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Coupling impedances of small discontinuities: A general approach

Sergey S. Kurennoy and Robert L. Gluckstern
Physics Department, University of Maryland, College Park, Maryland 20742

Gennady V. Stupakov
Stanford Linear Accelerator Center, Stanford, California 94309
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A general theory of the beam interaction with small discontinuities of the vacuum chamber of an accelerator is developed taking into account the reaction of radiated waves back on the discontinuity. The reactive impedance calculated earlier is reproduced as the first order and the resistive one as the second order of a perturbation theory based on this general approach. The theory also gives, in an easy and natural way, the analytical results for the frequencies and coupling impedances of the trapped modes due to small discontinuities on the vacuum chamber of a general cross section. Formulas for two important particular cases—a circular and a rectangular chamber—are presented.

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I. INTRODUCTION

A common tendency in the design of modern accelerators is to minimize beam-chamber coupling impedances to avoid beam instabilities and reduce heating. Even contributions from tiny discontinuities such as pumping holes have to be accounted for, due to their large number, which makes analytical methods for calculating the impedances of small discontinuities very important. According to the Bethe theory of diffraction by small holes [1], the fields diffracted by a hole can be found as those radiated by effective electric and magnetic dipoles. The coupling impedance of pumping holes in the vacuum chamber walls has been calculated earlier [2–4] using this idea. The imaginary part of the impedance is proportional to the difference of hole polarizabilities $(\psi-\chi)$, where the magnetic susceptibility ψ and the electric polarizability χ are small compared to the cubed typical dimension b^3 of the chamber cross section. From considerations of the energy radiated into the chamber and through the hole, the real part of the hole impedance comes out to be proportional to $(\psi^2+\chi^2)$, being usually much smaller than the reactance.

In the present article we develop this analytical approach by taking into account the reaction of radiated waves back on the discontinuity. It leads to a more general theory, which provides us with a general picture, in a wide frequency range, of the coupling impedance of a small discontinuity on the vacuum chamber with an arbitrary cross section. The theory gives analytical expressions for the real and the imaginary part of the coupling impedance. It also reproduces easily all previous results, including those about trapped modes due to small discontinuities of a circular waveguide [5].

While our consideration here is restricted to small holes, it can be readily applied to other small discontinuities such as enlargements or irises. The method remains valid because the idea of effective polarizabilities works equally well in these cases also, as shown in Ref. [6].

The paper is organized as follows. A general analysis of the fields in the chamber with a small discontinuity is given in Sec. II. Section III presents results for the coupling impedance and Sec. IV deals with the trapped modes. The formulas for the two particular cases of the vacuum chamber — with a circular cross section and with a rectangular one — are derived in Appendixes A and B.

II. FIELDS

Let us consider an infinite cylindrical pipe with an arbitrary cross section S and perfectly conducting walls. The z axis is directed along the pipe axis, a hole is located at the point $(\vec{b}, z=0)$, and a typical hole size h satisfies $h \ll b$. To evaluate the coupling impedance one has to calculate the fields induced in the chamber by a given current. If an ultrarelativistic point charge q moves parallel to the chamber axis with the transverse offset $\vec{s}$ from the axis, the fields harmonics $\vec{E}^b, \vec{H}^b$ produced by this charge on the chamber wall without hole would be

$$E^b_\nu(\vec{s},z;\omega) = Z_0 H^b_\tau(\vec{s},z;\omega) = -Z_0 q e^{ikz}\sum_{n,m} k_{nm}^{-2} e_{nm}(\vec{s})\nabla_\nu e_{nm}(\vec{b}) \ , \tag{1}$$

where $Z_0 = 120\pi\ \Omega$ is the impedance of free space and $k^2_{nm}, e_{nm}(\vec{r})$ are eigenvalues and orthonormalized eigenfunctions (EFs) of the two-dimensional (2D) boundary problem in S

$$\left(\nabla^2 + k^2_{nm}\right) e_{nm} = 0, \qquad e_{nm}\big|_{\partial S} = 0 \ . \tag{2}$$

Here $\vec{\nabla}$ is the 2D gradient in plane S, $k=\omega/c$, $\hat{\vec{\nu}}$ means an outward normal unit vector, $\hat{\vec{\tau}}$ is a unit vector tangent to the boundary ∂S of the chamber cross section S, and $\{\hat{\vec{\nu}}, \hat{\vec{\tau}}, \hat{\vec{z}}\}$ form a right-handed basis. The eigenvalues and EFs for particular cross sections are given in Appendixes A and B.

At distances l such that $h \ll l \ll b$, the fields radiated by the hole into the pipe are equal to those produced by effective dipoles [1,7]

$$P_\nu = -\chi\varepsilon_0 E_\nu^h/2, \quad M_\tau = (\psi_{\tau\tau}H_\tau^h + \psi_{\tau z}H_z^h)/2 \ ,$$
$$M_z = (\psi_{z\tau}H_\tau^h + \psi_{zz}H_z^h)/2 \ , \tag{3}$$

where the superscript h means that the fields are taken at the hole. Polarizabilities ψ, χ are related to the effective ones α_e, α_m used in [7,2] as $\alpha_e = -\chi/2$ and $\alpha_m = \psi/2$, so that for a circular hole of radius a, in a thin wall $\psi = 8a^3/3$ and $\chi = 4a^3/3$ [1]. In general, ψ is a symmetric 2D tensor, which can be diagonalized. If the hole is symmetric and its symmetry axis is parallel to $\hat{\vec{z}}$, the skew terms vanish, i.e., $\psi_{\tau z} = \psi_{z\tau} = 0$. In a more general case of a nonzero tilt angle α between the major symmetry axis and $\hat{\vec{z}}$,

$$\begin{aligned}\psi_{\tau\tau} &= \psi_\perp \cos^2\alpha + \psi_\parallel \sin^2\alpha \ ,\\ \psi_{\tau z} &= \psi_{z\tau} = (\psi_\parallel - \psi_\perp)\sin\alpha\cos\alpha \ ,\\ \psi_{zz} &= \psi_\perp \sin^2\alpha + \psi_\parallel \cos^2\alpha \ ,\end{aligned} \tag{4}$$

where $\psi_\parallel$ is the longitudinal magnetic susceptibility (for the external magnetic field along the major axis) and $\psi_\perp$ is the transverse one (the field is transverse to the major axis of the hole). When the effective dipoles are obtained, e.g., by substituting beam fields (1) into Eqs. (3), one can calculate the fields in the chamber as a sum of waveguide eigenmodes excited in the chamber by the dipoles and find the impedance. This approach has been carried out for a circular pipe in [2] and for an arbitrary chamber in [8].

However, a more refined theory should take into account the reaction of radiated waves back on the hole. The radiated fields in the chamber can be expanded in a series in TM and TE eigenmodes [7] as

$$\vec{F} = \sum_{n,m}\left[A_{nm}^+\vec{F}_{nm}^{(E)+}\theta(z) + A_{nm}^-\vec{F}_{nm}^{(E)-}\theta(-z)\right] + \sum_{n,m}\left[B_{nm}^+\vec{F}_{nm}^{(H)+}\theta(z) + B_{nm}^-\vec{F}_{nm}^{(H)-}\theta(-z)\right] \ , \tag{5}$$

where $\vec{F}$ means either $\vec{E}$ or $\vec{H}$, superscripts $\pm$ denote waves radiated, respectively, in the positive ($+$, $z>0$) or negative ($-$, $z<0$) direction, and $\theta(z)$ is the Heaviside step function. The fields $F_{nm}^{(E)}$ of $\{n,m\}$th TM eigenmode in Eq. (5) are expressed [7] in terms of EFs (2)

$$\begin{aligned}E_z^\mp &= k_{nm}^2 e_{nm}\exp(\pm\Gamma_{nm}z), \qquad H_z^\mp = 0,\\ \vec{E}_t^\mp &= \pm\Gamma_{nm}\vec{\nabla}e_{nm}\exp(\pm\Gamma_{nm}z),\\ \vec{H}_t^\mp &= \frac{ik}{Z_0}\hat{\vec{z}}\times\vec{\nabla}e_{nm}\exp(\pm\Gamma_{nm}z) \ ,\end{aligned} \tag{6}$$

where propagation factors $\Gamma_{nm} = (k_{nm}^2 - k^2)^{1/2}$ should be replaced by $-i\beta_{nm}$ with $\beta_{nm} = (k^2 - k_{nm}^2)^{1/2}$ for $k > k_{nm}$. For given values of dipoles (3) the unknown coefficients $A_{nm}^\pm$ can be found [2,8] using the Lorentz reciprocity theorem

$$A_{nm}^\pm = a_{nm}M_\tau \pm b_{nm}P_\nu \ , \tag{7}$$

with

$$a_{nm} = -\frac{ikZ_0}{2\Gamma_{nm}k_{nm}^2}\nabla_\nu e_{nm}^h, \qquad b_{nm} = \frac{1}{2\varepsilon_0 k_{nm}^2}\nabla_\nu e_{nm}^h \ . \tag{8}$$

The fields $F_{nm}^{(H)}$ of the TE_{nm} eigenmode in Eq. (5) are

$$\begin{aligned}H_z^\mp &= k_{nm}'^2 h_{nm}\exp(\pm\Gamma_{nm}'z), \qquad E_z^\mp = 0,\\ \vec{H}_t^\mp &= \pm\Gamma_{nm}'\vec{\nabla}h_{nm}\exp(\pm\Gamma_{nm}'z),\\ \vec{E}_t^\mp &= -ikZ_0\hat{\vec{z}}\times\vec{\nabla}h_{nm}\exp(\pm\Gamma_{nm}'z) \ ,\end{aligned} \tag{9}$$

with propagation factors $\Gamma_{nm}' = (k_{nm}'^2 - k^2)^{1/2}$ replaced by $-i\beta_{nm}' = -i(k^2 - k_{nm}'^2)^{1/2}$ when $k > k_{nm}'$. Here EFs h_{nm} satisfy the boundary problem (2) with the Neumann boundary condition $\nabla_\nu h_{nm}|_{\partial S} = 0$ and $k_{nm}'^2$ are the corresponding eigenvalues; see Appendixes A and B. The TE-mode excitation coefficients in the expansion (5) for the radiated fields are

$$B_{nm}^\pm = \pm c_{nm}M_\tau + d_{nm}P_\nu + q_{nm}M_z \ , \tag{10}$$

where

$$c_{nm} = \frac{1}{2k_{nm}'^2}\nabla_\tau h_{nm}^h, \qquad q_{nm} = \frac{1}{2\Gamma_{nm}'}h_{nm}^h,$$
$$d_{nm} = -\frac{ik}{2Z_0\varepsilon_0\Gamma_{nm}'k_{nm}'^2}\nabla_\tau h_{nm}^h \ . \tag{11}$$

Now we can add corrections to the beam fields (1) due to the radiated waves in the vicinity of the hole. It gives

$$E_\nu = \frac{E_\nu^b + \psi_{z\tau}\Sigma_x' Z_0 H_\tau + \psi_{zz}\Sigma_x' Z_0 H_z}{1 - \chi(\Sigma_1 - \Sigma_1')}, \tag{12}$$

$$H_\tau = \frac{H_\tau^b + \psi_{\tau z}(\Sigma_2 - \Sigma_2')H_z}{1 - \psi_{\tau\tau}(\Sigma_2 - \Sigma_2')}, \tag{13}$$

$$H_z = \frac{\chi\Sigma_x' E_\nu/Z_0 + \psi_{z\tau}\Sigma_3' H_\tau}{1 - \psi_{zz}\Sigma_3'}, \tag{14}$$

where ($s = \{n,m\}$ is a generalized index)

$$\Sigma_1 = \frac{1}{4}\sum_s \frac{\Gamma_s\left(\nabla_\nu e_s^h\right)^2}{k_s^2}, \quad \Sigma_2 = \frac{k^2}{4}\sum_s \frac{\left(\nabla_\nu e_s^h\right)^2}{\Gamma_s k_s^2},$$
$$\Sigma_1' = \frac{k^2}{4}\sum_s \frac{\left(\nabla_\tau h_s^h\right)^2}{\Gamma_s' k_s'^2}, \quad \Sigma_2' = \frac{1}{4}\sum_s \frac{\Gamma_s'\left(\nabla_\tau h_s^h\right)^2}{k_s'^2},$$
$$\Sigma_x' = i\frac{k}{4}\sum_s \frac{h_s^h\nabla_\tau h_s^h}{\Gamma_s'}, \quad \Sigma_3' = \frac{1}{4}\sum_s \frac{k_s'^2\left(h_s^h\right)^2}{\Gamma_s'} \ . \tag{15}$$

Since this consideration works at distances larger than h, one should restrict the summation in Eq. (15) to the values of $s = \{n,m\}$ such that $k_s h \le 1$ and $k_s' h \le 1$.

III. IMPEDANCE

A. Longitudinal impedance

The generalized longitudinal impedance of the hole depends on the transverse offsets from the chamber axis $\vec{s}$

of the leading particle and $\vec{t}$ of the test particle and is defined [9] as

$$Z(k;\vec{s},\vec{t}\,) = -\frac{1}{q}\int_{-\infty}^{\infty} dz e^{-ikz} E_z(\vec{t},z;\omega) \ , \tag{16}$$

where the longitudinal field $E_z(\vec{t},z;\omega)$ is taken along the test particle path. The displacements from the axis are assumed to be small $s \ll b$ and $t \ll b$. The impedance $Z(k;\vec{s},\vec{t}\,)$ includes higher multipole longitudinal impedances and in the limit $s,t \to 0$ gives the usual monopole one $Z(k) = Z(k;0,0)$. To calculate $E_z(\vec{t},z;\omega)$, we use Eq. (5) with coefficients (7) and (10) in which the corrected near-hole fields (12)–(14) are substituted [a dependence on $\vec{s}$ enters via beam fields (1)]. It yields

$$Z(k;\vec{s},\vec{t}\,) = -\frac{ikZ_0 e_\nu(\vec{s}\,)e_\nu(\vec{t}\,)}{2} \times \left[\frac{\psi_{\tau\tau}}{1-\psi_{\tau\tau}(\Sigma_2 - \Sigma_2')} + \psi_{\tau z}^2 \Sigma_3' - \frac{\chi}{1-\chi(\Sigma_1 - \Sigma_1')} \right] , \tag{17}$$

where

$$e_\nu(\vec{r}\,) \equiv \frac{E_\nu^b}{Z_0 q} = -\sum_s \frac{e_s(\vec{r}\,)\nabla_\nu e_s(\vec{b}\,)}{k_s^2} \tag{18}$$

is merely the normalized electrostatic field produced at the hole location by the filament charge displaced from the chamber axis by the distance $\vec{r}$; cf. Eq. (1). In practice, we are interested only in the monopole term $Z(k) = Z(k;0,0)$ and will mostly use below Eq. (17) with the replacement $e_\nu(\vec{s}\,)e_\nu(\vec{t}\,) \to \tilde{e}_\nu^2$, where $\tilde{e}_\nu \equiv e_\nu(0)$. In deriving Eq. (17) we have neglected the coupling terms between E_ν, H_τ, and H_z [cf. Eqs. (12)–(14)], which contribute to the third order of an expansion discussed below, and also have taken into account that $\psi_{\tau z} = \psi_{z\tau}$.

For a small discontinuity, polarizabilities $\psi, \chi = O(h^3)$ and are small compared to b^3. If we expand the impedance (17) in a perturbation series in polarizabilities, the first order gives

$$Z_1(k) = -\frac{ikZ_0\tilde{e}_\nu^2}{2}(\psi_{\tau\tau} - \chi) \ , \tag{19}$$

which is exactly the inductive impedance obtained in [8] for an arbitrary cross section of the chamber. For a particular case of a circular pipe, from either direct summation in (1) or applying the Gauss law, we get $\tilde{e}_\nu = 1/(2\pi b)$, the substitution of which into Eq. (19) leads to a well-known result [2,3]. From a physical point of view, keeping only the first-order term (19) corresponds to dropping all radiation corrections in Eqs. (12)–(14).

These corrections first reveal themselves in the second-order term

$$Z_2(k) = -\frac{ikZ_0\tilde{e}_\nu^2}{2}\left[\psi_{\tau\tau}^2(\Sigma_2 - \Sigma_2') + \psi_{\tau z}^2\Sigma_3' + \chi^2(\Sigma_1' - \Sigma_1)\right] , \tag{20}$$

which at frequencies above the chamber cutoff has both a real and an imaginary part. The real part of the impedance is

$$\mathrm{Re} Z_2(k) = \frac{k^3 Z_0 \tilde{e}_\nu^2}{8}\left\{ \psi_{\tau z}^2 \sum_s^{<} \frac{k_s'^2 \left(h_s^h\right)^2}{k^2\beta_s'} + \psi_{\tau\tau}^2\left[\sum_s^{<}\frac{\left(\nabla_\nu e_s^h\right)^2}{\beta_s k_s^2} + \sum_s^{<}\frac{\beta_s'\left(\nabla_\tau h_s^h\right)^2}{k^2 k_s'^2}\right] + \chi^2\left[\sum_s^{<}\frac{\beta_s\left(\nabla_\nu e_s^h\right)^2}{k^2k_s^2} + \sum_s^{<}\frac{\left(\nabla_\tau h_s^h\right)^2}{\beta_s' k_s'^2}\right]\right\} , \tag{21}$$

where the sums include only a finite number of the eigenmodes propagating in the chamber at a given frequency, i.e., those with $k_s < k$ or $k_s' < k$.

The dependence of Re Z on frequency is rather complicated; it has sharp peaks near the cutoffs of all propagating eigenmodes of the chamber and increases on average with the frequency increase. Well above the chamber cutoff, i.e., when $kb \gg 1$ (but still $kh \ll 1$ to justify the Bethe approach), this dependence can be derived as follows. If the waveguide cross section S is a simply connected region, the average number $n(k)$ of the eigenvalues k_s (or k_s') which are less than k for $kb \gg 1$, is proportional to k^2 [10]:

$$n(k) \simeq \frac{S}{4\pi}k^2 + O(k) \ ,$$

where S is the area of the cross section. Using this property and taking into account that $\nabla_\nu e_s^h \propto k_s e_s^h$ and $\nabla_\tau h_s^h \propto k_s' h_s^h$, we replace sums on the right-hand side (rhs) of Eq. (21) by integrals as $\sum_s^< \to \int^k dk \frac{d}{dk}n(k)$. It turns out that all sums in Eq. (21) have the same asymptotic behavior, being linear in k, and as a result, Re $Z \propto k^4$. Obtaining the exact coefficient in this dependence seems rather involved for a general S, but it can be easily done for a rectangular chamber; see Appendix B. The result is

$$\mathrm{Re}\, Z = \frac{Z_0 k^4 \tilde{e}_\nu^2}{12\pi}(\psi_{\tau\tau}^2 + \psi_{\tau z}^2 + \chi^2) \ . \tag{22}$$

Remarkably, the same answer (for $\psi_{\tau z} = 0$) has been obtained in Ref. [8] simply by calculating the energy radiated by the dipoles into a half-space. The physical reason for this coincidence is clear: at frequencies well above the cutoff the effective dipoles radiate into the waveguide the same energy as into an open half-space.

Strictly speaking, the real part of impedance is nonzero even below the chamber cutoff, due to radiation outside. In the case of a thin wall, Re Z below the cutoff can be estimated by Eq. (22) and twice that for high frequencies $kb \gg 1$. For a thick wall, the contribution of the radiation outside to Re Z is still given by Eq. (22), but with the outside polarizabilities substituted, and it decreases exponentially with the thickness increase [3].

The real part of the impedance is related to the power P scattered by the hole into the beam pipe as Re $Z = 2P/q^2$. These energy considerations can be used as an alternative way for the impedance calculation. The radiated power is

$$P = \sum_s \left[A_s^2 P_s^{(E)} + B_s^2 P_s^{(H)} \right] ,$$

where we sum over all propagating modes in both directions and P_s means the time-averaged power radiated in the sth eigenmode:

$$P_s^{(E)} = k\beta_s k_s^2/(2Z_0), \quad P_s^{(H)} = Z_0 k \beta_s' k_s'^2/2 .$$

Substituting beam fields (1) into Eqs. (7)–(11) for the coefficients A_s and B_s and performing calculations gives us exactly the result (21). Such an alternative derivation of the real part has been carried out in Ref. [4] for a circular pipe with a symmetric untilted hole ($\psi_{\tau z} = 0$). Our result (21) coincides, in this particular case, with that of Ref. [4]. It is appropriate to mention also that in this case at high frequencies the series has been summed approximately [4] using asymptotic expressions for roots of the Bessel functions and the result, of course, agrees with Eq. (22).

One should note that the additional $\psi_{\tau z}^2$ term in Eq. (21) is important in some particular cases. For example, this skew term gives a leading contribution to Re Z for a long and slightly tilted slot because $\psi_{\tau z}$ can be much larger than $\psi_{\tau\tau}$ in this case since $\psi_{\|} \gg \psi_{\perp}$; cf. Eqs. (4).

B. Transverse impedance

We will make use of the expression for the generalized longitudinal impedance $Z(k;\vec{s},\vec{t})$, Eq. (17). According to the Panofsky-Wenzel theorem, the transverse impedance can be derived as $\vec{Z}_\perp(k;\vec{s},\vec{t}) = \vec{\nabla} Z(k;\vec{s},\vec{t})/(ks)$; see, e.g., [9] for details. This way leads to the expression

$$\vec{Z}_\perp(k;\vec{s},\vec{t}) = -\frac{iZ_0 e_\nu^{\rm dip}(\vec{s})\vec{\nabla} e_\nu(\vec{t})}{2s} \times \left[\frac{\psi_{\tau\tau}}{1-\psi_{\tau\tau}(\Sigma_2 - \Sigma_2')} + \psi_{\tau z}^2 \Sigma_3' - \frac{\chi}{1-\chi(\Sigma_1 - \Sigma_1')} \right] , \tag{23}$$

where $e_\nu^{\rm dip}(\vec{s}) = \vec{s} \cdot \vec{\nabla} e_\nu(\vec{s})$.

Going to the limit $s \to t \to 0$, we get the usual dipole transverse impedance

$$\vec{Z}_\perp(k) = -iZ_0(d_x^2 + d_y^2)\ /2\vec{a}_d \cos(\varphi_b - \varphi_d) \times \left[\frac{\psi_{\tau\tau}}{1-\psi_{\tau\tau}(\Sigma_2 - \Sigma_2')} + \psi_{\tau z}^2 \Sigma_3' - \frac{\chi}{1-\chi(\Sigma_1 - \Sigma_1')} \right] . \tag{24}$$

Here x, y are the horizontal and vertical coordinates in the chamber cross section; $d_x \equiv \partial_x e_\nu(0)$, $d_y \equiv \partial_y e_\nu(0)$; $\varphi_b = \varphi_s = \varphi_t$ is the azimuthal angle of the beam position in the cross-section plane; and $\vec{a}_d = \vec{a}_x \cos\varphi_d + \vec{a}_y \sin\varphi_d$ is a unit vector in this plane in direction φ_d, which is defined by conditions $\cos\varphi_d = d_x/\sqrt{d_x^2 + d_y^2}$, $\sin\varphi_d = d_y/\sqrt{d_x^2 + d_y^2}$. It is seen from Eq. (24) that the angle φ_d shows the direction of the transverse-impedance vector $\vec{Z}_\perp$ and therefore of the beam-deflecting force. Moreover, the value of $Z_\perp$ is maximal when the beam is deflected along this direction and vanishes when the beam offset is perpendicular to it. For a circular pipe, $\varphi_d = \varphi_h$ and the deflecting force is directed toward (or opposite to) the hole. For a general cross section, this is not the case; see [8] for rectangular and elliptic chambers.

Equation (24) includes the corrections due to waves radiated by the hole into the chamber in exactly the same way as Eq. (17) for the longitudinal impedance. If we expand it in a series in the polarizabilities, the first order of the square brackets in (24) gives $(\psi_{\tau\tau} - \chi)$ and the resulting inductive impedance coincides with that obtained in [8]. The second-order term includes Re $Z_\perp$; cf. Sec. III A.

IV. TRAPPED MODES

So far we considered the perturbation expansion of Eq. (17) implicitly assuming that correction terms $O(\psi)$ and $O(\chi)$ in the denominators of its rhs are small compared to 1. Under certain conditions this assumption is incorrect and this situation leads to some nonperturbative results. Indeed, at frequencies slightly below the chamber cutoffs $0 < k_s - k \ll k_s$ (or the same with replacement $k_s \to k_s'$), a single term in sums Σ_1', Σ_2, or Σ_3' becomes very large, due to very small $\Gamma_s = (k_s^2 - k^2)^{1/2}$ (or Γ_s') in its denominator, and then the "corrections" $\psi\Sigma$ or $\chi\Sigma$ can be of the order of 1. As a result, one of the denominators of the rhs of Eq. (17) can vanish, which corresponds to a resonance of the coupling impedance. On the other hand, vanishing denominators in Eqs. (12)–(14) mean the existence of nonperturbative eigenmodes of the chamber with a hole, since nontrivial solutions $E, H \neq 0$ exist even for vanishing external (beam) fields $E^b, H^b = 0$. These eigenmodes are nothing but the trapped modes studied in [5] for a circular waveguide with a small discontinuity. In our approach, one can easily derive parameters of trapped modes for waveguides with an arbitrary cross section.

A. Frequency shifts

Let us for brevity restrict ourselves to the case $\psi_{\tau z} = 0$ and consider Eq. (13) in more detail. For $H^b = 0$ we have

$$H_\tau \left[1 - \psi_{\tau\tau} \frac{k^2 \left(\nabla_\nu e_s^h\right)^2}{4\Gamma_s k_s^2} + \cdots \right] = 0 , \tag{25}$$

where $s \equiv \{n, m\}$ is the generalized index, and the ellipsis denotes all other terms of the series Σ_2, Σ_2'. At frequency Ω_s slightly below the cutoff frequency $\omega_s = k_s c$ of the TM$_s$ mode, the fraction in Eq. (25) is large due to small Γ_s in its denominator and one can neglect the other terms. Then the condition for a nontrivial solution $H_\tau \neq 0$ to exist is

$$\Gamma_s \simeq \frac{1}{4}\psi_{\tau\tau}\left(\nabla_\nu e_s^h\right)^2 \ . \tag{26}$$

In other words, there is a solution of the homogeneous, i.e., without external currents, Maxwell equations for the chamber with the hole, having the frequency $\Omega_s < \omega_s$, the sth trapped TM mode. When Eq. (26) is satisfied, the series (5) is obviously dominated by the single term $A_s F_s^E$; hence the fields of the trapped mode have the form [cf. Eq. (6)]

$$\begin{aligned} \mathcal{E}_z &= k_s^2 e_s \exp(-\Gamma_s|z|), \qquad \mathcal{H}_z = 0, \\ \vec{\mathcal{E}}_t &= \mathrm{sgn}(z)\Gamma_s \vec{\nabla} e_s \exp(-\Gamma_s|z|) \ , \\ Z_0\vec{\mathcal{H}}_t &= ik\hat{\vec{z}} \times \vec{\nabla} e_s \exp(-\Gamma_s|z|) \ , \end{aligned} \tag{27}$$

up to some arbitrary amplitude. Strictly speaking, these expressions are valid at distances $|z| > b$ from the discontinuity. Typically, $\psi_{\tau\tau} = O(h^3)$ and $\nabla_\nu e_s^h = O(1/b)$ and, as a result, $\Gamma_s b \ll 1$. It follows that the field of the trapped mode extends along the vacuum chamber over the distance $1/\Gamma_s$, which is large compared to the chamber transverse dimension b.

The existence of the trapped modes in a circular waveguide with a small hole was proved in [5] and conditions similar to Eq. (26) for this particular case were obtained in [5,11] using the Lorentz reciprocity theorem. From the general approach presented here for the waveguide with an arbitrary cross section, their existence follows in a natural way. Moreover, in such a derivation, the physical mechanism of this phenomenon becomes quite clear: a tangential magnetic field induces a magnetic moment on the hole and the induced magnetic moment supports this field if the resonance condition (26) is satisfied, so that the mode can exist even without an external source. One should also note that the induced electric moment P_ν is negligible for the trapped TM mode since $P_\nu = O(\Gamma_s b) M_\tau$, as follows from Eq. (27).

Equation (26) gives the frequency shift $\Delta\omega_s \equiv \omega_s - \Omega_s$ of the trapped sth TM mode down from the cutoff ω_s

$$\frac{\Delta\omega_s}{\omega_s} \simeq \frac{1}{32k_s^2}\psi_{\tau\tau}^2\left(\nabla_\nu e_s^h\right)^4 \ . \tag{28}$$

In the case of a small hole this frequency shift is very small and for the trapped mode (27) to exist, the width of the resonance should be smaller than $\Delta\omega_s$. Contributions to the resonance width come from energy dissipation in the waveguide wall due to its finite conductivity and from energy radiation inside the waveguide and outside, through the hole. Radiation escaping through the hole is easy to estimate [5] and for a thick wall it is exponentially small; see, e.g., [3]. The damping rate due to a finite conductivity is $\gamma = P/(2W)$, where P is the time-averaged power dissipation and W is the total field energy in the trapped mode, which yields

$$\frac{\gamma_s}{\omega_s} = \frac{\delta}{4k_s^2}\oint dl \left(\nabla_\nu e_s\right)^2 \ , \tag{29}$$

where δ is the skin depth at frequency Ω_s and the integration is along the boundary ∂S. The evaluation of the radiation into the lower waveguide modes propagating in the chamber at given frequency Ω_s is also straightforward [4] if one makes use of the coefficients of mode excitation by effective dipoles on the hole Eqs. (7)–(11). The corresponding damping rate $\gamma_R = O(\psi^3)$ is small compared to $\Delta\omega_s$. For instance, if there is only one TE$_p$ mode with the frequency below that for the lowest TM$_s$ mode, like in a circular waveguide (H_{11} has a lower cutoff than E_{01}),

$$\frac{\gamma_R}{\Delta\omega_s} = \frac{\psi_{\tau\tau}\beta_p'}{k_p'^2}\left(\nabla_\nu h_s^h\right)^2 \ , \tag{30}$$

where $\beta_p' \simeq (k_s^2 - k_p'^2)^{1/2}$ because $k \simeq k_s$.

One can easily see that denominator $[1 - \chi(\Sigma_1 - \Sigma_1')]$ in Eq. (12) does not vanish because singular terms in Σ_1' have the "wrong" sign. However, due to the coupling between E_ν and H_z, a nontrivial solution $E_\nu, H_z \neq 0$ of simultaneous equations (12) and (14) can exist, even when $E^b = 0$. The corresponding condition has the form

$$\Gamma_{nm}' \simeq \frac{1}{4}\left[\psi_{zz}k_{nm}'^2\left(h_{nm}^h\right)^2 - \chi\left(\nabla_\tau h_{nm}^h\right)^2\right] \ , \tag{31}$$

which gives the frequency of the trapped TE$_{nm}$ mode, provided the rhs of Eq. (31) is positive.

B. Impedance

The trapped mode (27) gives a resonance contribution to the longitudinal coupling impedance at $\omega \approx \Omega_s$

$$Z_s(\omega) = \frac{2i\Omega_s\gamma_s R_s}{\omega^2 - (\Omega_s - i\gamma_s)^2} \ , \tag{32}$$

where the shunt impedance R_s can be calculated as that for a cavity with given eigenmodes, e.g. [9],

$$R_s = \frac{\sigma\delta\left|\int dz \exp(-i\Omega_s z/c)\mathcal{E}_z(z)\right|^2}{\int_{S_w} ds|\mathcal{H}_\tau|^2} \ . \tag{33}$$

The integral in the denominator is taken over the inner wall surface and we assume here that the power losses due to its finite conductivity dominate. Integrating in the numerator one should include all TM modes generated by the effective magnetic moment on the hole using Eqs. (7)–(11), in spite of a large amplitude of only the trapped TM$_s$ mode. While all other amplitudes are suppressed by a factor $\Gamma_s b \ll 1$, their contributions are comparable to that from TM$_s$ because this integration produces the factor $\Gamma_q b$ for any TM$_q$ mode. The integral in the denominator is obviously dominated by TM$_s$. Performing calculations yields

$$R_s = \frac{Z_0\tilde{e}_\nu^2\psi_{\tau\tau}^3 k_s \left(\nabla_\nu e_s^h\right)^4}{8\delta\oint dl\left(\nabla_\nu e_s\right)^2} \ , \tag{34}$$

where $\tilde{e}_\nu = e_\nu(0)$ is defined by Eq. (18).

Results for a particular shape of the chamber cross section can be obtained from the equations above by substituting the corresponding eigenfunctions (see Appendixes A and B).

One should note that typically the peak value R_s of the impedance resonance due to one small hole is rather small except for the limit of a perfectly conducting wall $\delta \to 0$; indeed, $R_s \propto (h/b)^9 b/\delta$ and $h \ll b$. However, for many not-so-far separated holes, the resulting impedance can be much larger. The trapped modes for many discontinuities on a circular waveguide have been studied in Ref. [11] and the results can be readily transferred to the considered case of an arbitrary shape of the chamber cross section. In particular, it was demonstrated that the resonance impedance in the extreme case can be as large as N^3 times that for a single discontinuity, where N is the number of discontinuities. It strongly depends on the distribution of discontinuities, or on the distance between them if a regular array is considered.

V. DISCUSSION

The analytical approach developed above provides a general picture for the coupling impedance of a small discontinuity on the vacuum chamber in a wide frequency range, up to frequencies well above the cutoff. The upper limit on the frequency is imposed by the applicability of the Bethe theory: the wavelength must be large compared to the typical size of the discontinuity.

The developed theory gives the real and the imaginary part of the impedance, as well as trapped modes. Results for specific shapes of the chamber cross section can be derived from the formulas obtained by substituting corresponding EFs; see Appendixes A and B for circular and rectangular cross sections, respectively. For a more complicated shape, the impedance dependence on the hole position can be easily obtained numerically by solving a 2D electrostatic problem for $\tilde{e}_\nu$; cf., for example, in Ref. [8] for an elliptical pipe.

We have not considered explicitly effects of the wall thickness, assuming that the hole polarizabilities are the inside ones [3] and they include these effects. We also briefly discussed the radiation escaping through the hole, of which contributions to the real part of the impedance are estimated [2,3,5] and usually are very small.

At high frequencies (above the chamber cutoff) the mutual interaction of many holes is important because it can cause resonances when the hole pattern is periodic. The effect strongly depends on the periodicity of the hole distribution in the longitudinal direction and even small random violations of the periodicity damp these resonances; see in [12,4]. The trapped modes in circular waveguides with many discontinuities have been studied in [11]. It was shown that the interaction of identical discontinuities can increase the resonances due to the trapped modes essentially, but the resonance strength is reduced drastically by a small randomization of the parameters of the discontinuities (such as the hole size or slot length, or the area of the longitudinal cross section for axisymmetric enlargements). The effects of the mutual interaction remain qualitatively the same for an arbitrary cross section and the results mentioned can be easily transferred to this case.

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APPENDIX A: CIRCULAR CHAMBER

For a circular cross section of radius b the eigenvalues $k_{nm} = \mu_{nm}/b$, where μ_{nm} is mth zero of the Bessel function $J_n(x)$ and the normalized EFs are

$$e_{nm}(r,\varphi) = \frac{J_n(k_{nm}r)}{\sqrt{N^E_{nm}}} \left\{ \begin{array}{c} \cos n\varphi \\ \sin n\varphi \end{array} \right\} , \tag{A1}$$

with $N^E_{nm} = \pi b^2 \epsilon_n J^2_{n+1}(\mu_{nm})/2$, where $\epsilon_0 = 2$ and $\epsilon_n = 1$ for $n \neq 0$. For TE modes $k'_{nm} = \mu'_{nm}/b$ with $J'_n(\mu'_{nm}) = 0$ and

$$h_{nm}(r,\varphi) = \frac{J_n(k'_{nm}r)}{\sqrt{N^H_{nm}}} \left\{ \begin{array}{c} \cos n\varphi \\ \sin n\varphi \end{array} \right\} , \tag{A2}$$

where $N^H_{nm} = \pi b^2 \epsilon_n (1 - n^2/\mu'^2_{nm}) J^2_n(\mu'_{nm})/2$. In this case $\tilde{e}_\nu = 1/(2\pi b)$, which also follows from the Gauss law, and formula (19) for the inductive impedance takes an especially simple form; cf. [2,3].

Assuming the hole is located at $\varphi = 0$, we get, for the trapped modes from Eq. (26),

$$\Gamma_{nm} = \frac{\psi_{\tau\tau}\mu^2_{nm}}{2\pi\epsilon_n b^4} \tag{A3}$$

and from Eq. (34)

$$R_{nm} = \frac{Z_0 \psi^3_{\tau\tau}\mu^3_{nm}}{32\pi^4 \epsilon_n \delta b^8} . \tag{A4}$$

For TE modes, from Eq. (31),

$$\Gamma'_{nm} = \frac{\psi_{zz}\mu'^4_{nm}}{2\pi\epsilon_n b^4 (\mu'^2_{nm} - n^2)} . \tag{A5}$$

Note that only the modes with $\cos n\varphi$ can be trapped, while sine modes just do not "see" the hole.

The results of this section coincide with those of [5,11], except R_s in [5], where the contribution of only the trapped mode to Eq. (33) was taken into account. Formulas for an axisymmetric enlargement with area A of the longitudinal cross section are easily obtained from Eqs. (A3) and (A4) with $n = 0$ by the substitution $\psi_\tau \to 4\pi b A$.

APPENDIX B: RECTANGULAR CHAMBER

For a rectangular chamber of width a and height b the eigenvalues are $k_{nm} = \pi\sqrt{n^2/a^2 + m^2/b^2}$, with $n, m =$

$1, 2, \ldots,$ and the normalized EFs are

$$e_{nm}(x,y) = \frac{2}{\sqrt{ab}} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{b} , \tag{B1}$$

with $0 \le x \le a$ and $0 \le y \le b$. Let a hole be located in the sidewall at $x = a$, $y = y_h$. From Eq. (18), after some algebra, follows

$$\tilde{e}_\nu = \frac{1}{b}\Sigma\left(\frac{a}{b}, \frac{y_h}{b}\right) , \tag{B2}$$

where

$$\Sigma(u,v) = \sum_{l=0}^{\infty} \frac{(-1)^l \sin[\pi(2l+1)v]}{\cosh[\pi(2l+1)u/2]} \tag{B3}$$

is a fast converging series; the behavior of $\Sigma(u,v)$ versus v for different values of the aspect ratio u is plotted in Ref. [8]. Substituting (B2) into the formulas of Sec. III gives the impedance of the hole in a rectangular chamber.

For a rectangular chamber, it is easy to derive the asymptotics of $\mathrm{Re}\, Z$ at high frequencies $kb \gg 1$. Let us take for simplicity $a = b$ and the hole be in the middle of the wall $y_h = b/2$. Then the sums on the rhs of Eq. (21) take the form

$$\sum_s^{<} \frac{\left(\nabla_\nu e_s^h\right)^2}{\beta_s k_s^2} = \frac{4\pi}{b} \sum_{m,n=1}^{<} \frac{n^2}{m^2+n^2} \times \frac{\sin^2(\pi m/2)}{\sqrt{(kb/\pi)^2 - m^2 - n^2}} ,$$

where the sum is restricted to $m^2 + n^2 < (kb/\pi)^2$. For large values of kb, one can replace $\sin^2(\pi m/2)$ by $1/2$ (only odd m's contribute) and approximate the remaining sum by an integral

$$\int_0\int_0^{x^2+y^2=(kb/\pi)^2} dx dy \frac{x^2}{(x^2+y^2)\sqrt{(kb/\pi)^2 - x^2 - y^2}} .$$

It is easily evaluated using polar coordinates (ρ, ϕ) such that $x = \rho\cos\phi$, $y = \rho\sin\phi$, with $0 < \rho < kb/\pi$ and $0 < \phi < \pi/2$, and the result is

$$\sum_s^{<} \frac{\left(\nabla_\nu e_s^h\right)^2}{\beta_s k_s^2} \simeq \frac{k}{2\pi} .$$

In a similar way,

$$\sum_s^{<} \frac{\beta_s' \left(\nabla_\tau h_s^h\right)^2}{k^2 k_s'^2} \simeq \sum_s^{<} \frac{\beta_s \left(\nabla_\nu e_s^h\right)^2}{k^2 k_s^2} \simeq \frac{k}{6\pi},$$

$$\sum_s^{<} \frac{\left(\nabla_\tau h_s^h\right)^2}{\beta_s' k_s'^2} \simeq \frac{k}{2\pi}, \qquad \sum_s^{<} \frac{k_s'^2 \left(h_s^h\right)^2}{k^2 \beta_s'} \simeq \frac{2k}{3\pi} .$$

Substituting these asymptotics into Eq. (21) leads to Eq. (22).

For the trapped modes, Eq. (26) gives

$$\Gamma_{nm} = \frac{\psi_{\tau\tau}\pi^2 n^2}{a^3 b} \sin^2\left(\frac{\pi m y_h}{b}\right) , \tag{B4}$$

and from Eq. (34) the impedance is

$$R_{nm} = \frac{Z_0 \psi_{\tau\tau}^3 \pi^3 n^2 \sqrt{n^2 b^2 + m^2 a^2}}{2\delta a^4 b^2 (n^2 b^3 + m^2 a^3)} \times \Sigma^2\left(\frac{a}{b}, \frac{y_h}{b}\right) \sin^4\left(\frac{\pi m y_h}{b}\right) . \tag{B5}$$

Both the frequency shift and especially the impedance decrease very fast if the hole is displaced closer to the corners of the chamber, i.e., when $y_h \to b$ or $y_h \to 0$.

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