Hydrodynamics and dynamic fluctuations of fluid membranes

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This paper develops a formalism for studying the renormalization of dynamic properties of fluid membranes under coarse graining produced by removal of high-wave-number degrees of freedom. It derives hydrodynamical equations for fluid membranes and shows that appropriately chosen noise sources can lead to a Fokker-Planck equation for the probability distribution that decays to the thermal equilibrium at long times that includes measure factor corrections to the usual Boltzmann weight. Membranes that are incompressible at short length scales are shown, via both static and dynamic renormalization calculations, to be compressible at long length scales. As a result, a flexible membrane will always have a density mode that is distinct from its height or shape mode. Dissipative coefficients in the membrane Rouse model are shown to renormalize whereas those in the Zimm model are shown not to.

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I. INTRODUCTION

Amphiphilic molecules in water spontaneously segregate into extended bilayer fluid membranes [1,2]. At length scales large compared to molecular dimensions, the details of molecular architecture and interactions become unimportant, and it is appropriate to describe the membrane as a structureless flexible surface. Static physical and statistical properties of membranes are well described by the Helfrich-Canham Hamiltonian [3,4], which depends only on membrane geometry via its local metric and curvature tensors. Dynamical properties of membranes at long-length scales can also be described phenomenologically. In this paper, we will derive stochastic hydrodynamical equations for fluid membranes and explore their consequences. Our primary goal is to develop a formalism that will allow us to calculate how dynamical as well as static parameters renormalize under removal of high-wave-number degrees of freedom. A preliminary report of our work appears in [5].

The stochastic hydrodynamical treatment of fluid membranes presents a number of unique problems. The equilibrium statistical mechanics of fluid membranes is complicated by measure factors [6,7] in partition traces. We show that these factors require that our noise sources have nonvanishing averages in spatially nonuniform configurations. Our hydrodynamical equations describe both shape changes and tangent plane motion. They are invariant with respect to arbitrary time-dependent reparametrizations. They have nondissipative couplings between variables with opposite sign under time reversal. These terms appear as Poisson brackets in flat-space stochastic equations [8–11]. We show that they can be derived from Poisson brackets in fluid membranes also. This derivation, like quantization of the electromagnetic field [12], must be carried out for a particular parametrization (or gauge) of the surface. Finally, the nonlinear dependence of both Poisson brackets and dissipative coefficients on fields leads to additional contributions to the average noise.

Static statistical properties of membranes can be calculated as though the membranes were isolated: an incompressible solvent does not induce static inter- or intramembrane interactions. Dynamic properties depend on the solvent: motion of the membrane at nonzero frequency induces motion in the solvent, which in turn exerts a force on the membranes at distant points [13–15]. Thus, the dynamics of physical membranes in solution is a three-dimensional problem depending on the dynamics of the solvent. Our hydrodynamical equations allow for friction forces between the solvent and the membrane in addition to intrinsic viscous forces within the membranes. We will consider renormalization in two simple models: the Rouse model [16] in which the membrane experiences a dissipative force proportional to its velocity relative to a rigidly fixed solvent and the Zimm model [16] in which the membrane induces shear modes in the solvent, which are treated at zero frequency. The effects of nonlinear interactions on the three-dimensional dynamics of membrane systems have been treated elsewhere [17,18].

Thermal fluctuations crumple [19,20] a membrane so that it is compressible after coarse graining via the removal of high-wave-number degrees of freedom even if it is incompressible at the molecular length scale. We derive this result using both static and dynamic renormalization procedures. Thus, a membrane will always have a hydrodynamic density mode distinct from the shape or height mode. Both the height and density modes are needed to reproduce the experimentally observed hydrodynamic modes of lamellar lyotropic systems [21,22]

Hydrodynamical equations for fluid membranes have been derived by other authors [23,24]. Our derivation, however, emphasizes reparametrization invariance. It is similar in spirit to the purely dissipative model used by Goldstein *et al.* [25] to study space curves in two dimensions. In fact, our Rouse model reduces to this model in two dimensions when motion in the tangent plane is prohibited. When expressed in the Monge gauge, our dynamical equations for the height are identical to the gauge invariant version of the Kardar-Parisi-Zhang

(KPZ) [26] equation used by Golubović and Wang [27]. We assume that relative motion of the two monolayers comprising the bilayer can be neglected and that the membrane can be modeled as a single fluid surface. In real membranes, sliding of monolayers across each other, though inhibited by friction, does occur [28] and gives rise to an experimentally observable mode with complex crossover structure [29]. Our treatment could easily be generalized to study the effects of renormalization on these modes.

This paper consists of nine sections, of which this is the first. Section II reviews differential geometry mostly to establish notation. It also introduces dynamic reparametrization. Section III reviews static energies of compressible membranes and the calculation of forces. Section IV considers static renormalization of membranes and shows how a a membrane that is incompressible at the molecular length scale become compressible at long-length scales. Section V derives the complete reparametrization invariant hydrodynamic equations for a fluid membrane. Section VI calculates dynamical renormalization in the Rouse model and shows that there are nontrivial renormalizations of the dynamic friction coefficients. Section VII does the same for the Zimm model. In this case, however, there is no renormalization of the friction coefficient because of its nonanalytic nature. Section VIII derives Poisson brackets in the Monge gauge and shows that they reproduce the nondissipative parts of the hydrodynamical equations derived in Sec. V. Finally, Sec. IX investigates the Fokker-Planck equation for the field probability distribution and shows that appropriately chosen noises will drive the system to equilibrium at long times.

II. SURFACE GEOMETRY, SCALARS, AND TENSORS

We are concerned with statistical and dynamical properties of fluctuating membranes. In this section, we will review concepts in differential geometry [30,7] needed to describe these membranes, mostly to establish notation. We will also review some important scalars and tensors.

A. Elememtary differential geometry of a surface

At length scales long compared to molecular lengths, membranes can be modeled as two-dimensional fluctuating surfaces embedded in three-dimensional Euclidean space, which we will often refer to as the embedding space. The medium (usually a solvent) in which the membrane moves lives in the embedding space. Points on a two-dimensional surface embedded in three-dimensional Euclidean space are specified by a three-dimensional vector $\mathbf{R}(\tilde{u})$ with components $R_i(\tilde{u})$, $i = 1,2,3$, as a function of a parameter $\tilde{u} = (u^1, u^2)$ in a two-dimensional manifold, which we will refer to as the parameter manifold. Covariant tangent-plane vectors are then defined as

$$\mathbf{e}_a = \partial_a \mathbf{R}, \qquad a = 1,2, \tag{2.1}$$

where $\partial_a = \partial/\partial u^a$. The metric tensor is

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b. \tag{2.2}$$

Its inverse g^{ab} satisfying

$$g^{ab} g_{bc} = \delta^a_c \tag{2.3}$$

allows us to define contravariant tangent-plane vectors $\mathbf{e}^a = g^{ab}\mathbf{e}_b$ satisfying $\mathbf{e}^a \cdot \mathbf{e}_b = \delta^a_b$. Any vector $\mathbf{v}$ in the tangent plane can be expressed as $\mathbf{v} = v^a \mathbf{e}_a = v_a \mathbf{e}^a$, where $v_a = \mathbf{e}_a \cdot \mathbf{v}$ and $v^a = \mathbf{e}^a \cdot \mathbf{v} = g^{ab} v_b$ are, respectively, its covariant and contravariant components. A unit normal $\mathbf{n}$ to the surface can be constructed from $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}. \tag{2.4}$$

The area of a surface element with sides du^1 and du^2 is $dS = \mathbf{n} \cdot (\mathbf{e}_1 du^1 \times \mathbf{e}_2 du^2) = \sqrt{g} d^2u$, where

$$g = \det g_{ab} \tag{2.5}$$

is the determinant of the metric tensor. The curvature tensor is then

$$K_{ab} = \mathbf{n} \cdot \partial_a \partial_b \mathbf{R}. \tag{2.6}$$

The mean curvature is

$$\frac{1}{2} H = \frac{1}{2} K^a_a = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{2.7}$$

and the Gaussian curvature is

$$S = \det K^a_b = \frac{1}{R_1}\frac{1}{R_2}, \tag{2.8}$$

where R_1 and R_2 are the principal radii of curvature at the point of the surface in question.

There are an infinite number of ways of parametrizing a surface, i.e., there an infinity of choices for the parameter $\tilde{u}$. A change of variables from $\tilde{u}$ to $\tilde{u}'$ is called a reparametrization. For reasons that will become apparent later, a choice of parametrization is closely analogous to fixing gauge in a gauge theory. We will, therefore, often refer to a particular parametrization as a gauge choice and a reparametrization transformation as a gauge transformation. We will be concerned with how functions on a surface transform under time-dependent reparametrizations. To describe these transformations, let us assume that there is some time-independent parametrization of $\tilde{u}$, and an associated time-independent grid defined by lines of constant u^1 and u^2 in the parameter manifold as shown in Fig. 1(a). Time-dependent parametrizations can then be defined in terms of an invertible map $\tilde{u} \to \tilde{u}'(t)$ defined via

$$\tilde{u}'(t) = \tilde{\Phi}(\tilde{u}, t) \equiv \tilde{u}'(\tilde{u}, t), \tag{2.9}$$

where $\tilde{\Phi}$ has components Φ^1 and Φ^2. Under this transformation, the fixed grid in Fig. 1(a) translates and distorts in time as depicted in Figs. 1(b) and 1(c). We can

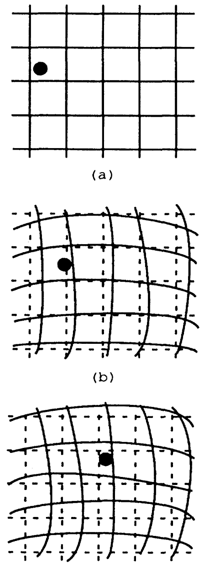

FIG. 1. (a) shows a grid on a surface formed by lines of constant constant u^1 and u^2. (b) and (c) show the grid at two successive later times. The lines of constant u^1 and u^2 can both move and distort with time. The dot in all three figures shows a moving particle on the surface. Its velocity in the embedding space is its velocity relative to the grid plus the velocity of the grid relative to some fixed configuration.

of course define other time-dependent parametrizations by applying another time-dependent map either to $\tilde{u}$ or $\tilde{u}'$.

B. The Monge Gauge

For nearly flat surfaces, the most convenient gauge is the Monge gauge in which surface coordinates are parametrized by their height $h(\mathbf{x})$ above a flat surface with Euclidean coordinates $(\mathbf{x}, 0)$. In this case, $\tilde{u} \equiv \mathbf{x} = (x, y)$, and

$$\mathbf{R}(\mathbf{x}) = (\mathbf{x}, h(\mathbf{x})). \tag{2.10}$$

The metric tensor and its inverse in this gauge are, respectively,

$$g_{ab} = \delta_{ab} + \partial_a h \partial_b h \tag{2.11}$$

and

$$g^{ab} = \delta^{ab} - \frac{\partial_a h \partial_b h}{g}, \tag{2.12}$$

where

$$g = 1 + (\boldsymbol{\nabla} h)^2. \tag{2.13}$$

Finally, the curvature tensor is

$$K_{ab} = \frac{\partial_a \partial_b h}{\sqrt{g}}. \tag{2.14}$$

All of our renormalization calculations will be carried out in the Monge gauge.

C. Scalars and tensors

A quantity that remains unchanged under a gauge transformation is a scalar. No observable quantity can depend on how we choose to parametrize the surface, provided we do not also transform the embedding space. Thus, all observables are gauge scalars. The vector $\mathbf{R}(\tilde{u}, t)$ specifies a point in Euclidean space. It is an observable and does not change under reparametrization. Thus, each of its Cartesian components is a scalar under reparametrization: $\mathbf{R}(\tilde{u}, t) \to \mathbf{R}'(\tilde{u}', t) = \mathbf{R}(\tilde{u}, t)$. Covariant components of vectors and tensors transform via $\partial u'^a/\partial u^b$ and contravariant components via $\partial u^a/\partial u'^b$:

$$A'^{ab\cdots}_{cd\cdots} = \frac{\partial u'^a}{\partial u^{a'}} \frac{\partial u'^b}{\partial u^{b'}} \frac{\partial u^{c'}}{\partial u'^c} \frac{\partial u^{d'}}{\partial u'^d} \cdots A^{a'b'\cdots}_{c'd'\cdots}. \tag{2.15}$$

The quantities g_{ab}, K_{ab}, etc., transform as tensors according to the above rules. The surface element $dS = d^2u\sqrt{g}$ is a scalar because

$$\sqrt{g'} = \det(\partial u^a/\partial u'^b)\sqrt{g} \tag{2.16}$$

and $d^2u' = \det(\partial u'^b/\partial u^a) d^2u$. Similarly $H = \mathrm{Tr} K^a_b$ and $S = \det K^a_b$ are both scalars.

The mass density $\rho(\tilde{u}, t)$ is also a scalar. Let particle α have coordinate $u_\alpha(t)$ in the parameter manifold. Its position in Euclidean space is the $\mathbf{R}_\alpha(t) = \mathbf{R}(\tilde{u}_\alpha(t), t)$. The integral of the mass density over the surface is the total mass M: $\int d^2u\sqrt{g}\rho(\tilde{u}, t) = M$. The microscopic expression,

$$\rho(\tilde{u}, t) = \frac{m}{\sqrt{g}} \sum_\alpha \delta(\tilde{u} - \tilde{u}_\alpha(t)), \tag{2.17}$$

for the density satisfies the above constraint. Under the transformation $\tilde{u} \to \tilde{\Phi}(\tilde{u}, t)$, the coordinates of particle α transform according to $\tilde{u}_\alpha(t) \to \tilde{u}'_\alpha = \tilde{\Phi}(\tilde{u}_\alpha(t), t)$. The quantity $\tilde{u}_\alpha(t)$ measures the position of particle α relative to the fixed grid (see Fig. 1) whereas $\tilde{u}'_\alpha(t)$ measures the position of the particle relative to the time-dependent grid. The transformation law for the density can be obtained from Eq. (2.17), $\rho(\tilde{u}, t) \to \rho'(\tilde{u}', t)$, where

$$\begin{aligned} \rho'(\tilde{u}', t) &= \frac{m}{\sqrt{g(\tilde{u}', t)}} \sum_\alpha \delta(\tilde{u}'(t) - \tilde{u}'_\alpha(t)) \\ &= \frac{m}{\sqrt{g(\tilde{u}', t)}} \sum_\alpha \delta(\tilde{\Phi}(\tilde{u}, t) - \tilde{\Phi}(\tilde{u}_\alpha(t), t)) \\ &= \frac{m}{\sqrt{g(\tilde{u}', t)}} \sum_\alpha \delta(\tilde{u} - \tilde{u}_\alpha(t))[\det \partial\Phi^a(\tilde{u}, t)/\partial u^b] \\ &= \rho(\tilde{u}, t), \end{aligned} \tag{2.18}$$

where we used Eq. (2.16) with $\Phi^a = u'^a$. This verifies explicitly that $\rho(\tilde{u}, t)$ is a scalar under gauge transformations.

Scalars involving derivatives with respect to u^a are constructed from the covariant derivative D_a. For a vector v^a, $D_a v^c = \partial_a v^c + \Gamma^c_{ab} v^b$ where $\Gamma^c_{ab} = \mathbf{e}^c \cdot \partial_a \mathbf{e}_b$ is the connection. The quantity $D_a v^a = g^{-1/2}\partial_a(g^{1/2}v^a)$ is a scalar. The curvature tensor can be expressed compactly in terms of covariant derivatives:

$$D_a \mathbf{e}_b = K_{ab}\mathbf{n}. \tag{2.19}$$

Thus, the mean curvature is obtained from

$$D_a \mathbf{e}^a = H\mathbf{n}. \tag{2.20}$$

This equation will prove useful in the analysis that follows.

III. ENERGIES AND FORCES

The free energy or Landau-Ginzburg-Wilson (LGW) Hamiltonian $\mathcal{H}$ for a fluctuating surface can be constructed from the scalars discussed in the preceding section. We consider first terms in $\mathcal{H}$ depending only on the mass density. As in flat space [11], we can introduce a Helmholtz free energy density $f(\rho)$ and a chemical potential μ on a curved surface [31]. The associated LGW Hamiltonian is

$$\mathcal{H}_\rho = \int d^2u\sqrt{g}[f(\rho) - \mu\rho]. \tag{3.1}$$

In general, there will also be terms depending on covariant derivatives of ρ (e.g., $D^a\rho D_a\rho$). They will, however, not concern us here. The equilibrium value, ρ_0, of ρ in mean-field theory is determined by minimizing $\mathcal{H}_\rho$ with respect to ρ: $\partial f/\partial\rho - \mu = 0$ at $\rho = \rho_0$. We can expand $\mathcal{H}_\rho$ in a power series in $\delta\rho = \rho - \rho_0$:

$$\mathcal{H}_\rho = \sigma \int d^2u\sqrt{g} + \tfrac{1}{2}\chi_0^{-1}\int d^2u\sqrt{g}(\delta\rho)^2, \tag{3.2}$$

where χ_0/ρ_0^2 is the compressibility and

$$\sigma \equiv \sigma(\rho_0) = f(\rho_0) - \mu\rho_0 \tag{3.3}$$

is the surface tension or energy per unit area of the surface at the equilibrium density. At density ρ, the membrane pressure is

$$p(\rho) = -[f(\rho) - \rho\partial f(\rho)/\partial\rho]. \tag{3.4}$$

In equilibrium at chemical potential μ, $p = -\sigma(\rho_0)$. For ρ near ρ_0, we can expand p in $\delta\rho$:

$$p(\rho) = -\sigma - \rho_0\chi_0^{-1}\delta\rho. \tag{3.5}$$

In the absence of external forces, whose contributions are included in $f(\rho)$, p is normally positive in equilibrium systems. If the membrane is under external tension, then p can be negative and σ positive. In free membranes, σ is effectively zero [32].

There are energies associated with distorting a membrane away from its preferred shape. The Helfrich-Canham Hamiltonian [3,4],

$$\mathcal{H}_c = \tfrac{1}{2}\kappa\int d^2u\sqrt{g}(H - H_0)^2 + \kappa_g\int d^2u\sqrt{g}S, \tag{3.6}$$

describes the energy of shape variations that are slow on a scale set by the membrane width. Here κ and κ_g are, respectively, the bending and Gaussian curvature rigidities with units of energy, and $H_0 = 2/R_0$ is twice the spontaneous curvature. In what follows, we will focus on nearly flat membranes of fixed topology for which the spontaneous curvature is zero and for which the Gaussian curvature term can be ignored. The bending rigidity κ depends on density. In Eq. (3.6), we assumed that $\kappa = \kappa(\rho = \rho_0)$ is independent of ρ. When ρ deviates from ρ_0, there are additional terms in $\mathcal{H}$ involving both $\delta\rho$ and curvature. The leading such term is

$$\mathcal{H}_{c-\rho} = \lambda\int d^2u\sqrt{g}\delta\rho H^2, \tag{3.7}$$

where $\lambda = \frac{1}{2}(\partial\kappa/\partial\rho)|_{\rho=\rho_0}$. This term is subdominant compared to others in our present calculations, and we will ignore it. Our model membrane Hamiltonian is thus

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_\rho \tag{3.8}$$

with $H_0 = 0$.

The force density $\mathbf{f}_s(\tilde{u})$ acting on a given point on a membrane arising from membrane stresses can be calculated by taking the functional derivative of $\mathcal{H}$ with respect to membrane displacements at constant mass in each volume element in the parameter manifold. This is equivalent to a derivative at constant $\sqrt{g}\rho$ [33]:

$$\begin{aligned}\mathbf{f}_s(\tilde{u}) &= -\left.\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\mathbf{R}(\tilde{u})}\right|_{\sqrt{g}\rho} \\ &= -\mathbf{e}^a\partial_a p - [\kappa Q + p(\rho)H]\mathbf{n},\end{aligned} \tag{3.9}$$

where

$$Q = H(K^a_b - \tfrac{1}{2}\delta^a_b H)(K^a_b - \tfrac{1}{2}\delta^a_b H) + D^2 H \tag{3.10}$$

arises from $\mathcal{H}_c$. Alternative forms of $\mathbf{f}_s$, expressed in terms of derivatives of $\mathcal{H}$ with respect to $\mathbf{R}$ at constant ρ and with respect to ρ at constant $\mathbf{R}$, will be useful in what follows. The derivative of $\mathcal{H}$ at constant ρ is

$$\left.\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\mathbf{R}(\tilde{u})}\right|_\rho = [\kappa Q + p(\rho)H]\mathbf{n} - \frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\rho}D_a(\rho\mathbf{e}^a), \tag{3.11}$$

where we used

$$\begin{aligned}D_a[(\mu\rho - f)\mathbf{e}^a] &= D_a\{[p - (\rho/\sqrt{g})(\delta\mathcal{H}/\delta\rho)]\mathbf{e}^a\} \\ &= pH\mathbf{n} - (1/\sqrt{g})(\delta\mathcal{H}/\delta\rho)D_a(\rho\mathbf{e}^a).\end{aligned} \tag{3.12}$$

We can calculate $\delta\mathcal{H}/\delta h(\mathbf{x})|_\rho$ in the Monge gauge by taking the z component of Eq. (3.11):

$$\left.\frac{\delta\mathcal{H}}{\delta h(\mathbf{x})}\right|_\rho = (\kappa Q + pH) - \frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\rho(\mathbf{x})}\partial_a\left(\rho\frac{\partial_a h}{\sqrt{g}}\right). \tag{3.13}$$

Combining Eqs. (3.9) and (3.13), we obtain

$$\mathbf{f}_s(\mathbf{x}) = -\left[\left.\frac{\delta\mathcal{H}}{\delta h}\right|_\rho + \frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\rho}\partial_a\left(\rho\frac{\partial_a h}{\sqrt{g}}\right)\right]\mathbf{n}$$

$$-\rho\partial_a\left(\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\rho}\right)\mathbf{e}^a \tag{3.14}$$

for the force density in the Monge gauge.

IV. STATIC RENORMALIZATION

Before proceeding to the derivation and study of membrane hydrodynamics, we will in this section consider renormalization of membranes in static equilibrium under removal of high-wave-number degrees of freedom. The renormalization of κ and σ have been extensively studied [34–36,6]. Here we will be interested principally in how the compressibility χ renormalizes. We will also review the measure for integrating over different inequivalent shapes in partition sums. Naive measures do not preserve rotational invariance and Ward identities. Though it possible to obtain the correct one-loop renormalization of κ and σ in static calculations without serious attention to measure [36], it is not possible to obtain correctly the renormalization of σ in the dynamical calculations we present here.

A fluctuating membrane has shape fluctuations with wavelengths up to a cutoff $\Lambda \sim 2\pi/a$ where a is a molecular length. After removal of degrees of freedom with wave number $\Lambda/b < q < \Lambda$ the resulting coarse-grained membrane will have shape fluctuations with maximum wave number Λ/b as depicted schematically in Fig. 2. Under this transformation, the determinant of the metric tensor changes from g to $g' < g$, and the effective area of a patch of membrane associated with an area element d^2u in the parameter manifold decreases: $d^2u\sqrt{g'} < d^2u\sqrt{g}$. On the other hand, the mass associated with an area element in the parameter manifold cannot change under coarse graining. Thus, the local mass density $\rho(\tilde{u})$ transforms under coarse graining to a larger mass density ρ' satisfying

$$\sqrt{g'}\rho' = \sqrt{g}\rho. \tag{4.1}$$

The additional mass density arises because of membrane crumpling, which is not visible at the new length scale. Figure 2 suggests strongly that the compressibility of the coarse-grained membrane is larger than that of the original membrane: the density of the coarse-grained membrane can easily be increased by increasing the local crumpling. Thus, a membrane that is incompressible at the molecular scale becomes compressible at the coarse-grain scale $a' = 2\pi/\Lambda'$. This important fact, which we will derive below, implies that a membrane can never be considered as incompressible at long-length scales. As a consequence, there will always be hydrodynamic density modes.

FIG. 2. Schematic representation of a fluctuating surface. The full line represents the surface in which fluctuations with wave number up to Λ are allowed. The dotted line represents the coarse-grained surface with maximum wave number of fluctuations less than Λ/b. Each element of area of the coarse-grained surface has greater mass than the original surface because of surface crumpling. The degree of crumpling can change locally causing the effective coarse-grained density to change and rendering the membrane compressible.

A. Partition function and measure

The partition function for membranes can be written as

$$Z = \sum_{\text{shapes}} \int \mathcal{D}[\rho] e^{-\mathcal{H}/T}, \tag{4.2}$$

where

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_\rho. \tag{4.3}$$

The sum over shapes is a sum over all physically distinct realizations of the surface. Naively, this is an integral over all positions $\mathbf{R}(\tilde{u})$ of local points on the surface. However, translations of $\mathbf{R}$ parallel to the surface do not change the membrane shape. As a result, shape changes are parametrized by one variable rather than the apparent three in $\mathbf{R}(\tilde{u})$. Different choices of this variable correspond to different gauges. In this paper, we will be concerned only with what are called normal gauges in which different surfaces are specified by the normal distance of their points from some reference surface $\mathbf{R}_0(\tilde{u})$: $\mathbf{R}(\tilde{u}) = \mathbf{R}_0(\tilde{u}) + \nu(\tilde{u})\mathbf{n}_0(\tilde{u})$ where $\mathbf{n}_0$ is the unit normal to the surface $\mathbf{R}_0$. In the Monge gauge, which we will use most extensively, the reference surface is the flat xy plane: $\tilde{u} = \mathbf{x} \equiv (x, y)$, $\mathbf{R}_0(\tilde{u}) = (x, y, 0)$, $\mathbf{n}_0 = \mathbf{e}_z$, and $\mathbf{R}(\mathbf{x})$ is given by Eq. (2.10).

The sum over shapes will involve an integral over $\nu(\tilde{u})$ or over $h(\mathbf{x})$ in the Monge Gauge. As in all gauge theories [37], there is a Fadeev-Popov (FP) determinant $\Delta_{FP} \equiv \exp(-\mathcal{H}_{FP}/T)$ associated with gauge fixing, and we would naively expect $\sum_{\text{shapes}} = \int \mathcal{D}[\nu(\tilde{u})]\Delta_{FP}$. The FP determinant, evaluated in Ref. [6], is $\Delta_{FP} = \prod_{\tilde{u}} \mathbf{n}(\tilde{u}) \cdot \mathbf{n}_0(\tilde{u})$, where $\mathbf{n}(\tilde{u})$ is the unit normal to the actual surface. The product is over cells in the parameter space whose number tends to infinity and volume tends to zero in the continuum limit. Associated with each of these cells, which can be constructed from lines of constant u^1 and u^2 as shown in Fig. 1, is a surface element each with the same physical area in the embedding space. In the Monge gauge [6,38], this reduces to $\Delta_{FP} = \prod_{\mathbf{x}} g^{-1/2}(\mathbf{x})$. The FP determinant can be reexpressed as $\Delta_{FP} = e^{-\mathcal{H}_{FP}/T}$, where $\mathcal{H}_{FP} = \frac{T}{2}\sum_{\mathbf{x}} \ln g$, the factor of $\sqrt{g}$ in the integral arising from the constraint of identical physical area for each cell. If we were dealing with a standard gauge-field theory, $\mathcal{D}[\nu]\Delta_{FP}$ would be the complete and correct measure for integrating over physically different states. In membranes, however, there is an additional complication. As just discussed, the grid in parameter space should be constructed so that the area of the surface associated with each of its cells is identical. In normal gauges, the parameter manifold is chosen to be a fixed reference surface. The physical surface has greater area than the reference surface unless the two surfaces coincide exactly. We are thus faced with a problem. We either have to allow the grid in the parameter manifold to change as the membrane fluctuates in order to maintain constant membrane area per cell, or we have to find some way of dealing with extra membrane area if the grid of the parameter manifold is fixed. For practical calculations, it is very desirable to keep the grid in the parameter manifold fixed in order, for example, to have simply defined Fourier transforms. To correct for the missing area, it is necessary to introduce an additional factor, $e^{-\mathcal{H}_L/T}$, in the measure. This factor, analogous to the Liouville correction in string theory [39,40], was calculated in Ref. [6]. The sum over shapes is now $\sum_{\text{shapes}} = \int \mathcal{D}[\nu(\tilde{u})]e^{-(\mathcal{H}_L+\mathcal{H}_{FP})/T}$ where the grid of points in the parameter space remains fixed. The combined Liouville and FP factors can be expressed in terms of an effective measure Hamiltonian, which in the Monge gauge to lowest order in curvature is

$$\begin{aligned}\mathcal{H}_M &= \mathcal{H}_{FP} + \mathcal{H}_L \\ &= \left(\int_{BZ'} - \int_{BZ}\right) \frac{T}{2} \ln[\kappa(q_a q_b g^{ab})^2 + \sigma q_a q_b g^{ab}] \\ &\quad \times (\lambda^2/\Lambda^2 T) - \tfrac{1}{2}T \int_{BZ} \ln 1/\sqrt{g} \\ &= \tfrac{1}{2}\mu_M \int d^2x (\nabla h)^2 + O((\nabla h)^4),\end{aligned} \tag{4.4}$$

where g_{ab} is the metric tensor of the reference surface, $\int_{BZ} = \int_0^\Lambda d^2q/(2\pi)^2$ is the integral over the symmetric Brillouin zone of the reference surface, $\int_{BZ'}$ is the integral over the distorted Brillouin zone of the tilted surface with nonzero ∇h, and

$$\mu_M = \tfrac{1}{2}T \int^\Lambda \frac{d^2q}{(2\pi)^2} \left[\ln\left(\sigma q^2 + \kappa q^4\right) \frac{\lambda^2}{\Lambda^2 T} + \frac{3\kappa q^4 + 2\sigma q^2}{\kappa q^4 + \sigma q^2}\right]. \tag{4.5}$$

The full partition function in the Monge gauge is thus

$$Z = \int \mathcal{D}[h] \int \mathcal{D}[\rho] e^{-(\mathcal{H}_M + \mathcal{H})/T}. \tag{4.6}$$

The measures $\mathcal{D}[h]$ and $\mathcal{D}[\rho]$ are now the naive unitless measures defined on the flat reference surface: $\mathcal{D}[h] = \prod_{\mathbf{x}} dh[\mathbf{x}]/\lambda$ and $\mathcal{D}[\rho] = \prod_{\mathbf{x}} d\rho(\mathbf{x})/\overline{\rho}$, where λ is the thermal wavelength [41] and $\overline{\rho} = m/\lambda^2$.

B. Momentum shell coarse graining

We can now carry out the momentum shell coarse-graining procedure. We divide h into slow and fast components $h = h^< + h^>$ where $h^<(\mathbf{q}) = 0$ if $\Lambda/b < q < \Lambda$ and $h^>(\mathbf{q}) = 0$ if $0 < q < \Lambda/b$. In addition, we introduce [following Eq. (4.1)] a coarse-grained density ρ' satisfying $\sqrt{g'}\rho' = \sqrt{g}\rho$, where $g' = 1 + (\nabla h^<)^2$ and $g = 1 + (\nabla h)^2$. The partition function is then

$$\begin{aligned}Z &= \int \mathcal{D}[h^<]\mathcal{D}[h^>]\mathcal{D}[\rho]\mathcal{D}[\rho'] \\ &\quad \times \prod_{\mathbf{x}} \delta(\rho' - (g/g')^{1/2}\rho) e^{-(\mathcal{H}+\mathcal{H}_M)/T} \\ &= \int \mathcal{D}[h^<]\mathcal{D}[\rho']\mathcal{D}[h^>]\mathcal{D}[\rho]\mathcal{D}[\theta] \\ &\quad \times \exp[-(\mathcal{H} + \mathcal{H}_M)/T \\ &\quad -i \int \theta(\mathbf{x})(\rho' - \rho(g/g')^{1/2}) d^2x].\end{aligned} \tag{4.7}$$

Integrating over $h^>$ with $b = 1+\epsilon$ and $\epsilon \ll 1$, and keeping only harmonic terms in $h^<$, we find

$$Z = \int \mathcal{D}[h^<]\mathcal{D}[\rho]\mathcal{D}[\rho']\mathcal{D}[\theta] e^{-\mathcal{H}_{\text{mid}}/T}, \tag{4.8}$$

where

$$\begin{aligned}\mathcal{H}_{\text{mid}} &= \int d^2x \{\sigma'[1 + \tfrac{1}{2}(\nabla h^<)^2] + \tfrac{1}{2}\kappa'(\nabla^2 h^<)^2\} \\ &\quad + \int d^2x \{\tfrac{1}{2}(\rho_0^2 T^2/\Lambda^2)\Delta_2 \theta^2 \\ &\quad -iT(1+\Delta_1)\theta\delta\rho + iT\theta[\rho' - (1+\Delta_1)\rho_0]\} \\ &\quad + \tfrac{1}{2}\chi_0^{-1}(1+\Delta_1) \int d^2x (\delta\rho)^2 + \cdots,\end{aligned} \tag{4.9}$$

where

$$\Delta_1 = \frac{\epsilon}{4\pi} \frac{T\Lambda^2}{\kappa\Lambda^2 + \sigma}, \tag{4.10}$$

$$\Delta_2 = \frac{\epsilon}{4\pi} \frac{T\Lambda^4}{(\kappa\Lambda^2 + \sigma)^2}, \tag{4.11}$$

$$\kappa' = \kappa(1 - 3\Delta_1), \tag{4.12}$$

$$\sigma' = \sigma\left[1 + \frac{\epsilon}{4\pi} T\Lambda^2 \ln(\kappa\Lambda^2 + \sigma)\frac{\lambda^2}{T}\right]. \tag{4.13}$$

Equations (4.12) and (4.13) could also be expressed as differential flow equations. Finally, we integrate over θ and $\delta\rho$ to obtain

$$Z = \int \mathcal{D}[h']\mathcal{D}[\rho'] e^{-\mathcal{H}'/T}, \tag{4.14}$$

where

$$\mathcal{H}' = \int d^2u \sqrt{g'}[\sigma' + \tfrac{1}{2}\kappa'(H')^2 + \tfrac{1}{2}(\chi')^{-1}(\delta\rho')^2], \tag{4.15}$$

where $H' \approx (\nabla^2 h^<)^2$, $g' = 1 + (\nabla h^<)^2$, and

$$(\chi')^{-1} = \chi_0^{-1}[1 + \Delta_1 + (\rho_0^2/\Lambda^2)\Delta_2 \chi_0^{-1}]^{-1} \tag{4.16}$$

or

$$\chi' = \chi_0(1 + \Delta_1) + (\rho_0^2/\Lambda^2)\Delta_2. \tag{4.17}$$

The last equation tells us that the compressibility increases upon coarse graining as suggested by our discussion at the beginning of this section. Even if the membrane is incompressible at the microscopic length scale, i.e., even if $\chi_0 = 0$, it is compressible at longer length scales (i.e., $\chi' \neq 0$).

The new surface tension σ' in Eq. (4.13) is the coefficient of the coarse-grained area in $\mathcal{H}'$. It is thus the coefficient of $\frac{1}{2}(\nabla h^<)^2$ when $(g')^{1/2}$ is expanded in $(\nabla h^<)^2$. The equality of both coefficients is guaranteed to the order we are calculating by the Liouville-Fadeev-Popov factor of Eq. (4.4). Thus, the coarse-grained height-height correlation function is

$$\langle h^<(\mathbf{q})h^<(-\mathbf{q})\rangle = \frac{T}{\sigma' q^2 + \kappa' q^4} \tag{4.18}$$

as required by rotational invariance.

V. HYDRODYNAMICS

The long-wavelength, low-frequency dynamics of any system is controlled by hydrodynamical equations relating conserved and broken-symmetry variables. In isolated two-dimensional (2D) fluid membranes, there are four intrinsic conserved quantities: energy, mass, and two tangent-plane components of momentum. If the membrane is free to move in a three-dimensional embedding space, the component of momentum normal to the surface is also conserved, leading to a total of five conserved quantities. In addition, uniform rigid translations of the membrane in its embedding space leave its energy unchanged, and the membrane position variable $\mathbf{R}(\tilde{u})$ is a hydrodynamic variable. As in static problems, reparametrization invariance reduces the number of physically independent components of $\mathbf{R}$ from 3 to 1. Thus, an isolated membrane has 6 hydrodynamic variables and 6 associated hydrodynamic modes: one heat, one tangent-plane transverse momentum, two propagating longitudinal tangent-plane sound modes, and two propagating capillary or height modes. When a membrane interacts with its environment, its momentum is no longer conserved. There are then only three hydrodynamic modes: energy, mass (provided the membrane does not exchange mass with its environment), and height.

In this section, we will derive the complete covariant hydrodynamic equations both for an isolated membrane and for one that interacts with its environment. To keep our discussion as simple as possible, we will consider only isothermal processes so that we do not have to consider energy conservation. Though membrane hydrodynamical equations have been derived elsewhere [23,24], we present here a derivation that emphasizes reparametrization invariance and clearly identifies physical quantities, which do not change under reparametrization.

We begin with the mass conservation law, which will give us the tangent-plane momentum density. Using Eq. (2.17), we obtain

$$\begin{aligned}\partial_t[\sqrt{g(\tilde{u},t)}\rho(\tilde{u},t)] &= m\partial_t \sum_\alpha \delta(\tilde{u}-\tilde{u}_\alpha(t))\\ &= -m\partial_a \sum_\alpha \frac{du^a_\alpha}{dt}\delta(\tilde{u}-\tilde{u}_\alpha)\\ &= -\partial_a[\sqrt{g}j^a(\tilde{u},t)],\end{aligned} \tag{5.1}$$

where ∂_t is a derivative at constant $\tilde{u}$, and

$$j^a(\tilde{u},t) = m\sum_\alpha \frac{du^a_\alpha}{dt}\frac{\delta(\tilde{u}-\tilde{u}_\alpha)}{\sqrt{g}} \tag{5.2}$$

is the momentum density. Equation (5.1) can be rewritten in covariant form as

$$D_t\rho + D_a j^a = 0, \tag{5.3}$$

where $D_t = g^{-1/2}\partial_t g^{1/2}$ and D_a is the covariant derivative. Though j^a transforms as a vector under time-independent gauge transformations, its transformation properties under time-dependent gauge transformations are more complicated:

$$\begin{aligned}j'^a(\tilde{u}',t) &= \sum_\alpha \frac{du'^a_\alpha}{dt}\frac{\delta(\tilde{u}'-\tilde{u}'_\alpha)}{\sqrt{g(\tilde{u}',t)}}\\ &= \sum_\alpha\left(\frac{\partial u'^a}{\partial u^b}\frac{du^b_\alpha}{dt} + \left.\frac{\partial u'^a}{\partial t}\right|_{\tilde{u}}\right)\frac{\delta(\tilde{u}-\tilde{u}_\alpha)}{\sqrt{g(\tilde{u},t)}}\\ &= \frac{\partial u'^a}{\partial u^b}j^b(\tilde{u},t) + \rho(\tilde{u},t)\left.\frac{\partial u'^a}{\partial t}\right|_{\tilde{u}}.\end{aligned} \tag{5.4}$$

The second term in this equation is familiar from Galilean transformations in flat space where the momentum density $\mathbf{j}(\mathbf{r},t)$ transforms according to $\mathbf{j}'(\mathbf{r}',t) = \mathbf{j}(\mathbf{r},t) + \rho\mathbf{v}$ under $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \mathbf{v}t$.

The momentum density j^a depends on parametrization and, therefore, cannot be a physical observable. To determine the gauge invariant momentum density, we consider the three-dimensional mass density $\rho_3(\mathbf{r},t)$ at point $\mathbf{r}$ in the embedding space:

$$\rho_3(\mathbf{r},t) = \int d^2u\sqrt{g}\rho(\tilde{u},t)\delta(\mathbf{r}-\mathbf{R}(\tilde{u},t)), \tag{5.5}$$

which is a conserved quantity. The 3D momentum density $\mathbf{J}_3(\mathbf{r},t) = \rho_3\mathbf{v}_3$, where $\mathbf{v}_3$ is the velocity, is the current associated with ρ_3. To determine $\mathbf{J}_3$, we calculate

$$\begin{aligned}\partial_t\rho_3(\mathbf{r},t) &= \int d^2u\partial_t(\sqrt{g}\rho)\delta(\mathbf{r}-\mathbf{R}(\tilde{u},t))\\ &\quad - \int d^2u\sqrt{g}\rho\partial_t\mathbf{R}\cdot\nabla\delta(\mathbf{r}-\mathbf{R})\\ &= -\int d^2u[\partial_a(\sqrt{g}j^a) + \sqrt{g}\rho\partial_t\mathbf{R}\cdot\nabla]\delta(\mathbf{r}-\mathbf{R}),\end{aligned} \tag{5.6}$$

where ∇ operates on the Euclidean space position $\mathbf{r}$. Then integrating by parts and using

$$\partial_a\delta(\mathbf{r}-\mathbf{R}) = -\mathbf{e}_a\cdot\nabla\delta(\mathbf{r}-\mathbf{R}) \tag{5.7}$$

we obtain

$$\begin{aligned}\partial_t\rho_3(\mathbf{r},t) &= -\nabla\cdot\int d^2u\sqrt{g}(j^a\mathbf{e}_a + \rho\partial_t\mathbf{R})\delta(\mathbf{r}-\mathbf{R})\\ &= -\nabla\cdot\mathbf{J}_3(\mathbf{r}),\end{aligned} \tag{5.8}$$

where

$$\mathbf{J}_3(\mathbf{r}) = \int d^2\tilde{u}\sqrt{g}\mathbf{J}(\tilde{u})\delta(\mathbf{r}-\mathbf{R}(\tilde{u})) \tag{5.9}$$

with

$$\mathbf{J}(\tilde{u}) = j^a \mathbf{e}_a + \rho \partial_t \mathbf{R}. \tag{5.10}$$

In the above, $\partial_t \mathbf{R}(\tilde{u}, t) = \partial_t \mathbf{R}(\tilde{u}, t)/\partial t|_{\tilde{u}}$ as before. Under a time-dependent gauge transformation, $\partial_t \mathbf{R}(\tilde{u}', t)|_{\tilde{u}} = \partial_t \mathbf{R}|_{\tilde{u}'} + (\partial u'^a/\partial t)\partial_a \mathbf{R}$, and we can see with the aid of Eq. (5.4) that $\mathbf{J}(\tilde{u}, t)$ is gauge invariant. Thus, $\mathbf{J}(\tilde{u}, t)$ is the physical momentum density of the 2D membrane in the 3D embedding space. Note that the component of $\mathbf{J}$ normal to the membrane, $J_n = \rho \mathbf{n} \cdot \partial_t \mathbf{R}$, does not depend on j^a. The component of $\mathbf{J}$ parallel to the membrane, $J^a = \mathbf{e}^a \cdot \mathbf{J} = j^a + \rho \mathbf{e}^a \cdot \partial_t \mathbf{R}$, has contributions from both j^a, the momentum relative to the grid $\tilde{u}$, and $\mathbf{e}^a \cdot \partial_t \mathbf{R}$, the rate of change of the grid marks in Euclidean space. From $\mathbf{J}(\tilde{u}, t)$, we can define a local membrane velocity $\mathbf{v}(\tilde{u})$ via

$$\mathbf{J}(\tilde{u}, t) = \rho(\tilde{u}, t)\mathbf{v}(\tilde{u}, t). \tag{5.11}$$

The normal velocity is simply

$$v_n = \mathbf{n} \cdot \partial_t \mathbf{R}, \tag{5.12}$$

and the tangential velocity is $v^a = (j^a/\rho) + \mathbf{e}^a \cdot \partial_t \mathbf{R}$. Both $\mathbf{v}(\tilde{u}, t)$ and $\mathbf{v}_3(\mathbf{R}(\tilde{u}, t), t)$ are the local membrane velocity and must, therefore, equal each other:

$$\mathbf{v}(\tilde{u}, t) = \mathbf{v}_3(\mathbf{R}(\tilde{u}, t), t). \tag{5.13}$$

Having determined the momentum density, we need to calculate its equation of motion. This is most easily done by looking first at the equation for $\mathbf{J}_3$. The material time derivative $d\mathbf{J}_3(\mathbf{r}, t)/dt$ is equal to the total force density,

$$\mathbf{F}_T(\mathbf{r}, t) = \int d^2u \sqrt{g} \mathbf{f}_T(\tilde{u}, t) \delta(\mathbf{r} - \mathbf{R}(\tilde{u}, t)), \tag{5.14}$$

on the membrane at point $\mathbf{r}$, where

$$\mathbf{f}_T = \mathbf{f}_s + \mathbf{f}_{\text{dis}} + \mathbf{f}_{\text{ext}} \tag{5.15}$$

is the total force per unit area on the membrane at the parameter point $\tilde{u}$. The latter consists of an external part $\mathbf{f}_{\text{ext}}$, a dissipative part $\mathbf{f}_{\text{dis}}$, and a reactive internal part $-g^{-1/2}\delta\mathcal{H}/\delta\mathbf{R}$ [Eq. (3.9)]. The equation of motion for $\mathbf{J}$ is thus

$$\frac{d\mathbf{J}_3}{dt} = \partial_t \mathbf{J}_3 + \nabla_i(v_i \mathbf{J}_3) = \int d^2u \sqrt{g} \mathbf{f}_T \delta(\mathbf{r} - \mathbf{R}). \tag{5.16}$$

The left-hand side of this equation can be rewritten as an integral over the surface using Eq. (5.9):

$$\begin{aligned}
\partial_t \mathbf{J}_3 &+ \boldsymbol{\nabla}_i(v_i \mathbf{J}_3) \\
&= \partial_t \int d^2u \sqrt{g} \mathbf{J} \delta(\mathbf{r} - \mathbf{R}) + \boldsymbol{\nabla}_i \int d^2u \sqrt{g}(J_i \mathbf{J}/\rho) \\
&\quad \times \delta(\mathbf{r} - \mathbf{R}) \\
&= \int d^2u [\partial_t(\sqrt{g}\mathbf{J}) - \sqrt{g}\mathbf{J}(\partial_t \mathbf{R} - \mathbf{J}/\rho) \cdot \boldsymbol{\nabla}] \delta(\mathbf{r} - \mathbf{R}) \\
&= \int d^2u [\partial_t(\sqrt{g}\mathbf{J}) + \sqrt{g}(j^a \mathbf{J}/\rho)\mathbf{e}_a \cdot \boldsymbol{\nabla}] \delta(\mathbf{r} - \mathbf{R}) \\
&= \int d^2u [\partial_t(\sqrt{g}\mathbf{J}) + \partial_a(\sqrt{g} j^a \mathbf{J}/\rho)] \delta(\mathbf{r} - \mathbf{R}),
\end{aligned} \tag{5.17}$$

where we used Eq. (5.13) relating $\mathbf{v}_3(\mathbf{R}(\tilde{u}), t)$ to $\mathbf{v}(\tilde{u}, t) = \mathbf{J}/\rho$ and Eq. (5.7). Now, using Eqs. (5.17) and (5.14), we have

$$D_t \mathbf{J} + D_a(j^a \mathbf{J}/\rho) = -\frac{1}{\sqrt{g}} \frac{\delta \mathcal{H}}{\delta \mathbf{R}}\bigg|_{\sqrt{g}\rho} + \mathbf{f}_{\text{dis}} + \mathbf{f}_{\text{ext}}, \tag{5.18}$$

where $\delta\mathcal{H}/\delta\mathbf{R}$ is given by Eq. (3.9).

The dissipative force density $\mathbf{f}_{\text{dis}} = \mathbf{f}_{\text{vis}} + \mathbf{f}_{\text{m}}$ has a contribution $\mathbf{f}_{\text{vis}}$ from intramembrane viscous forces, and a contribution $\mathbf{f}_{\text{m}}$ from interactions with the surrounding medium. It can be decomposed into a normal and a tangent plane part: $\mathbf{f}_{\text{dis}} = f_{\text{dis},n}\mathbf{n} + f^a_{\text{dis}}\mathbf{e}_a$. The tangent-plane part of $\mathbf{f}_{\text{vis}}$ can be expressed in terms of a tangent-plane viscosity tensor and covariant derivatives of the membrane velocity:

$$f^a_{\text{vis}} = \eta^{abcd} D_b D_c v_d, \tag{5.19}$$

where

$$\eta^{abcd} = \eta_s(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd}) + \zeta_s g^{ab}g^{cd} \tag{5.20}$$

with η_s and ζ_s, respectively, the surface shear and bulk viscosities. The normal component of the viscous force is

$$f_{\text{vis},n} = \eta_n D_a D^a v_n. \tag{5.21}$$

The force $\mathbf{f}_{\text{m}}$ arises from interactions with the embedding fluid. It can be modeled phenomenologically as a friction force proportional to the difference between the membrane velocity $\mathbf{v}$ and the medium velocity $\mathbf{V} = V_n \mathbf{n} + V^a \mathbf{e}_a$ at the membrane:

$$\mathbf{f}_{\text{m}} = -\gamma_n(v_n - V_n)\mathbf{n} - \gamma_t(v^a - V^a)\mathbf{e}_a \tag{5.22}$$

with different friction coefficients γ_n and γ_t for flow respectively perpendicular to and parallel to the membrane. There can only be flow perpendicular to the membrane if there are pores in the membrane allowing the embedding fluid to pass through it. If the membrane has no pores, $v_n = V_n$, a condition that can be imposed by setting $\gamma_n = \infty$. No slip boundary conditions $v^a = V^a$ parallel to the membrane can be imposed by $\gamma_t = \infty$.

VI. THE ROUSE MODEL

A. Model and harmonic modes

The hydrodynamical equations derived in the preceding section, though perfectly general, are quite complex. In this section, we will consider a simplified model, the membrane version of the Rouse model [16], in which the inertial and nonlinear terms on the left-hand side of Eq. (5.18) are neglected and in which the background fluid is rigidly fixed so that

$$\mathbf{f}_{\text{m}} = -\gamma_n v_n \mathbf{n} - \gamma_t v^a \mathbf{e}_a. \tag{6.1}$$

The intramembrane viscous force $\mathbf{f}_{\text{vis}}$ is subdominant compared to $\mathbf{f}_{\rm m}$, and we can ignore it in $\mathbf{f}_{\rm dis} \approx \mathbf{f}_{\rm m}$. We will be interested in how short-wavelength fluctuations modify long-wavelength properties so we will add a random thermal noise source $\boldsymbol{\zeta}(\tilde{u},t) = \zeta_n \mathbf{n} + \zeta^a \mathbf{e}_a$ whose statistics we will discuss shortly. The dynamical equations in the Rouse model are then

$$\gamma_n \partial_t \mathbf{R}\cdot\mathbf{n} = -\frac{1}{\sqrt{g}}\left.\frac{\delta\mathcal{H}}{\delta\mathbf{R}}\right|_{\rho\sqrt{g}}\cdot\mathbf{n} + \zeta_n, \tag{6.2}$$

$$\gamma_t\left(j^a/\rho + \partial_t\mathbf{R}\cdot\mathbf{e}^a\right) = -\frac{1}{\sqrt{g}}\left.\frac{\delta\mathcal{H}}{\delta\mathbf{R}}\right|_{\rho\sqrt{g}}\cdot\mathbf{e}^a + \zeta^a. \tag{6.3}$$

The mass current j^a can be removed from Eq. (6.3) by applying the covariant derivative D_a to both sides and using the mass conservation equation, Eq. (5.3):

$$D_t\rho - D_a(\rho\partial_t\mathbf{R}\cdot\mathbf{e}^a) + \gamma_t^{-1}D_a(\rho\mathbf{f}_s\cdot\mathbf{e}^a) + \gamma_t^{-1}D_a(\rho\zeta^a) = 0, \tag{6.4}$$

where $\mathbf{f}_s$ is the membrane force Eq. (3.14). In the Monge gauge, Eqs. (6.2) and (6.4), respectively, yield equations for $\partial_t h$ and $\partial_t \rho$:

$$\partial_t h = -\frac{\sqrt{g}}{\gamma_n}\left.\frac{\delta\mathcal{H}}{\delta h}\right|_\rho - \frac{1}{\gamma_n}\partial_a\left(\rho\frac{\partial_a h}{\sqrt{g}}\right)\frac{\delta\mathcal{H}}{\delta\rho} + \frac{\sqrt{g}}{\gamma_n}\zeta_n, \tag{6.5}$$

$$\partial_t\rho = \frac{1}{\gamma_t}D_a\left(\rho^2 D^a\frac{1}{\sqrt{g}}\frac{\delta\mathcal{H}}{\delta\rho}\right) + \frac{1}{\sqrt{g}}\partial_a\left(\frac{1}{\sqrt{g}}\rho\partial_a h\right)\partial_t h - \frac{1}{\gamma_t}\frac{1}{\sqrt{g}}\partial_a(\rho\sqrt{g}\zeta^a). \tag{6.6}$$

Our principal interest in this section is how removal of high-wave-number degrees of freedom in $h(\mathbf{q})$ renormalizes static and dynamic coefficients to lowest order in T/κ. We can, therefore, linearize these equations in ρ by setting $\rho = \rho_0 + \delta\rho$. To first order in $\delta\rho$, Eqs. (6.5) and (6.6) can be written as $F_h = 0$ and $F_\rho = 0$ where

$$F_h = \partial_t h + \sqrt{g}(\kappa Q - \sigma H)/\gamma_n + \sqrt{g}(\rho_0\chi_0^{-1}/\gamma_n)H\delta\rho - \zeta_h = 0, \tag{6.7}$$

$$F_\rho = D_t\rho - (\rho_0^2\chi_0^{-1}/\gamma_t)D^2\delta\rho - \rho_0 D_a(\partial_t\mathbf{R}\cdot\mathbf{e}^a) + (\rho_0/\gamma_t)D_a\zeta^a = 0, \tag{6.8}$$

where $\zeta_h = \sqrt{g}\zeta_n/\gamma_n$. We will use Eqs. (6.7) and (6.8) in our renormalization calculations.

The noiseless linearized Rouse model equations are

$$\partial_t\rho - \frac{\rho_0^2\chi_0^{-1}}{\gamma_t}\nabla^2\delta\rho = 0, \tag{6.9}$$

$$\partial_t h + \frac{1}{\gamma_n}(\kappa\nabla^4 - \sigma\nabla^2)h = 0. \tag{6.10}$$

Note that ρ and h are completely decoupled. There are, therefore, independent height (capillary) and density modes with dispersion

$$\omega_\rho = -i(\rho^2\chi_0^{-1}/\gamma_t)q^2,\qquad \omega_h = -i(\sigma q^2 + \kappa q^4)/\gamma_n. \tag{6.11}$$

The density mode is identical to that of a fluid membrane on a rigid substrate [42].

B. Noise

Noise fluctuations should be chosen so that the membrane will be driven towards thermodynamic equilibrium at long times. Normally a white noise spectrum with zero mean (provided dissipative coefficients are frequency independent) will accomplish this purpose. Our system, as we have seen, has complexities not encountered in most models. The equilibrium partition function is not $Z = \int\mathcal{D}[h]\mathcal{D}[\rho]\exp[-\mathcal{H}/T]$ but $Z = \int\mathcal{D}[h]\mathcal{D}[\rho]\exp[-(\mathcal{H}+\mathcal{H}_M)/T]$, where $\mathcal{H}_M$ is the Liouville-Fadeev-Popov Hamiltonian of Eq. (4.4). Thus, whereas dynamical forces in our dynamical equations are determined by the physical Hamiltonian $\mathcal{H}$, equilibrium statistical weights are determined by the effective Hamiltonian $\mathcal{H}+\mathcal{H}_M$. If we insisted that the average noise $\langle\boldsymbol{\zeta}\rangle$ be zero for every configuration of $\mathcal{H}$, then Eq. (63) would decay to a distribution with weight $e^{-\mathcal{H}/T}$. In Sec. IX, we will show that in order to ensure that the correct equilibrium distribution $e^{-(\mathcal{H}+\mathcal{H}_M)/T}$ is reached at long times, it is necessary to choose

$$\langle\zeta_h\rangle = -\sqrt{g}\frac{\delta\mathcal{H}_M}{\delta h(\mathbf{x})}, \tag{6.12}$$

$\langle\zeta_i\rangle = 0$, $i = x, y$, and

$$\langle\delta\zeta_i(\mathbf{x},t)\delta\zeta_j(\mathbf{x}',t')\rangle = 2T\gamma_{ij}\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g}}\delta(t-t'), \tag{6.13}$$

where $\gamma_{ij} = \gamma_n n_i n_j + \gamma_t e_i^a e_{aj}$. The existence of a nonvanishing average noise can be given further justification. In stochastic models such as we are considering, noise arises from degrees of freedom with wave number greater than the cutoff Λ. Normally, these modes are symmetrically distributed (e.g., in a spherical shell with $\Lambda < q < c\Lambda$ with $c > 1$) and contribute no average force. In our system, however, the number of high-wave-number degrees of freedom depends on membrane configuration [as can be seen from $\mathcal{H}_M$ in Eq. (4.4)] and becomes asymmetric if $\boldsymbol{\nabla}h$ is not spatially uniform, and we should in fact expect there to be forces proportional to $\nabla^2 h$ arising from degrees of freedom with wave number greater than our cutoff.

C. Dynamic renormalization

We can now use standard procedures to study how dynamical quantities renormalize under removal of high-wave-number degrees of freedom. We follow the field-theoretic version of the Martin-Siggia-Rose formalism [43,44] and introduce a generating function enforcing $\sqrt{g}F_\rho = 0$ and $F_h = 0$:

$$G = \left\langle\int D[\sqrt{g}F_\rho]D[F_h]\prod_{\mathbf{x}}\delta(\sqrt{g}F_\rho)\delta(F_h)\right\rangle, \tag{6.14}$$

where the average is over fluctuations of the noise. Introducing integral representations for the two δ functions and transforming from integrals over $\sqrt{g}F_\rho$ and F_h we obtain

$$G = \int D[\hat\rho]D[\hat h]D[\delta\rho]D[h]\exp\left(-\int dt\int d^2\mathbf{x}\mathcal{L}\right), \tag{6.15}$$

where

$$\mathcal{L} = \hat\rho\sqrt{g}[D_t\rho - (\rho_0^2\chi_0^{-1}/\gamma_t)D^2\delta\rho - \rho_0 D_a(\partial_t\mathbf{R}\cdot\mathbf{e}^a)] - \sqrt{g}(\rho_0^2T/\gamma_t)D_a\hat\rho D^a\hat\rho$$
$$+\hat h[\partial_t h + \sqrt{g}(\kappa Q - \sigma H + \rho_0\chi_0^{-1}H\delta\rho + \delta\mathcal{H}_M/\delta h)/\gamma_n] - T\sqrt{g}\hat h^2/\gamma_n - \ln J. \tag{6.16}$$

Here J is the Jacobian of the transformation from $\sqrt{g}F_\rho$ and F_h to $\delta\rho$ and h. It does not contribute to the one-loop calculations we present here. The action $\mathcal{L}$ can be divided into a harmonic part:

$$\mathcal{L}_0 = \chi_0\hat\rho\partial_t\delta\rho - (\rho_0^2/\gamma_t)\hat\rho\nabla^2\delta\rho - T(\rho_0^2/\gamma_t)(\nabla\hat\rho)^2 - \rho_0\partial_t\hat\rho + \hat h\partial_t h + \hat h(\kappa\nabla^4 - \sigma\nabla^2)h/\gamma_n - T\hat h^2/\gamma_n \tag{6.17}$$

and a nonlinear part

$$\mathcal{L}_v = -(\rho_0 + \chi_0\delta\rho)\partial_t\hat\rho(\nabla h)^2/2 - (\rho_0^2/\gamma_t)\partial_a\hat\rho\partial_b h(\partial_b\delta\rho\partial_a h - \partial_a\delta\rho\partial_b h/2)$$
$$+\rho_0\nabla\hat\rho\cdot\nabla h\partial_t h + T(\rho_0^2/\gamma_t)\partial_a\hat\rho\partial_b h(\partial_b\hat\rho\partial_a h - \partial_a\hat\rho\partial_b h/2)$$
$$+(\kappa/\gamma_n)\hat h[\nabla^2 h\partial_a\partial_b h\partial_a\partial_b h - \partial_a h\partial_b h(\partial_a\partial_b\nabla^2 h - \delta_{ab}\nabla^4 h/2) + (\kappa/2\gamma_n)\partial_a h[\nabla^2\partial_a\hat h(\nabla h)^2 + \partial_a\hat h(\nabla^2 h)^2]$$
$$-(\sigma/2\gamma_n)(\nabla h)^2(\partial_a\hat h\partial_a h + \hat h\nabla^2 h) + (\rho_0/\gamma_n)\hat h\delta\rho\nabla^2 h - T(\nabla h)^2\hat h^2/2\gamma_n - \mu_M\hat h\nabla^2 h/\gamma_n. \tag{6.18}$$

Recall that μ_M [Eq. (4.4)] is expressed as an integral over q of the form $\int_0^\Lambda$, which can be broken up into an integral over large and small q: $\int_0^{\Lambda/b} + \int_{\Lambda/b}^\Lambda$. It can thus be treated in much the same way as measure factors are treated in the nonlinear σ model [45]. We can now use standard diagrammatic perturbation theory to calculate corrections to the various vertices appearing in $\mathcal{L}_0$ resulting from the removal of high-wave-number degrees of freedom in h. The one-loop diagrams contributing to these vertices are shown in Fig. 3. Dotted lines represent the response function

$$G_{\hat h h} = \frac{1}{-i\omega + g(k)} \tag{6.19}$$

and unbroken lines represent the correlation function

$$G_{hh} = \frac{2}{\gamma_n}\frac{T}{\omega^2 + g^2(k)}, \tag{6.20}$$

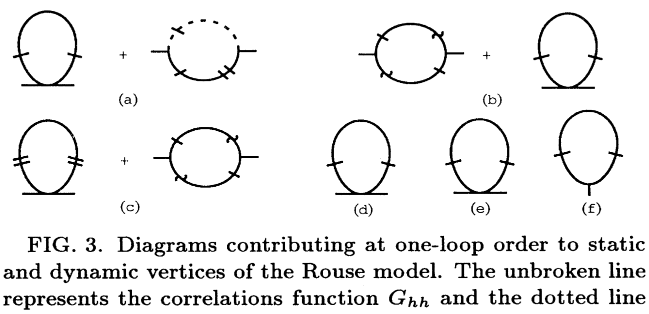

FIG. 3. Diagrams contributing at one-loop order to static and dynamic vertices of the Rouse model. The unbroken line represents the correlations function G_{hh} and the dotted line the response function $G_{\hat h h}$. Each straight cross line represents a spatial gradient, and each curved cross line represents a time derivative. The diagrams contribute to the following vertices: (a) $\hat\rho\partial_t\rho$ or χ, (b) $\nabla\hat\rho\cdot\nabla\hat\rho$ or ρ_0^2/γ_t, (c) $\hat h\nabla^2 h$ or σ/γ_n, (d) $\hat h\nabla^4 h$ or κ/γ_n, (e) $\hat h\hat h$ or $1/\gamma_n$, and (f) $\partial_t\hat\rho$ or ρ_0.

where

$$g(k) = (\kappa k^4 + \sigma k^2)/\gamma_n. \tag{6.21}$$

Corrections to χ are obtained from the $\hat\rho$-$\partial_t\delta\rho$ vertex shown in Fig. 3(a):

$$\Delta\chi = \frac{1}{2}\chi_0\int\frac{d\omega}{2\pi}\int^> k^2 G_{hh}(k,\omega) + \frac{\rho_0^2}{\gamma_n}\int\frac{d\omega}{2\pi}\int^> k^4 G_{hh}G_{\hat h h} = \Delta_1\chi_0 + (\rho_0^2/\Lambda^2)\Delta_2, \tag{6.22}$$

where Δ_1, Δ_2 are defined in Eqs. (4.10) and (4.11), and $\int^> = \int_{\Lambda/b}^\Lambda d^2k/(2\pi)^2$. Corrections to ρ_0^2/γ_t can be obtained either from the $\nabla^2\hat\rho$-$\delta\rho$ or the $\nabla\hat\rho$-$\nabla\hat\rho$ vertex [Fig. 3(b)]:

$$\Delta(\rho_0^2/\gamma_t) = \frac{\rho_0^2}{2T}\int\frac{d\omega}{2\pi}\int^> \omega^2k^2G_{hh}G_{hh} = (\rho_0^2/\gamma_n)\Delta_1. \tag{6.23}$$

Corrections to κ/γ_n, σ/γ_n, and γ_n^{-1} are obtained, respectively from the $\hat h$-$\nabla^4 h$ [Fig. 3(d)], the $\hat h$-$\nabla^2 h$ [Fig. 3(c)], and $\hat h$-$\hat h$ [Fig. 3(e)] vertices:

$$\Delta(\kappa/\gamma_n) = -(\kappa/\gamma_n)\int\frac{d\omega}{2\pi}\int^> k^2G_{hh} = -2\kappa/\gamma_n\Delta_1, \tag{6.24}$$

$$\Delta(\sigma/\gamma_n) = -(\kappa/\gamma_n)\int\frac{d\omega}{2\pi}\int^>(3k^4/2)G_{hh} - (\sigma/\gamma_n)\int\frac{d\omega}{2\pi}\int^> k^2G_{hh} + \mu_M = \epsilon(T\Lambda^2/4\pi)\log(\kappa\Lambda^2+\sigma)\lambda^2/T, \tag{6.25}$$

$$\Delta\gamma_n^{-1} = (1/\gamma_n)\int \frac{d\omega}{2\pi}\int^{>} (k^2/2)G_{hh} = \Delta_1/\gamma_n. \tag{6.26}$$

Finally, the equilibrium in-plane density ρ_0 changes under coarse graining. Its shift can be obtained from the vertex linear in $\partial_t\hat{\rho}$ [Fig. 3(f)]:

$$\Delta\rho_0 = \int \frac{d\omega}{2\pi}\int^{>} (k^2/2)G_{hh} = \rho_0\Delta_1. \tag{6.27}$$

The resulting coarse-grained generating function is

$$G = \int D[\hat{\rho}]D[\hat{h}^<]D[\delta\rho]D[h^<]\exp\left(-\int dt\int d^2\mathbf{x}\mathcal{L}^<\right) \tag{6.28}$$

with

$$\begin{aligned}\mathcal{L}^< = &-\partial_t\hat{\rho}(\rho_0' + \chi'\delta\rho) - (\rho'^2/\gamma_t')\hat{\rho}\nabla^2\delta\rho - T(\rho'^2/\gamma_t')(\nabla\hat{\rho})^2 \\ &+\hat{h}\partial_t h^< + \hat{h}(\kappa'\nabla^4 - \sigma'\nabla^2)h^</\gamma_n' - T\hat{h}^2/\gamma_n' + \cdots,\end{aligned} \tag{6.29}$$

where

$$\kappa' = \kappa(1 - 3\Delta_1), \tag{6.30}$$

$$\sigma' = \sigma\left[1 + \frac{\epsilon}{4\pi}T\Lambda^2\ln(\kappa\Lambda^2 + \sigma)\frac{\lambda^2}{T}\right], \tag{6.31}$$

$$\chi' = \chi_0\left(1 + \Delta_1 + \frac{\rho_0^2}{\Lambda^2}\Delta_2\chi_0^{-1}\right), \tag{6.32}$$

$$\gamma_n' = \gamma_n(1 - \Delta_1), \tag{6.33}$$

$$\gamma_t' = \gamma_t\left[1 + \Delta_1\left(2 - \frac{\gamma_t}{\gamma_n}\right)\right], \tag{6.34}$$

$$\rho_0' = \rho_0(1 + \Delta_1). \tag{6.35}$$

The equations for κ', σ', and χ' agree with those calculated statically in Eqs. (4.12), (4.13), and (4.16). The equation for ρ_0' is a consequence of mass conservation.

VII. THE ZIMM MODEL

Membrane motion will in general excite modes in the solvent in which it is embedded. The Rousse model treats the solvent as rigid and ignores these modes. In this section, we will consider the membrane generalization of the Zimm model familiar from polymer dynamics [16]. In this model, modes excited in the incompressible solvent lead to an effective long-range hydrodynamic interaction between points on the membrane. To keep our discussion simple, we will ignore density and consider only height fluctuations, though, as we have seen, the membrane is compressible at long wavelength.

We are interested in the long-wavelength, low-frequency dynamics of the membrane. We can therefore ignore inertial and nonlinear terms in Eq. (5.18). We can also ignore $\mathbf{f}_{\text{dis}}$ relative to $\mathbf{f}_{\text{m}}$. We can thus obtain the force equation

$$-[v_i(\mathbf{R}) - V_i(\mathbf{R})] = -\gamma_{ij}^{-1}f_{sj} \tag{7.1}$$

from Eqs. (5.18) and (5.22), where $\mathbf{f}_s$ is given by Eqs. (3.9) or (3.14). The membrane exerts a force on the embedding fluid, and the equation for the fluid velocity is

$$\partial_t\mathbf{V}(\mathbf{r}) - \frac{\eta}{\rho_s}\nabla^2\mathbf{V}(\mathbf{r}) = -\frac{1}{\rho_s}\boldsymbol{\nabla}p + \frac{1}{\rho_s}\int d^2u\sqrt{g}\delta(\mathbf{r} - \mathbf{R})\mathbf{f}_s(\mathbf{R}(\tilde{u})), \tag{7.2}$$

where ρ_s is the solvent mass density, and $\boldsymbol{\nabla}$ is a gradient in 3D Euclidean space. The solvent is incompressible so that Eq. (7.2) must be supplemented with

$$\boldsymbol{\nabla}\cdot\mathbf{V} = 0. \tag{7.3}$$

Using Eq. (7.2), we can solve for $\mathbf{V}$ in terms of $\mathbf{f}_s$. In Fourier space,

$$V_i(\mathbf{q},\omega) = G_{ij}(\mathbf{q},\omega)\int d^2u\sqrt{g}e^{i\mathbf{q}\cdot\mathbf{R}(\tilde{u})}f_{sj}(\mathbf{R}(\tilde{u})), \tag{7.4}$$

where

$$G_{ij}(\mathbf{q},\omega) = (\delta_{ij} - \hat{q}_i\hat{q}_j)\frac{1}{-i\omega\rho_s + \eta q^2}, \tag{7.5}$$

where $\hat{\mathbf{q}} = \mathbf{q}/q$. In the low-frequency limit, we can set $\omega = 0$ in this equation so that

$$V_i(\mathbf{R}) = \int d^2u'\sqrt{g(u')}O_{ij}(\mathbf{S}(\tilde{u},\tilde{u}'))f_{sj}(\mathbf{R}(\tilde{u}')), \tag{7.6}$$

where $\mathbf{S}(\tilde{u},\tilde{u}') = \mathbf{R}(\tilde{u}) - \mathbf{R}(\tilde{u}')$ and

$$O_{ij}(\mathbf{S}) = \frac{1}{8\pi\eta|\mathbf{S}|}\left(\delta_{ij} + \frac{S_iS_j}{S^2}\right) \tag{7.7}$$

is the Oseen tensor [16].

Equation (7.4) expresses $\mathbf{V}$ as a function of $\mathbf{f}_s$, which in turn depends only on membrane parameters. We can thus use Eq. (7.1) to obtain an equation for the membrane velocity in terms of membrane coordinates alone. In particular, the velocity normal to the membrane is

$$\begin{aligned}\partial_t\mathbf{R}\cdot\mathbf{n} &= \mathbf{V}\cdot\mathbf{n} \\ &= \int d^2u'\sqrt{g(u')}n_i(\tilde{u})O_{ij}(\mathbf{S})f_{sj}(\mathbf{R}(\tilde{u}')) \\ &\quad + n_i\gamma_{ij}^{-1}f_j.\end{aligned} \tag{7.8}$$

The second term on the right-hand side of this equation is unimportant compared to the first and can be ignored. Furthermore,

$$\mathbf{f}_s = -\frac{\partial\mathcal{H}}{\partial h}\mathbf{n} \tag{7.9}$$

in the Monge gauge when the density is constant. Thus, the equation of motion for the height in the Monge gauge is

$$\partial_t h(\mathbf{x}) = -\int d^2x'\Gamma(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}}{\delta h(\mathbf{x}')} + \zeta_h, \tag{7.10}$$

where

$$\Gamma(\mathbf{x},\mathbf{x}') = \sqrt{g(\mathbf{x})}n_i(\mathbf{x})O_{ij}(\mathbf{S})\sqrt{g(\mathbf{x}')}n_j(\mathbf{x}') = O_{ab}\partial_a h\partial_b' h - O_{az}\partial_a h - O_{za}\partial_a' h + O_{zz}. \quad (7.11)$$

As in the Rouse model, the noise must be chosen so that the equilibrium probability distribution $e^{-(\mathcal{H}+\mathcal{H}_M)/T}$ is approached at long times. To this end, we set $\zeta_h = \langle\zeta_h\rangle + \delta\zeta_h$ with

$$\langle\zeta_h\rangle = -\int d^2x'\Gamma(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}_M}{\delta h(\mathbf{x}')} + T\int d^2x\frac{\delta\Gamma(\mathbf{x},\mathbf{x}')}{\delta h(\mathbf{x}')} \quad (7.12)$$

and

$$\langle\delta\zeta_h(\mathbf{x},t)\delta\zeta_h(\mathbf{x}',t)\rangle = 2T\Gamma(\mathbf{x},\mathbf{x}')\delta(t-t'). \quad (7.13)$$

The second term in Eq. (7.12) arises because $\Gamma(\mathbf{x},\mathbf{x}')$ depends on $h(\mathbf{x})$. We will discuss this point more thoroughly in Sec. IX. Equations (7.10)–(7.13) now define a complete stochastic dynamical theory, whose harmonic and nonlinear properties we can study.

To line order in h, Eq. (7.10) reduces to

$$\partial_t h = -\int d^2x\frac{1}{8\pi\eta|\mathbf{x}-\mathbf{x}'|}(\kappa\nabla'^4 - \sigma\nabla'^2)h(\mathbf{x}') \quad (7.14)$$

when $\zeta_h = 0$. This leads to the familiar [14] dispersion law

$$\omega(\mathbf{q}) = -\frac{i}{4\eta q}(\kappa q^4 + \sigma q^2), \quad (7.15)$$

which produces $\omega(\mathbf{q}) \sim -iq^3$ when $\sigma = 0$.

To study static and dynamic renormalization, we again construct a generating function $Z = \int \mathcal{D}\hat{h}\mathcal{D}h\exp(-\int dt\mathcal{L})$, where

$$\mathcal{L} = \int d^2x\hat{h}\partial_t h + \int d^2x d^2x'\hat{h}(\mathbf{x})\Gamma(\mathbf{x},\mathbf{x}')\left[\frac{\delta(\mathcal{H}+\mathcal{H}_M)}{\delta h(\mathbf{x}')} - T\hat{h}(\mathbf{x}')\right], \quad (7.16)$$

where we have ignored the $\delta\Gamma(\mathbf{x},\mathbf{x}')/\delta h(\mathbf{x})$ contribution to the noise because it does not affect our one-loop calculations. The action $\mathcal{L}$ can be decomposed into harmonic and nonlinear parts, $\mathcal{L} = \mathcal{L}_{\rm har} + \mathcal{L}_{\rm nl}$, with

$$\mathcal{L}_{\rm har} = \int d^2x\left\{\hat{h}\partial_t h + \hat{h}\int d^2x'\frac{1}{8\pi\eta|\mathbf{x}-\mathbf{x}'|}[(\kappa\nabla'^4 h' - \sigma\nabla'^2 h' - T\hat{h}']\right\} \quad (7.17)$$

and

$$\begin{aligned}\mathcal{L}_{\rm nl} = \int d^2x\int d^2x'\frac{1}{8\pi\eta|\mathbf{x}-\mathbf{x}'|}\Bigg\{&\kappa\hat{h}'[\nabla^2 h\partial_a\partial_b h\partial_a\partial_b h - \partial_a h\partial_b h(\partial_a\partial_b\nabla^2 h - \delta_{ab}\nabla^4 h/2)]\\ &+(\kappa/2)\partial_a h[\nabla'^2\partial_a'\hat{h}'(\nabla h)^2 + \partial_a'\hat{h}'(\nabla^2 h)^2] - (\sigma/2)(\nabla h)^2(\partial_a'\hat{h}'\partial_a h + \hat{h}'\nabla^2 h)\\ &+\hat{h}'(\kappa\nabla^4 h + \sigma\nabla^2 h - T\hat{h})\Bigg(\partial_a h\partial_a' h' + \frac{(x-x')_a(x-x')_b\partial_a h\partial_b' h'}{|\mathbf{x}-\mathbf{x}'|^2}\\ &-\frac{(x-x')_a\partial_a' h'(h-h')}{|\mathbf{x}-\mathbf{x}'|^2} - \frac{(x-x')_a\partial_a h(h-h')}{|\mathbf{x}-\mathbf{x}'|^2} + \frac{(h-h')^2}{2|\mathbf{x}-\mathbf{x}'|^2}\Bigg)\Bigg\}.\end{aligned} \quad (7.18)$$

Following the same procedures as for the Rouse model, we find that the static coefficients κ and σ are renormalized exactly as in the Rouse model or in static calculations. The dynamical dissipative coefficient η^{-1}, however, is not renormalized because $\Gamma(\mathbf{q})$ is nonanalytic in $\mathbf{q}$ and there can be no diagrammatic corrections to it.

VIII. POISSON BRACKETS

In Sec. IV, we derived the hydrodynamical equations for fluid membranes using standard approaches based on conservation laws and Newton's laws. In this section, we will recast these equations in terms of Poisson brackets. This will allow us to show that the dynamical probability distribution will decay to equilibrium at long times in the presence of appropriately chosen noise sources. As we have already seen, the physical properties of fluid membranes are gauge invariant implying that the membrane Lagrangian must also be. As a result, we can expect to encounter the same difficulties in specifying canonical momenta and Poisson brackets that one encounters in quantizing the electromagnetic field [12]. Though our final equations are gauge invariant, we will need to specify a gauge to derive our Poisson bracket relations. Here we will use the Monge gauge.

Implementing the constraint that particles remain within the membrane as they move and as the shape of the membrane changes presents additional difficulties. It is of course interactions among membrane particles and between membrane particles and their solvent that causes the membrane to form. One could in principle describe membrane dynamics in terms of the Lagrangian of all membrane and solvent particles. In this approach, however, the existence of the membrane is totally obscured. We, therefore, need to find a formulation of the Lagrangian and a choice of independent variables in which the constraint that membrane particles always remain in the membrane is satisfied. Our approach will be to choose the particle coordinates $\tilde{u}_\alpha$ in the parameter manifold as canonical coordinates *and* the single independent components of the position vector field $\mathbf{R}(\tilde{u},t)$ as a canonical field. Thus, in the Monge gauge, we will construct a Lagrangian in terms of $\mathbf{x}_\alpha$, $\dot{\mathbf{x}}_\alpha$, $h(\mathbf{x},t)$, and $\partial_t h(\mathbf{x},t)$.

The construction of the Lagrangian begins with the kinetic energy. The position of particle α in Euclidean space is $\mathbf{R}_\alpha(t) = \mathbf{R}(\tilde{u}_\alpha,t)$, and its velocity is $\dot{\mathbf{R}}_\alpha = \dot{u}^a_\alpha \mathbf{e}_{\alpha,a} + \partial_t \mathbf{R}_\alpha$ where $\mathbf{e}_{\alpha,a} = \partial_a \mathbf{R}(\tilde{u}_\alpha,t)$ is the covariant tangent-plane vector at the position of particle α. The velocity $\dot{\mathbf{R}}_\alpha$ is gauge invariant. The particles interact via a potential $V[\{\mathbf{R}_\alpha\}]$ that depends only on their Euclidean space coordinates. The Lagrangian is thus

$$\mathcal{L} = \tfrac{1}{2}m\sum_\alpha [g_{\alpha,ab}(\dot{u}^a_\alpha + \mathbf{e}^a_\alpha\cdot\partial_t\mathbf{R}_\alpha)(\dot{u}^b_\alpha + \mathbf{e}^b_\alpha\cdot\partial_t\mathbf{R}_\alpha) + (\mathbf{n}_\alpha\cdot\partial_t\mathbf{R}_\alpha)^2] - V[\{\mathbf{R}_\alpha\}], \tag{8.1}$$

where $g_{\alpha,ab} = \mathbf{e}_{\alpha,a}\cdot\mathbf{e}_{\alpha,b}$. Since $\mathcal{L}$ is constructed from gauge invariant quantities, it is gauge invariant as required.

We now specialize to the Monge gauge for which $\mathbf{R}_\alpha = (\mathbf{x}_\alpha, h(\mathbf{x}_\alpha,t))$. As discussed above, we regard $\mathbf{x}_\alpha$ and the field $h(\mathbf{x},t)$ as our canonical coordinates and express $\mathcal{L}$ as

$$\mathcal{L} = \tfrac{1}{2}m\sum_\alpha[\dot{\mathbf{x}}^2_\alpha + (\partial_t h_\alpha + \nabla h_\alpha\cdot\dot{\mathbf{x}}_\alpha)^2] - V. \tag{8.2}$$

The momentum conjugate to $\mathbf{x}_\alpha$ and $h(\mathbf{x},t)$ are then

$$\mathbf{p}_\alpha \equiv \frac{\delta\mathcal{L}}{\delta\dot{\mathbf{x}}_\alpha} \tag{8.3}$$

$$= m\dot{\mathbf{x}}_\alpha + m[\partial_t h(\mathbf{x}_\alpha) + \nabla_\alpha h(\mathbf{x}_\alpha)\cdot\dot{\mathbf{x}}_\alpha]\nabla_\alpha h(\mathbf{x}_\alpha) \tag{8.4}$$

and

$$j_h(\mathbf{x}) \equiv \frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta(\partial_t h(\mathbf{x}))} \tag{8.5}$$

$$= \frac{m}{\sqrt{g}}\sum_\alpha(\partial_t h_\alpha + \nabla h\cdot\dot{\mathbf{x}}_\alpha)\delta(\mathbf{x}-\mathbf{x}_\alpha),$$

where we used $\delta\partial_t h(\mathbf{x}_\alpha,t)/\delta\partial_t h(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{x}_\alpha)$. The fundamental Poisson bracket relations are

$$\{p_{\alpha,a}, x_{\beta,b}\} = \delta_{\alpha\beta}\delta_{ab}, \tag{8.6}$$

$$\{j_h(\mathbf{x}), h(\mathbf{x}')\} = \delta(\mathbf{x}-\mathbf{x}')/\sqrt{g}. \tag{8.7}$$

The microscopic Hamiltonian associated with $\mathcal{L}$ is

$$\mathcal{H} = -\mathcal{L} + \sum_\alpha \mathbf{p}_\alpha\cdot\mathbf{x}_\alpha + \int d^2x\sqrt{g}j_h(\mathbf{x})\partial_t h(\mathbf{x})$$

$$= \tfrac{1}{2}m\sum_\alpha[\dot{\mathbf{x}}^2_\alpha + (\nabla h_\alpha\cdot\dot{\mathbf{x}}_\alpha + \partial_t h_\alpha)^2] + V. \tag{8.8}$$

This Hamiltonian is not expressed, as it should in principle be, as a function of the canonical momentum $\mathbf{p}_\alpha$ and $j_h(\mathbf{x})$. This is because we cannot directly solve for $\partial_t h_\alpha$ in terms of $j_h(\mathbf{x})$. We are, however, interested in a coarse grained Hamiltonian, which is a function of coarse-grained momentum densities, rather than a particle Hamiltonian expressed as a function of particle momenta. The Hamiltonian in Eq. (8.8) can be coarse-grained in the standard way. We introduce the momentum density

$$\mathbf{j}(\mathbf{x},t) = \frac{m}{\sqrt{g}}\sum_\alpha\dot{\mathbf{x}}_\alpha\delta(\mathbf{x}-\mathbf{x}_\alpha) = \mathbf{P}(\mathbf{x}) - \nabla h(\mathbf{x},t)j_h(\mathbf{x},t), \tag{8.9}$$

where

$$\mathbf{P}(\mathbf{x}) = \frac{1}{\sqrt{g}}\sum_\alpha\mathbf{p}_\alpha\delta(\mathbf{x}-\mathbf{x}_\alpha). \tag{8.10}$$

$\mathbf{P}(\mathbf{x})$ and $j_h(\mathbf{x})$ are the canonical momentum densities, whereas $\mathbf{j}(\mathbf{x},t)$ is the component of the physical momentum density [Eq. (5.10)] in the xy plane. To keep our notation compact, we will treat $\mathbf{j} = (j_x, j_y, 0)$ as a vector in three-dimensional Euclidean space with components only in the xy plane. Also $j_x = j^1$ and $j_y = j^2$ are the two components of the momentum density j^a in the Monge gauge. The canonical momentum j_h satisfies

$$j_h(\mathbf{x}) = \rho(\mathbf{x})\partial_t h(\mathbf{x}) + \nabla h\cdot\mathbf{j}, \tag{8.11}$$

as can be seen directly from Eq. (8.5). Thus, j_h is the component of the physical momentum density along $\mathbf{n}_0 = \mathbf{e}_z$ as can be seen by taking the z component of Eq. (5.10). Thus the physical momentum density is

$$\mathbf{J}(\mathbf{x}) = \mathbf{j}(\mathbf{x}) + j_h(\mathbf{x})\mathbf{e}_z. \tag{8.12}$$

The coarse-grained kinetic energy part of the Hamiltonian is

$$\mathcal{H}_K = \tfrac{1}{2}\int d^2x\sqrt{g}\frac{1}{\rho(\mathbf{x})}(\mathbf{j}\cdot\mathbf{j} + j_h^2)$$

$$= \tfrac{1}{2}\int d^2x\sqrt{g}\frac{1}{\rho(\mathbf{x})}\mathbf{J}\cdot\mathbf{J}. \tag{8.13}$$

Though this expression was derived in the Monge gauge, it is independent of gauge since $\mathbf{J}$ is. Finally, the coarse-grained potential energy is $\mathcal{H}$ [Eq. (4.3)], and the total coarse-grained Hamiltonian is

$$\mathcal{H}_T = \mathcal{H}_K + \mathcal{H}. \tag{8.14}$$

The equilibrium partition function is

$$Z_T = \int\mathcal{D}[\mathbf{j}]\mathcal{D}[j_h]\mathcal{D}[\rho]\mathcal{D}[h]\mathcal{P}_{\rm eq}, \tag{8.15}$$

where

$$\mathcal{P}_{\rm eq} = e^{-(\mathcal{H}_T+\mathcal{H}_M)/T} \tag{8.16}$$

is the equilibrium probability distribution including measure factors.

Poisson-bracket relations among the fields $\mathbf{j}$, j_h, h, and ρ can be calculated with the aid of Eqs. (8.5), (8.6), (8.7), and (8.9). The results are

$$\{\mathbf{j}(\mathbf{x}),\rho(\mathbf{x}')\} = \rho(\mathbf{x})\nabla\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right) + \nabla h(\mathbf{x})\rho(\mathbf{x}')\frac{\nabla' h}{g(\mathbf{x}')}\cdot\nabla'\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x}')}}\right), \tag{8.17}$$

$$\{j_a(\mathbf{x}),j_b(\mathbf{x}')\} = j_b(\mathbf{x})\partial_a\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right) - j_a(\mathbf{x})\partial'_b h\frac{\nabla h}{g(\mathbf{x})}\cdot\nabla\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right)$$
$$+j_h(\mathbf{x})(\partial_b h - \partial'_b h)\partial_a\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right) - (a\leftrightarrow b,\mathbf{x}\leftrightarrow\mathbf{x}'), \tag{8.18}$$

$$\{\mathbf{j}(\mathbf{x}),h(\mathbf{x}')\} = -\nabla h\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}, \tag{8.19}$$

$$\{\mathbf{j}(\mathbf{x}),j_h(\mathbf{x}')\} = \mathbf{j}(\mathbf{x})\frac{\nabla h}{g(\mathbf{x})}\cdot\nabla\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right) + j_h(\mathbf{x})\nabla\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right)$$
$$+j_h(\mathbf{x}')\nabla h\frac{\nabla' h}{g(\mathbf{x}')}\cdot\nabla'\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right), \tag{8.20}$$

$$\{j_h(\mathbf{x}),h(\mathbf{x}')\} = \frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}, \tag{8.21}$$

$$\{j_h(\mathbf{x}),\rho(\mathbf{x}')\} = -\rho(\mathbf{x}')\frac{\nabla' h}{g(\mathbf{x}')}\cdot\nabla'\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right), \tag{8.22}$$

$$\{j_h(\mathbf{x}),j_h(\mathbf{x}')\} = j_h(\mathbf{x})\frac{\nabla h}{g(\mathbf{x})}\cdot\nabla\left(\frac{\delta(\mathbf{x}-\mathbf{x}')}{\sqrt{g(\mathbf{x})}}\right) - (\mathbf{x}\leftrightarrow\mathbf{x}'). \tag{8.23}$$

It is a straightforward exercise to show that the Poisson-bracket equations

$$\partial_t h(\mathbf{x}) = -\{h(\mathbf{x}),\mathcal{H}\}, \tag{8.24}$$
$$\partial_t \rho(\mathbf{x}) = -\{\rho(\mathbf{x}),\mathcal{H}\}, \tag{8.25}$$
$$\partial_t \mathbf{j}(\mathbf{x}) = -\{\mathbf{j}(\mathbf{x}),\mathcal{H}\}, \tag{8.26}$$
$$\partial_t j_h(\mathbf{x}) = -\{j_h(\mathbf{x}),\mathcal{H}\}, \tag{8.27}$$

reproduce the nondissipative parts of the hydrodynamical equations in the Monge gauge. For example, Eq. (8.24) is

$$\partial_t h = -\int d^2x'\{h(\mathbf{x}),j_h(\mathbf{x}')\}\frac{\delta\mathcal{H}}{\delta j_h(\mathbf{x}')} - \int d^2x'\{h(\mathbf{x}),j^a(\mathbf{x}')\}\frac{\delta\mathcal{H}}{\delta j^a(\mathbf{x}')}$$
$$= -\rho^{-1}j^a\partial_a h + \rho^{-1}j_h, \tag{8.28}$$

in agreement with Eq. (8.11), and Eq. (8.25) is

$$\partial_t\rho = -\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}j^a) - \rho\frac{\partial_a}{g}\left(\frac{j_h - \partial_b j^b}{\rho}\right), \tag{8.29}$$

which, with the aid of Eq. (8.28), reduces to the mass conservation law of Eq. (5.3).

IX. THE FOKKER-PLANCK EQUATION

As we have indicated many times, noises in stochastic equations must be chosen to guarantee that the probability distribution for observables must decay to the equilibrium distribution $\mathcal{P}_{\rm eq}$ at long times. Membrane dynamics presents two problems not generally encountered in flat-space stochastic models. The first, discussed in Sec. IV, is that there are measure corrections in the partition function trace leading to $\mathcal{P}_{\rm eq} \sim e^{-(\mathcal{H}+\mathcal{H}_M)/T}$ rather than $\mathcal{P}_{\rm eq} \sim e^{-\mathcal{H}/T}$. The second is that the effective dissipative coefficients as well as Poisson brackets are nonlinear functions of fields in any given gauge. In this section, we will generalize standard derivations of the Fokker-Planck equation to include these complications for arbitrary field $\phi_\alpha(\mathbf{x})$. We will then show in particular that the Rouse model studied in Sec. VI does decay to equilibrium.

Let the general equations of motion for arbitrary field $\phi_\alpha(\mathbf{x})$ [which could be $h(\mathbf{x})$, $\mathbf{j}(\mathbf{x})$, etc.] be

$$\partial_t\phi_\alpha(\mathbf{x},t) = -\sum_\beta\int d^2x' Q_{\alpha\beta}(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}}{\delta\phi_\beta(\mathbf{x}')} - \sum_\beta\int d^2x'\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}}{\delta\phi_\beta(\mathbf{x}')} + \zeta_\alpha(\mathbf{x}), \tag{9.1}$$

where both the Poisson bracket $Q_{\alpha\beta}(\mathbf{x},\mathbf{x}') = \{\phi_\alpha(\mathbf{x}),\phi_\beta(\mathbf{x}')\}$ and the dissipative coefficient $\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}')$ can depend on the field $\phi_\alpha(\mathbf{x})$. In addition, $\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}')$ can either be a local operator [i.e., proportional to $\delta(\mathbf{x}-\mathbf{x}')$], or it can be a nonlocal operator, involving gradients as in the Zimm model. Note that it is the physical Hamiltonian $\mathcal{H}$, uncorrected by measure terms that appears in Eq. (9.1). We assume that the noise can have a nonvanishing expectation value $\langle\zeta_\alpha(\mathbf{x},t)\rangle$ with fluctuations $\delta\zeta_\alpha(\mathbf{x},t) = \zeta_\alpha(\mathbf{x},t) - \langle\zeta_\alpha(\mathbf{x},t)\rangle$ characterized by a Gaussian distribution with variance

$$\langle \delta\zeta_\alpha(\mathbf{x},t)\delta\zeta_\beta(\mathbf{x}',t')\rangle = A_{\alpha\beta}(\mathbf{x},\mathbf{x}')\delta(t-t'). \quad (9.2)$$

Both $\langle \zeta_\alpha(\mathbf{x},t)\rangle$ and $A_{\alpha\beta}(\mathbf{x},\mathbf{x}')$ will be chosen so that equilibrium is reached at long times. As discussed in Sec. VI, the rationale for a nonvanishing value of $\langle \zeta_\alpha(\mathbf{x},t)\rangle$ is that high-wave-number degrees of freedom may in fact exert a nonvanishing force in spatially inhomogeneous, nonequilibrium situations.

We now follow standard procedures to calculate the time derivative of the probability,

$$\mathcal{P}(\{\phi_\alpha(\mathbf{x})\},t) = \left\langle \prod_{\mathbf{x}}\prod_\alpha \delta(\phi_\alpha(\mathbf{x}) - \phi_\alpha(\mathbf{x},t)) \right\rangle_{\{\phi^0_\alpha(\mathbf{x})\},t_0}, \quad (9.3)$$

that $\phi_\alpha(\mathbf{x},t)$ has a value $\phi_\alpha(\mathbf{x})$ at time t given that it had a value of $\phi^0_\alpha(\mathbf{x})$ at time $t=t_0$. The average in this equation is over the noise subject to the initial condition $\phi_\alpha(\mathbf{x},t_0) = \phi^0_\alpha(\mathbf{x})$. The result is

$$\partial_t\mathcal{P} = \sum_{\alpha\beta}\int d^2x \frac{\delta}{\delta\phi_\alpha(\mathbf{x})}\left(\int d^2x'[Q_{\alpha\beta}(\mathbf{x},\mathbf{x}') + \Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}')]\frac{\delta\mathcal{H}}{\delta\phi_\beta(\mathbf{x}')}\mathcal{P} - \langle\zeta_\alpha(\mathbf{x},t)\rangle\mathcal{P}\right)$$
$$+\frac{1}{2}\sum_{\alpha\beta}\int d^2x d^2x' \frac{\delta}{\delta\phi_\alpha(\mathbf{x})}\frac{\delta}{\delta\phi_\beta(\mathbf{x}')}[A_{\alpha\beta}(\mathbf{x},\mathbf{x}')\mathcal{P}]. \quad (9.4)$$

If we choose

$$\langle\zeta_\alpha(\mathbf{x},t)\rangle = -\sum_\beta\int d^dx'[\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}') + Q_{\alpha\beta}(\mathbf{x},\mathbf{x}')]\frac{\delta\mathcal{H}_M}{\delta\phi_\beta(\mathbf{x}')} + T\sum_\beta\int d^2x'\frac{\delta}{\delta\phi_\beta(\mathbf{x}')}[\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}') + Q_{\alpha\beta}(\mathbf{x},\mathbf{x}')] \quad (9.5)$$

and

$$A_{\alpha\beta}(\mathbf{x},\mathbf{x}') = 2T\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}'), \quad (9.6)$$

then $\partial_t\mathcal{P} = 0$ when $\mathcal{P} \sim e^{-(\mathcal{H}+\mathcal{H}_M)/T}$ and $\mathcal{P}$ decays to this form for any initial state. The term proportional to $\delta Q_{\alpha\beta}(\mathbf{x},\mathbf{x}')/\delta\phi_\beta(\mathbf{x}')$ is familiar from flat-space stochastic models with nontrivial Poisson brackets [9–11]. It is normally included in the original stochastic equations with the requirement that $\langle\zeta_\alpha(\mathbf{x},t)\rangle = 0$. The term proportional to $\delta\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}')/\delta\phi_\beta(\mathbf{x}')$ is similar to that encountered in systems with a spatially varying diffusion constant [16]. Here we argue that it really should be taken to arise from the noise.

The complete hydrodynamical equations for a fluctuating membrane can clearly be cast in the form of Eq. (9.1). The form of $\Gamma_{\alpha\beta}(\mathbf{x},\mathbf{x}')$ will depend, of course, on the particular dissipative model we chose. The Rouse model can also be cast in the form of Eq. (9.1). Equation (6.5) for h is already in the form of Eq. (9.1). Equation (6.6) for ρ can be recast in this form by substituting Eq. (6.5) for $\partial_t h$. The result is

$$\partial_t h = -\int d^2x'\Gamma_{hh}(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}}{\delta h(\mathbf{x}')} - \int d^2x'\Gamma_{h\rho}(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}}{\delta\rho(\mathbf{x}')} + \zeta_h(\mathbf{x},t), \quad (9.7)$$

$$\partial_t\rho = -\int d^2x'\Gamma_{\rho\rho}(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}}{\delta\rho(\mathbf{x}')} - \int d^2x'\Gamma_{\rho h}(\mathbf{x},\mathbf{x}')\frac{\delta\mathcal{H}}{\delta h(\mathbf{x}')} + \zeta_\rho(\mathbf{x},t), \quad (9.8)$$

where

$$\Gamma_{hh}(\mathbf{x},\mathbf{x}') = \frac{\sqrt{g}}{\gamma_n}\delta(\mathbf{x}-\mathbf{x}'), \quad (9.9)$$

$$\Gamma_{h\rho}(\mathbf{x},\mathbf{x}') = \Gamma_{\rho h}(\mathbf{x},\mathbf{x}') = \frac{1}{\gamma_n}\partial_a\left(\rho\frac{\partial_a h}{\sqrt{g}}\right)\delta(\mathbf{x}-\mathbf{x}'), \quad (9.10)$$

$$\Gamma_{\rho\rho}(\mathbf{x},\mathbf{x}') = \frac{\gamma_n}{\sqrt{g}}\left[\frac{1}{\gamma_n}\partial_a\left(\rho\frac{\partial_a h}{\sqrt{g}}\right)\right]^2\delta(\mathbf{x}-\mathbf{x}') - \frac{1}{\gamma_t\sqrt{g}}\partial_a\sqrt{g}\rho^2 g^{ab}\partial_b\frac{1}{\sqrt{g}}\delta(\mathbf{x}-\mathbf{x}'). \quad (9.11)$$

Note that $\Gamma_{hh}(\mathbf{x},\mathbf{x}')$ and $\Gamma_{h\rho}(\mathbf{x},\mathbf{x}')$ are local but $\mathbf{x}$-dependent quantities whereas $\Gamma_{\rho\rho}(\mathbf{x},\mathbf{x}')$ is nonlocal. The noise ζ_h is equal to $\sqrt{g}\zeta_n/\gamma_n$, and

$$\zeta_\rho(\mathbf{x},t) = \int d^2x'\Gamma_{\rho h}(\mathbf{x},\mathbf{x}')\zeta_n(\mathbf{x}',t) - \frac{1}{\gamma_t}\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\rho\zeta^a). \quad (9.12)$$

One can show that $\delta\Gamma_{hh}/\delta\rho = \delta\Gamma_{hh}/\delta h = 0$ but that both the h and ρ derivatives of $\Gamma_{h\rho}$ and $\Gamma_{\rho\rho}$ are nonzero. Fortunately, the contributions that these derivatives make to $\langle\zeta_h\rangle$ and $\langle\zeta_\rho\rangle$ do not affect the renormalization equations to lowest order in T, and they were ignored in Sec. VI. The measure Hamiltonian depends on h and not on ρ. Thus, we have $\langle\zeta_h\rangle = -\sqrt{g}\delta\mathcal{H}_M/\delta h$ to lowest order in T in agreement with Eq. (6.12) . In addition, the variance $\langle\delta\zeta_\alpha\delta\zeta_\beta\rangle$ satisfies Eq. (9.6) with α and β running over h and ρ and $\Gamma_{\alpha\beta}$ given by Eqs. (9.9)–(9.11).

X. SUMMARY

In this paper, we have developed a formalism for treating hydrodynamics and dynamic fluctuations of deformable membranes. We derived gauge invariant hydrodynamical equations describing both tangent-plane motion and shape changes. We showed that noise sources can be chosen so that the probability distribution for hydrodynamic variables decays at long times to the equilibrium distribution with the required measure factor corrections to the usual Boltzmann factor. We were thus able to treat dynamical mode-mode coupling and renormalization using the same techniques that were developed for flat space. We calculated the renormalization of both static and dynamic coefficients under removal of high-wave number shape fluctuations in the Rouse and Zimm models.

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