: no noise ($\Delta^2 = 0$).

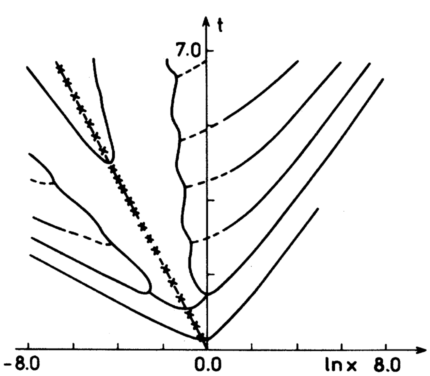

FIG. 3. Temporal evolution of locally most probable states from Fig. 2. Crosses denote the Markovian-driven case, $\gamma_0 = 1$, $\gamma_1 = 0$.

in bistable systems subject to both periodic and random driving. The most popular signature of this effect being commonly in use is the peak in the signal-to-noise ratio (SNR) as a function of the input noise strength, although the physics of this phenomenon is the transfer of energy into some physical process from the stochastic field (*noise*) with the assistance of the regular field (*pumping, signal*). The idea of SR was introduced as a plausible explanation, based on numerical simulations, of the recurrence of ice ages [2], and later discussed in many aspects, both purely theoretical and in application to several specific physical systems. SR is presently one of the most popular subjects in the theory of stochastic processes, and the relevant literature is vast [3,4].

Consider now the process (1) with $a = a_0$, $b = \beta \cos \omega t$. The solution for $x(t)$ is obtained easily in terms of the function (5). The oscillatory character of the process, seen in Figs. 1–3, is now sustained by the external pumping and new transient noise-induced states are being born in the course of time even for $t \to \infty$. This gives rise to the possibility of phenomena of the type of stochastic resonance.

The solution can be written in the form

$$\langle x(t) \rangle = f(t) + A\cos(\omega t + \phi)\,, \tag{9}$$

where $f(t)$ describes the transient behavior, similar to that given by Eq. (5). Elementary calculations give

$$A^2 = \omega^2 \alpha_0^2 + \alpha_1^2\,, \quad \phi = \arctan(-\omega\alpha_0/\alpha_1)\,, \tag{10}$$

$$\alpha_m = \beta \sum_{j=1}^{3} \frac{s_j (a_0 - z_j)^m}{\omega^2 + (a_0 - z_j)^2}. \tag{11}$$

The amplification of the incoming signal is given by the ratio of amplitudes A and β. The signal-to-noise ratio can then be defined as $R = A/\beta\Delta^2$. Figure 4 shows this measure of SNR as the function of the noise strength Δ^2 for a few values of ν. The dashed line shows the Markovian-driven case. The linearity of the process (1) is reflected in the property that the relative amplification does not depend on the amplitude β of the incoming signal.

The system considered exhibits four different time scales: $T_0 = 1/a_0$, $T_m = 1/\nu$, $T_c = 1/\lambda$, and $T_e = 2\pi/\omega$. The analysis of the dependence of R on noise parameters shows that the conditions for the stochastic resonance are present when all these time scales are roughly of the same order of magnitude. Moreover, the smaller the external frequency, and the longer T_m, the stronger the effect. For some combinations of parameters the process becomes divergent, especially for stronger noise. In such cases the SR can be observed only at the beginning of the evolution, before the increase of $\langle x(t) \rangle$ will obscure the effect.

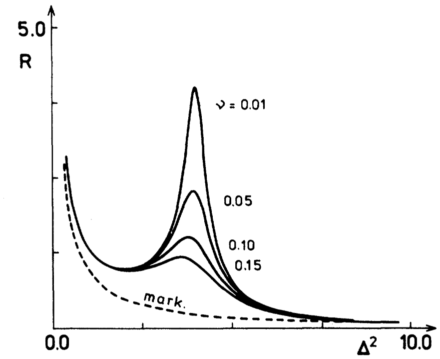

FIG. 4. Signal-to-noise ratio R vs Δ^2 and ν. $a_0 = 0.1$, $x_0 = 1$, $\omega = 1$, $\gamma_0 = 0$, $\gamma_1 = 1$, $\lambda = 2.5$.

The effect is damped by the addition of the Markovian contribution ($\gamma_0 > 0$) and vanishes when there is no non-Markovian contribution ($\gamma_1 = 0$). Therefore the stochastic resonance in a linear system is a purely non-Markovian phenomenon.

The SR described above is certainly atypical in comparison with the literature. In this respect, it is worthwhile to note that recently other unconventional cases of SR have been reported: SR without external periodic force (coherent oscillations of the system are induced either by the noise or are present as a deterministic limit cycle) [9], SR in nonlinear monostable oscillators [10], SR without symmetry breaking [11], or noise-enhanced heterodyning [12].

The behavior of the linear process (1) driven by *non-Markovian* dichotomic noise, described in this paper, resembles in some aspects the behavior of bistable processes. This is in contrast to the recently found effect that sufficiently strong *Markovian* noise may linearize the response of a nonlinear dynamical system [13].

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