Generalized relaxation theory and vortices in plasmas

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We present a generalization of the relaxation theory based on the canonical momentum of each species fluid in a multicomponent plasma. The generalized helicity, as a topological quantity, has a lifetime larger than the lifetime of the energy. We then propose a simple variational principle that suggests the existence of vortex structures. We study localized solutions, assuming the existence of a separatrix. Two-dimensional and three-dimensional solutions are studied for an electron-positron-proton plasma. Ideal magnetohydrodynamic three-dimensional localized vortices are studied as well. Possible cosmological implications are discussed.

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I. INTRODUCTION

The observations of galactic magnetic fields (GMF's) of order 10^{-6} G in contrast to the intergalactic magnetic field $<2\times10^{-8}$ G for $N_e<4\times10^{-5}$ electrons/cm^3 brought to the astrophysicist an outstanding problem which concerns the origin and nature of these fields. For a review, see, for example, the reports of Asseo and Sol [1] and Kronberg [2] and references therein. Essentially there are two schools of thought, one believing in a primordial cosmic magnetic field from the big bang and the other one in a generating magnetic field from a certain dynamical mechanism such as dynamos. In any case, it is of considerable appeal to verify if plasma phenomena, including cosmic magnetic fields, played a role in the formation of structures that led to seeds of what we see today, such as galaxies and clusters of galaxies. Kulsrud and Anderson [3] showed that it is difficult to make presently observed galactic magnetic fields through the dynamo mechanism after the recombination epoch. Ratra [4] discussed the possibility of magnetic field generation during the inflation epoch. Tajima *et al.* [5,6] and Coles [7] discussed the possibility of magnetic field generation during the plasma epoch. In particular, Tajima *et al.* [5] found that, for a primordial plasma (such as $t\sim10^{-2}$ sec) in a thermodynamic equilibrium, a large amount of magnetic field energy is stored in configurations with a minuscule spatial size, i.e., magnetic fields favor the formation of tiny "bubbles."

It is thus of interest to consider what are the natural forms of magnetic and plasma topologies, as, for example, in a relaxed state plasma [8,9]. Therefore we consider structures such as vortices and/or solitary waves of magnetic fields. In this work we present some exact static solutions at different spatial scales. For that purpose we assume that there is a slight asymmetry between the electron and the positron density, balanced by a background fluid of protons. We also assume that the fluid velocities of electrons and positrons are equal and that the plasma is in isothermic equilibrium. Finally, as one of the main features of the cosmological plasma we assume no external field boundary conditions (no static field at infinity).

Thus we seek localized stationary solutions of low frequency electromagnetic fields that do not cause charge separation. The localizability of the fields is required by virtue of the no external field boundary condition. That is, at spatial infinity the fields must vanish. We may require the existence of a separatrix beyond which the fields decrease fast enough so that the total field energy is finite. We show that these solutions are stable against linear perturbations.

The paper is organized as follows. Section II presents a generalized theory of relaxation for a multicomponent plasma and we conclude that the formation of vortices is a possible equilibrium configuration. In Sec. III we study static and stationary vortices in an electron-positron-proton (ee^+P, abbreviated *eeP*) plasma and present possible solutions. That is, we assume the existence of a separatrix beyond which the magnetic field vanishes (in some cases asymptotically vanishes). A similar technique [10] has been proposed to find vortices of the electron magnetohydrodynamic (EMHD) fluid in a uniform background proton plasma. In Sec. IV we present possible three-dimensional solitary vortices in spatial scales for which ideal magnetohydrodynamics (MHD) equations hold.

In this way the theory provides a possible framework that there is a hierarchy of formation of large-scale structures in a plasma, beginning with the spatial scale of thermal fluctuations in an *eeP* plasma up to large scales in a MHD plasma. Nevertheless, in this work, we do not address either the time evolution of each structure or the possible evolution of one kind of structure into another.

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II. GENERALIZED RELAXATION THEORY FOR MULTICOMPONENT PLASMAS

A. Vorticity equation

The macroscopic equations of a plasma with N species are

$$\nabla\cdot\mathbf{B}=0 , \tag{1}$$

$$\nabla\cdot\mathbf{E}=\frac{4\pi}{c}\sum_{a=1}^{N} q_a n_a , \tag{2}$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\sum_{a=1}^{N} q_a n_a \mathbf{v}_a+\frac{\partial}{c\partial t}\mathbf{E} , \tag{3}$$

$$\nabla\times\mathbf{E}=-\frac{\partial}{c\partial t}\mathbf{B} , \tag{4}$$

$$\frac{\partial}{\partial t}n_a+\nabla\cdot(n_a\mathbf{v}_a)=0 , \tag{5}$$

$$\nabla\cdot\mathbf{v}_a=0 , \tag{6}$$

$$n_a m_a\left[\frac{d}{dt}\right]_a \mathbf{v}_a=q_a n_a\left[\mathbf{E}+\frac{\mathbf{v}_a}{c}\times\mathbf{B}\right] -n_a m_a\nabla\phi_G-\nabla P_a+\mathbf{R}_a , \tag{7}$$

where $(d/dt)_a\equiv\partial/\partial t+\mathbf{v}_a\cdot\nabla$ and

$$R_a\equiv\mu_a\nabla^2\mathbf{v}_a-m_a n_a\sum_b \nu_{ab}^c(\mathbf{v}_a-\mathbf{v}_b) \tag{8}$$

with the viscosity μ_a and the collision frequency for different fluids ν_{ab}^c. We assume an equation of state $P_a=P_a(n_a)$.

Let the electric and magnetic fields be given by their potentials $\mathbf{E}=-\nabla\phi-(\partial/c\partial t)\mathbf{A}$ and $\mathbf{B}=\nabla\times\mathbf{A}$. If we use the canonical momentum

$$p_a\equiv m_a\mathbf{v}_a+\frac{q_a}{c}\mathbf{A} \tag{9}$$

of each of the species of the plasma and eliminate $\mathbf{A}$ in favor of $\mathbf{p}_a$ and $\mathbf{v}_a$, we get from the equation of motion (7)

$$\frac{\partial}{\partial t}\mathbf{p}_a=-\mathbf{v}_a\times\boldsymbol{\Omega}_a-\nabla\epsilon_a+\mathbf{r}_a , \tag{10}$$

where the generalized vorticity is

$$\boldsymbol{\Omega}_a\equiv-\nabla\times\mathbf{p}_a=-m_a\nabla\times\mathbf{v}_a-\frac{q_a}{c}\mathbf{B} , \tag{11}$$

ϵ_a is the energy of the component a,

$$\epsilon_a\equiv\tfrac{1}{2}m_a\mathbf{v}_a^2+q_a\phi+m_a\phi_G+\frac{P_a}{n_a} , \tag{12}$$

and

$$r_a\equiv\frac{\mathbf{R}_a}{n_a}-\frac{P_a}{n_a}\nabla(\ln n_a) . \tag{13}$$

Applying the curl on (10) we get an equation for the vorticity:

$$\frac{\partial}{\partial t}\boldsymbol{\Omega}_a=\nabla\times[\mathbf{v}_a\times\boldsymbol{\Omega}_a]-\nabla\times\left[\frac{R_a}{n_a}\right] . \tag{14}$$

In the limit of low viscosity and/or high density and low interfluid collision frequency

$$\frac{\mu_a}{n_a}\ll\min\left[L m_a v_a ; L^2\frac{q_a}{c}B\right] , \tag{15}$$

$$\nu_{ab}^c\ll\min\left[\frac{v_a}{L},\frac{v_b}{L};\frac{q_a B}{m_a c}\right] , \tag{16}$$

where L is a typical length in the problem, we can neglect $\mathbf{r}_a$ and $\mathbf{R}_a$. In this limit the equilibrium configuration will be the one in which ϵ_a is the level surface function for field lines of $\mathbf{v}_a$ and Ω_a:

$$\mathbf{v}_a\times\boldsymbol{\Omega}_a=\nabla\epsilon_a , \tag{17}$$

$$\mathbf{v}_a\cdot\nabla\epsilon_a=0 , \tag{18}$$

$$\boldsymbol{\Omega}_a\cdot\nabla\epsilon_a=0 . \tag{19}$$

The energy is constant along the streamlines of both the velocity and the vorticity fields.

Except possibly in subdomains where $\nabla\epsilon_a=\mathbf{0}$, the streamlines of $\mathbf{v}_a$ and $\boldsymbol{\Omega}_a$ lie on surfaces $\epsilon_a=$const. The topology of these surfaces is determined by the topology of the sets of points at which $\nabla\epsilon_a=0$: these points may be isolated, or they may fill three-dimensional subdomains [11].

B. Generalized relaxation theory

Now we relate the equations obtained above to a generalized version of the relaxation theory. The evolution of the fields, determined by (10) in the limit (15)–(16) and spatially constant density, preserves the generalized helicity

$$I_a^h=\int\mathbf{p}_a\cdot\boldsymbol{\Omega}_a d^3x-\oint\mathbf{p}_a\cdot dl_1\oint\mathbf{p}_a\cdot dl_2 , \tag{20}$$

where the integration is over the whole spatial volume and the line integrals appear for multiply connected spaces. This definition is gauge independent [8]. Indeed, the time derivative of (20) is

$$\frac{\partial}{\partial t}I_a^h=\oint[-\epsilon_a\boldsymbol{\Omega}_a+(\mathbf{p}_a\cdot\boldsymbol{\Omega}_a)\mathbf{v}_a-(\mathbf{p}_a\cdot\mathbf{v}_a)\boldsymbol{\Omega}_a]\cdot d\mathbf{S} . \tag{21}$$

So, for the boundary conditions $\boldsymbol{\Omega}_a\cdot\mathbf{n}=0$ and $\mathbf{v}_a\cdot\mathbf{n}=0$ we have $\partial/\partial t I_a^h=0$.

If we include the dissipation terms we find, assuming the above boundary conditions,

$$\frac{\partial}{\partial t}I_a^h=2\int\mathbf{r}_a\cdot\boldsymbol{\Omega}_a d^3x . \tag{22}$$

We promptly notice a special case for which $\mathbf{r}_a\cdot\boldsymbol{\Omega}_a=\lambda_a\mathbf{p}_a\cdot\boldsymbol{\Omega}_a$ so that the evolution of the helicity is $I_a^h(t)=I_a^h(0)\exp[2\int^t\lambda_a(t')dt']$. Therefore it can increase, decrease, or be constant, depending on the behavior of $\lambda_a(t)$ in time.

The time derivative of the total energy

$$E_{\text{total}} \equiv \int \frac{1}{2} \left[\sum_a n_a m_a v_a^2 + \frac{1}{4\pi}(E^2+B^2) \right] d^3x \quad (23)$$

is

$$\frac{\partial}{\partial t} E_{\text{total}} = \int \sum_a \mathbf{R}_a \cdot \mathbf{v}_a d^3x \ . \quad (24)$$

The total energy decreases in time mainly because of the viscosity term in (8). In the ideal limit neglecting $\mathbf{R}_a$ the total energy is, of course, conserved. But in general both helicity and total energy can decay in time. From the expressions (22) and (24) we conclude that the rates of decay of helicity and total energy may be different. I_a^h are topological quantities and we have some reason [8] to believe that the helicity does not decay as fast as the total energy does [10]. Indeed, since changing the helicity involves changing the topology of the lines, breaking and reconnecting them, it takes some time to happen while the dissipation of energy does not have such a constraint.

We estimate phenomenologically the lifetime for decreasing the total energy and the helicity as

$$\tau_{\text{energy}} = \min[1/\nu_a^c ; n_a m_a L^2/\mu_a] \ , \quad (25)$$

$$\tau_{\text{helicity}} = \max[1/\nu_a^c ; n_a m_a L^2/\mu_a] \ . \quad (26)$$

The case $1/\nu_a^c \ll n_a m_a L^2/\mu_a$ is one in which the dissipation of the energy is through interfluid collisions, usually at small scales compared to the other case $1/\nu_a^c \gg n_a m_a L^2/\mu_a$, in which the energy is dissipated through the viscosity of each fluid species.

In any case, given the motivations above, a variational principle is proposed as follows. Minimize E_{total}, subject to the constraint that $\sum_a I_a^h = \text{const}$. Let $\delta\phi$, $\delta\mathbf{A}$, and $\delta\mathbf{p}_a$ be the general variations of the electrostatic potential, the vector potential, and the canonical momentum, respectively. Then the "stationarity" condition

$$\delta \left[E_{\text{total}} - \lambda \sum_a I_a^h \right] = 0 \quad (27)$$

leads to

$$\nabla \cdot \mathbf{E} = 0 \ , \quad (28)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \sum_{a=1}^{N} q_a n_a \mathbf{v}_a \ , \quad (29)$$

$$\mathbf{\Omega}_a = -\frac{n_a}{2\lambda} \mathbf{v}_a \ . \quad (30)$$

The first two equations above are of no surprise. The last equation is a special case solution for Eq. (17). In this case the generalized vorticity field lines are frozen in the fluid.

To check the stability of these configurations, we make a second variation on (27), use Eqs. (28)–(30), and integrate by parts. We get

$$\delta^2 \left[E_{\text{total}} - \lambda \sum_a I_a^h \right] \Bigg|_{\text{extreme}} = \int d^3x \left[\sum_a \frac{n_a m_a}{2} (\delta v_a)^2 + \frac{1}{4\pi} \left[(\nabla \delta\phi)^2 + \sum_{ij} (\partial_j \delta A_1)^2 \right] - 2\lambda \sum_a \delta(\nabla \times \mathbf{p}_a) \cdot \delta \mathbf{p}_a \right] \ . \quad (31)$$

If the last term does not change sign, we can make the configuration stable by appropriate choice of λ. This term is called the average perturbation spirality in connection with amplifications of vortex disturbances in planetary atmospheres [12].

Combining (29), (30), and (10), we get

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \left[\left[2\lambda \sum_a \frac{q_a^2}{c} \right] \mathbf{B} + 2\lambda \sum_a (m_a q_a \nabla \times \mathbf{v}_a) \right] \ . \quad (32)$$

We promptly notice that this equation is a particular case of Eq. (17). It is very gratifying to see that the generalized vortex equation can have, in principle, relaxed state solutions. In the next section we study the more general equilibrium equation (17) with planar and axial symmetry for an electron-positron-proton plasma. We have been unsuccessful so far in finding an explicit solution for three-dimensional (3D) configurations. It may be that localized filamentary structures are preferred for this spatial scale.

III. ELECTRON-POSITRON-PROTON PLASMA

We assume that the electron and positron fluids have the same velocity field $\mathbf{v}$ due to the strong coupling with the isothermal high frequency photons. (Such a model has been discussed and justified in Ref. [13]. Further, Tajima *et al.* [5] have pointed out that in a thermal plasma there are mainly two components in the photon energy spectrum: The high frequency ones and the low frequency ones. The high frequency photons are the familiar blackbody radiation photons that permeate and interact strongly with the opaque plasma.) Therefore the canonical momentum (8) and the generalized vorticity (10) are [for the electrons ($-$), for the positrons ($+$), and for the protons, or ions (i)] ($\mathbf{v}_+ = \mathbf{v}_- = \mathbf{v}$)

$$\mathbf{p}_\pm = m\mathbf{v} \pm \frac{e}{c} \mathbf{A} \ , \quad (33)$$

$$\mathbf{\Omega}_\pm = -\nabla \times \mathbf{p}_\pm = -m \nabla \times \mathbf{v} \mp \frac{e}{c} \mathbf{B} \ , \quad (34)$$

and similarly for the protons

$$\mathbf{p}_i = m_i \mathbf{v}_i + \frac{e}{c} \mathbf{A} \ , \quad (35)$$

$$\mathbf{\Omega}_i = -\nabla \times \mathbf{p}_i = -m_i \nabla \times \mathbf{v}_i - \frac{e}{c} \mathbf{B} \ . \quad (36)$$

We neglect the displacement current $[(\partial/c\partial t)\mathbf{E} \ll \nabla \times \mathbf{B}]$ and assume that the quasineutrality condition ($n_i = n_- - n_+ \equiv \delta n^*$) holds. For simplicity we consider

δn^* constant and the ion velocity much smaller than the electron-positron velocity: $\mathbf{v}_i \ll \mathbf{v}$. Therefore from Ampère's law (3)

$$\mathbf{v} = -\frac{c}{4\pi e \delta n^*} \nabla \times \mathbf{B} . \tag{37}$$

Configurations in which this relationship between v and B holds are called "magnetic vortices" [14]. Let us use some appropriate units:

$$[\text{spatial coordinates}] = \frac{c}{\omega_p^*} , \tag{38}$$

$$[\text{time}] = \frac{mc}{eB_0} , \tag{39}$$

$$\omega_p^* \equiv \left[\frac{4\pi \delta n^* e^2}{m} \right]^{1/2} , \tag{40}$$

$$\mathbf{H} \equiv \frac{\mathbf{B}}{B_0} . \tag{41}$$

With these units the electron and positron vorticities (34) are

$$\Omega_{\pm} = \mp \mathbf{H} - \nabla^2 \mathbf{H} , \tag{42}$$

$$\frac{\partial}{\partial t} \Omega_{\pm} = \nabla \times [\Omega_{\pm} \times (\nabla \times \mathbf{H})] . \tag{43}$$

Adding and subtracting the previous equations we get

$$\frac{\partial}{\partial t} \mathbf{H} = \nabla \times [\mathbf{H} \times (\nabla \times \mathbf{H})] , \tag{44}$$

$$\frac{\partial}{\partial t} \nabla^2 \mathbf{H} = \nabla \times [(\nabla^2 \mathbf{H}) \times (\nabla \times \mathbf{H})] . \tag{45}$$

It may appear that this is an overdetermined set of equations for $\mathbf{H}$, but it is not. Note that the first equation above tells us that $c\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ and therefore it holds whenever we neglect the displacement current, that is, the assumption $(\partial / c \partial t)\mathbf{E} \ll \nabla \times \mathbf{B}$ is validated. Therefore the equations are self-consistent with the hypothesis. Equation (43) gives us also, in a more precise way, the range of validity for the neutrality condition:

$$\frac{1}{4\pi e \delta n^*} \nabla \cdot \mathbf{E} = -\frac{B_0^2}{4\pi \delta n^* mc^2} [\mathbf{H} \cdot \nabla^2 \mathbf{H} + (\nabla \times \mathbf{H})^2] . \tag{46}$$

Therefore the neutrality condition can hold either in an approximate way for $B_0^2 / 4\pi \ll \delta n^* mc^2$ or in an exact way for $\mathbf{H} \cdot \nabla^2 \mathbf{H} = -(\nabla \times \mathbf{H})^2$. Beltrami fields (usually known by the relation $\nabla \times \mathbf{H} = \alpha \mathbf{H}$), are fields in which the neutrality condition holds exactly.

A. 2D case

Let us now solve Eq. (44) for the case in which one spatial coordinate, say z, is ignorable. Physically it means that the typical length in the z direction is much larger than the typical length in the x-y plane ($\partial A_x / \partial z \ll \partial A_x / \partial y$ and $\partial A_y / \partial z \ll \partial A_y / \partial x$). Therefore we can write the magnetic field as

$$\mathbf{H} = [\nabla a(x,y,t) \times \hat{\mathbf{z}}] + h(x,y,t)\hat{\mathbf{z}} , \tag{47}$$

where $a(x,y,t)$ and $h(x,y,t)$ are functions of x,y, and t. Thus Eqs. (44) and (45) for the magnetic field (47) become

$$\frac{\partial}{\partial t} h = 0 , \tag{48}$$

$$\frac{\partial}{\partial t} \nabla^2 h + (h, \nabla^2 h) = 0 , \tag{49}$$

$$\frac{\partial}{\partial t} a + (h,a) = 0 , \tag{50}$$

$$\frac{\partial}{\partial t} \nabla^2 a + (h, \nabla^2 a) = 0 , \tag{51}$$

where $(f,g) \equiv \hat{\mathbf{z}} \cdot [\nabla f \times \nabla g]$. From (48) and (49) we get

$$\nabla^2 h = P[h] \tag{52}$$

for an arbitrary functional $P[h]$. For a vortex propagating in the x-y plane with velocity u, Eqs. (50) and (51) give us

$$\nabla^2 a = Q[h + \hat{\mathbf{z}} \cdot (\mathbf{u} \times \mathbf{r})] , \tag{53}$$

$$a = R[h + \hat{\mathbf{z}} \cdot (\mathbf{u} \times \mathbf{r})] , \tag{54}$$

where P, Q, and R are arbitrary functions of the arguments in the square brackets.

Let us solve a "linear" case $a = h + uy$, in which $\mathbf{u}$ is the x direction and Q is given by

$$Q[a] = \begin{cases} -c^2(h+uy) & \text{for } r < r_0 \\ +d^2(h+uy) & \text{for } r > r_0 , \end{cases} \tag{55}$$

where c and d are constants. So we have to solve just

$$\nabla^2 h = \begin{cases} -c^2(h+uy) & \text{for } r < r_0 \\ +d^2(h+uy) & \text{for } r > r_0 . \end{cases} \tag{56}$$

The general solution for h (continuous up to the first derivative) is given by

$$h = \begin{cases} u \left[\left[\frac{dr_0 K_2(dr_0)}{K_1(dr_0)} + B_1 d K_1'(dr_0) \right] \frac{J_1(cr)}{cJ_1'(cr_0)} - r \right] \sin\phi & \text{for } r < r_0 \quad (57) \\ u \left[\frac{r_0}{K_1(dr_0)} + B_1 \right] K_1(dr) \sin\phi & \text{for } r > r_0 , \quad (58) \end{cases}$$

where $J_1(cr_0)=0$ and J_1 and K_1 are Bessel functions.

Another possible choice of the arbitrary $Q[a]$ is

$$Q[a]=\begin{cases}-c^2(h+uy) & \text{for } r<r_0\\ 0 & \text{for } r>r_0\ ,\end{cases} \tag{59}$$

which yields the solution

$$h=\begin{cases}u\left[\dfrac{2}{cJ_1'(cr_0)}J_1(cr)-r\right]\sin\phi & \text{for } r<r_0\\ -ur_0^2\sin\phi/r & \text{for } r>r_0\end{cases} \tag{60}$$

with $J_1(cr_0)=0$.

Both solutions above are dipolelike solutions and decay with a power law of the distance. Therefore they are, in some sense, localized. The first solution has finite total energy while the second has a logarithmic divergence.

We emphasize that these solutions physically represent filamentary vortex structures. At large enough scales these solutions are thin (r_0 very small) "strings" that may eventually close themselves. A good ensemble of these filaments may form more complex structures in this large scale. As Petviashvili has shown [14], MHD equations resemble a set of equations for filamentary vortices in unmagnetized plasmas.

B. 3D case

Now we present the basic steps toward a 3D solution with azimuthal angle symmetry. Let us assume an axially symmetric 3D configuration in which the magnetic field is given by

$$\mathbf{H}=\frac{1}{r}[\nabla\psi(r,z,t)\times\hat{\boldsymbol{\phi}}]+\frac{1}{r}f(r,z,t)\hat{\boldsymbol{\phi}} \tag{61}$$

for some functions $\psi(r,z,t)$ and $f(r,z,t)$. Equations (44) and (45) become

$$\frac{d}{dt}(\mp\psi-\Delta^*\psi)=0\ , \tag{62}$$

$$\frac{d}{dt}\left[\frac{\mp f-\Delta^* f}{r^2}\right]=\frac{1}{r}\left[\nabla\left[\frac{1}{r^2}\Delta^*\psi\right]\times\nabla\left[\mp\psi-\Delta^*\psi\right]\right]_\phi\ , \tag{63}$$

where

$$\Delta^*\equiv r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\ , \tag{64}$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\frac{1}{r}[\nabla f\times\nabla]_\phi\ . \tag{65}$$

For a stationary configuration, moving in the z direction with speed u we have

$$\frac{d}{dt}=\frac{1}{r}[\nabla\tilde{f}\times\nabla]_\phi\ , \tag{66}$$

$$\tilde{f}=f+\frac{u}{2}r^2\ , \tag{67}$$

$$\Delta^*\psi\pm\psi=F_\pm[\tilde{f}]\ , \tag{68}$$

$$\Delta^* f\pm f+F'_\pm[\tilde{f}]\Delta^*\psi=r^2P_\pm[\tilde{f}]\ . \tag{69}$$

$F_\pm[\tilde{f}]$ and $P_\pm[\tilde{f}]$ are arbitrary functions of their arguments.

If we add and subtract the equations above we get

$$\Delta^*\psi=\frac{F_+[\tilde{f}]+F_-[\tilde{f}]}{2}\equiv G[\tilde{f}]\ , \tag{70}$$

$$\psi=\frac{F_+[\tilde{f}]-F_-[\tilde{f}]}{2}\equiv K[\tilde{f}]\ , \tag{71}$$

$$\Delta^* f+G[\tilde{f}]G'[\tilde{f}]=r^2\frac{P_+[\tilde{f}]+P_-[\tilde{f}]}{2}\equiv r^2T[\tilde{f}]\ , \tag{72}$$

$$f+K'[\tilde{f}]G[\tilde{f}]=r^2\frac{P_+[\tilde{f}]-P_-[\tilde{f}]}{2}\equiv r^2Q[\tilde{f}]\ , \tag{73}$$

which can be combined to form

$$\Delta^* f=r^2\left[T[\tilde{f}]-\frac{G'[\tilde{f}]}{K'[\tilde{f}]}Q[\tilde{f}]\right]+f\frac{G'[\tilde{f}]}{K'[\tilde{f}]}\ , \tag{74}$$

$$\Delta^*\psi=G[K^{-1}[\psi]]\equiv S[\psi]\ , \tag{75}$$

where K^{-1} is the inverse function of K.

We were not successful in finding an explicit nontrivial solution for these equations. Note that one needs to solve Eq. (75) only for ψ because $\tilde{f}$ becomes determined from ψ.

C. Relaxed state

The relaxed state configuration obeys the variational principle given in Eqs. (29) and (30). We conclude that for the electron-positron-proton plasma case

$$\mathbf{H}=-\frac{\delta n^*}{4\lambda}\nabla\times\mathbf{H}\ , \tag{76}$$

$$\nabla^2\mathbf{H}=\frac{n_++n_-}{4\lambda}\nabla\times\mathbf{H}\ . \tag{77}$$

Compatibility of these equations gives $(4\lambda)^2=\delta n^*(n_++n_-)$. Then we get a Helmholtz-like equation for $\mathbf{H}$:

$$\nabla^2\mathbf{H}=-\frac{n_-+n_+}{n_--n_+}\mathbf{H}\ . \tag{78}$$

The scales of these solutions are $\sqrt{(n_--n_+)/(n_-+n_+)}(c/\omega^*)=c/\omega_p$ where ω_p is the plasma frequency for the density n_-+n_+, which means the size is $\sqrt{(n_-+n_+)/n_--n_+)}$ times the collisionless skin depth of the ee^+ plasma.

It is well known that the general solution for the divergenceless fields satisfying (78) is

$$\mathbf{H}=\nabla\times(\hat{\mathbf{m}}u)+\frac{1}{\alpha}\nabla\times[\nabla\times(\hat{\mathbf{m}}u)]\ , \tag{79}$$

where $\hat{\mathbf{m}}$ is a unitary vector, and u satisfies the scalar Helmholtz equation

$$\nabla^2 u=-\alpha^2 u\ ,$$

where $\alpha \equiv \sqrt{(n_- + n_+)/(n_- - n_+)}$.

For the cosmological *eeP*, $\alpha \approx 10^4$. This estimate is based on the observed limits for the asymmetry of matter over antimatter. Therefore the spatial scale of the solution is, in principle, small.

IV. IDEAL MHD

As we discussed before, the *eeP* structures can combine to form larger-scale structures in MHD. Therefore it is appropriate to investigate localized solutions in MHD. The set of equations used is

$$\mathbf{B}\cdot\nabla_n = \mathbf{v}\cdot\nabla n = 0 , \tag{80}$$

$$\nabla\cdot\mathbf{v} = 0 , \tag{81}$$

$$\nabla\cdot\mathbf{B} = 0 , \tag{82}$$

$$nm \left[\frac{d}{dt} \right] \mathbf{v} = \frac{\mathbf{J}}{c} \times \mathbf{B} - \nabla P - mn\nabla\phi_G , \tag{83}$$

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} , \tag{84}$$

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) , \tag{85}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) . \tag{86}$$

The equation of motion (83) can be rewritten as

$$\frac{\partial}{\partial t}\mathbf{v} - [\mathbf{v}\times(\nabla\times\mathbf{v})] + \frac{1}{4\pi mn}[\mathbf{B}\times(\nabla\times\mathbf{B})] = -\nabla\left[\frac{P}{mn} + \tfrac{1}{2}v^2 + \phi_G \right] . \tag{87}$$

We assume the density is constant. Then the time evolution equations have the following three conserved integrals:

$$I_\pm = \int \left[\mathbf{v} \pm \frac{B}{\sqrt{4\pi mn}} \right]^2 d^3\mathbf{r} , \tag{88}$$

$$I_h = \int \mathbf{A}\cdot\mathbf{B}\, d^3\mathbf{r} . \tag{89}$$

Let us look for static solutions

$$[\mathbf{v}\times(\nabla\times\mathbf{v})] - \frac{1}{4\pi mn}[\mathbf{B}\times(\nabla\times\mathbf{B})] = \nabla\left[\frac{P}{mn} + \tfrac{1}{2}v^2 + \phi_G \right] , \tag{90}$$

$$\nabla\times[\mathbf{v}\times\mathbf{B}] = \mathbf{0} . \tag{91}$$

There are three possible vortices, depending on how **v** and **B** are related. They are called parallel, magnetic, and dynamic vortices. For parallel vortices

$$\mathbf{v} = \pm\frac{M}{\sqrt{4\pi mn}}\mathbf{B} , \tag{92}$$

$$\frac{M^2-1}{4\pi mn}[\mathbf{B}\times(\nabla\times\mathbf{B})] = \nabla\left[\frac{P}{mn} + \tfrac{1}{2}v^2 + \phi_G \right] . \tag{93}$$

There is a degeneracy when the constant $M=1$ and $P/mn + \frac{1}{2}v^2 + \phi_G = \text{const}$ which corresponds to Alfvén vortices.

Notice that

$$(M^2-1)\epsilon \equiv 4\pi(P + mn\tfrac{1}{2}v^2 + mn\phi_G) \tag{94}$$

is constant along the streamlines of the magnetic field. Let it be a 3D axially symmetric field:

$$r\mathbf{B} = \hat{\boldsymbol{\phi}}\times\nabla\psi + \hat{\boldsymbol{\phi}} f[\psi] \tag{95}$$

and $\epsilon = \epsilon[\psi]$.

So we get the Grad-Shafranov equations

$$\Delta^*\psi = -ff' - r^2\epsilon' , \tag{96}$$

where the prime means derivatives with respect to the argument and $\Delta^* \equiv (r\partial/\partial r)(1/r)\partial/\partial r + \partial^2/\partial z^2$.

Let the separatrix be a sphere of radius a. Then we take $\epsilon[\psi]$ and $f[\psi]$ to be linear inside the sphere and zero outside it. It turns out that ψ vanishes outside the separatrix. The inside equation and general solution [15] are as follows:

$$\Delta^*\psi = -k^2\psi + cr^2 , \tag{97}$$

$$\psi = \frac{c}{k^2}r^2 + \sum_{n=2}^{\infty} A_n C_n^{-1/2}\left[\frac{z}{R} \right]\sqrt{R}\, j_{n-1/2}(kR) , \tag{98}$$

where $R \equiv \sqrt{(r^2+z^2)}$ and $C_n^{-1/2}$ are Gegenbauer functions and $j_{n-1/2}$ are spherical Bessel functions. We impose continuity for ψ and its first derivative. This procedure leads us to

$$A_n = -\delta_{n,2}\frac{c}{k^2}1/j(ka) , \tag{99}$$

$$\tan ka = \frac{3ka}{3-(ka)^2} . \tag{100}$$

Therefore

$$\psi = \frac{c}{k^2}\left[1 - \frac{j(kR)}{j(ka)} \right]^2 r^2 , \tag{101}$$

where

$$j(\xi) = \frac{(\sin\xi - \xi\cos\xi)}{\xi^3} . \tag{102}$$

The first two roots of the transcendental are $ka = 5.76$ and $ka = 9.11$. This is an example of a localized solution in MHD. It is continuous up to the second derivative of ψ which is zero outside the separatrix.

Other localized numerical solutions were found [14] for $f = \sqrt{2/(n+1)}\psi^{n+1/2}$ and $\epsilon' = -\psi$ for $n = 2,3$. These solutions have a preferred direction of paramagnetic interaction along, say, the z axis, and of antiparamagnetic in the plane normal to the axis. Therefore these solitary vortices have a tendency to form linear polymerlike structure. In turn these "polymers" may form even larger structures, and so on [16].

V. DISCUSSION

We have found a series of localized relaxed solutions relevant for plasma structure formation. We obtained localized solutions for *eeP* in the form of long strings (2D solutions) and vortices in MHD scales. We can then set up a hierarchy in which a filamentary *eeP* may form a localized 3D solution in MHD. The localized solutions in MHD may also form larger-scale structures in a polymer-like shape. These solutions represent additional and perhaps more natural equilibrium structures than the ones found in earlier work [13].

In the quasi-two-dimensional limit of three dimensions, i.e., with the structure being a long string but not strictly a straight cylinder, the localized structures can meander in and weave through the plasma and occasionally crisscross each other. It is known [17,18] that the directions of such crisscrosses and thus the presence or lack of the strong magnetic field in the plane of contact and perpendicular to the reconnecting field lines are a crucial factor in determining the speed of possible reconnection of magnetic field lines. It is thus of much interest to pursue the study of the evolutionary outcome of such preferential reconnection in structure formation. Such interaction may be well described by the approach of Pumir and Siggia [19] in hydrodynamics and by Kinney *et al.* [20] in MHD. It is possible to speculate that a particular meandering and linking of such strings which originally did not carry an overall helicity can emerge with a directed helicity as a result of reconnection [21].

These structures may be of great importance to formation of isothermal perturbations during the radiation epoch of the universe. This scenario provides one possible way for formation of structures of later epochs that is consistent with the observed uniformity and isotropy of the microwave background radiation [5].

Moreover, it is often said that the effort of achieving a thermonuclear burning plasma is to copy astrophysical thermonuclear burning. (Conversely, the recent experimental progress [22] in tokamak fusion plasmas finds the presence of strong flows, a study of which may lead to more understanding of plasma vortices with flows as discussed in the present paper.) The absence of external magnetic fields for these localized vortices suggests a possible path toward a fusion reactor without (so many) external coils. An attempt in that direction is being investigated [23]. More analysis is necessary, however, to check the feasibility of this option as a reactor. The observed solar bubbles [24] may also be related to the MHD solutions presented in this paper.

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