Crack instabilities of a heated glass strip

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Recently, Yuse and Sano [Nature (London) **362**, 329 (1993)] have observed that a crack traveling in a glass strip submitted to a nonuniform thermal diffusion field undergoes numerous instabilities. We study two cases of quasistatic crack propagation. The crack extension condition in straight propagation is determined. An asymptotic analysis of the elastic free energy is introduced and scaling laws are derived. A linear stability analysis of the straight propagation is performed, based on the assumption that the crack tip propagation deviates from the centered straight one as soon as it is submitted to a "physical" singular shear stress. It is shown that a straight propagation can become unstable after which a wavy instability appears. The condition for instability as well as the selected wavelength is calculated quantitatively. The results are compared with experiments and the agreement is favorable.

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I. INTRODUCTION

The study of crack propagation often follows two approaches. The first is for dynamical fracture formation, where the cracked surfaces are created at a velocity of the order of the Rayleigh wave speed [1]. The second one is for slow or quasiequilibrium cracks. For the second case, the work of Griffith in 1921 [2] is often seen as the beginning of equilibrium fracture mechanics as a quantitative science of material behavior. However, from that time on the progress was mainly made in the fields of engineering. Recently, a renewal of interest has been caused by the work of Yuse and Sano [3]. They have carried out an experiment making reproducible sequences of crack patterns. This is an important step in the understanding of crack instabilities because well-controlled experiments in this field are uncommon.

As shown in Fig. 1, the experiment [3] is performed by pulling a thin glass strip from a hot region (heater) to a cold one (water bath) at a slow and constant velocity V. The control parameters are [3,4] the pulling velocity V, the strip width $2b$, and the temperature variation ΔT between the heater and the cold bath. When these pa-

rameters are small enough the strip does not break. By increasing essentially b or ΔT, a centered straight crack appears and extends at a velocity $-V$ in the frame of the strip. By further increasing these parameters, the straight crack becomes unstable and the fracture follows an oscillating path.

The experiment has been simulated numerically with spring models [5] and by doing a complete numerical resolution of the corresponding thermoelastic problem but with discontinuous incrementation of the crack path [6]. On the other hand, theoretical treatments [7,8] have been undertaken to explain the bifurcation from the straight crack propagation to the wavy one. Although the noncracked-cracked plate transition was not studied quantitatively in [3], the results [6–8] of this bifurcation analysis are quite unanimous and agree with the qualitative measurements in [3]. In fact, the condition of existence of stably advancing straight cracks is based on a criterion of energy minimization [2] that is well understood. On the contrary, the physical origin of the instability straight-undulating crack remains unclear. Even in the asymptotic regions, results of Refs. [6–8] do not agree with each other, although they use the same criterion to explain this transition. Moreover, a comparison of these results with experiment [3] cannot be done rigorously because of the uncertainty in the value of the so-called *fracture energy* [9]. The quantitative experimental study of the no crack–straight crack transition could solve this problem [4].

Much effort has been devoted to the study of the oscillatory instability by using the "*criterion of local symmetry*" [10]. It states that the path taken by a crack in brittle homogeneous isotropic material is the one for which the local stress field at the tip is of mode I type. Let us recall that mode I loading causes an opening of the fracture while mode II loading causes a shearing off. The local analysis in the neighborhood of a crack tip shows that the asymptotic stress tensor field $\overline{\overline{\Sigma}}$, in the polar coordinate system (r,ϕ), takes the universal form [1]
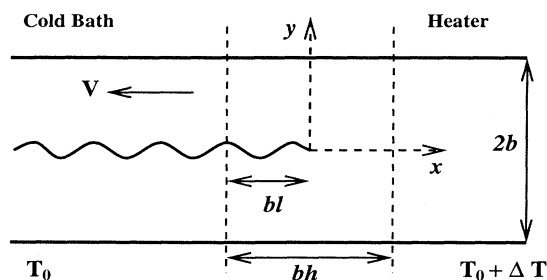

FIG. 1. Schematic representation of the experiment of Yuse and Sano [3].

$$\Sigma_{ij}(r,\phi) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} f_{ij}^{\mathrm{I}}(\phi) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}} f_{ij}^{\mathrm{II}}(\phi), \qquad (1)$$

where $f_{ij}^{\mathrm{I}}(\phi)$ and $f_{ij}^{\mathrm{II}}(\phi)$ are universal functions common to all configurations and loading conditions. The influence of configuration and loading are included in the asymptotic description of stress only through the scalar multipliers K_{I} and K_{II}, which are the elastic stress intensity factors of mode I and mode II loadings, respectively. The criterion of local symmetry features that, if a shear loading exists at the crack tip, $K_{\mathrm{II}} \neq 0$ and the crack will move by changing the orientation of the tangent to the path.

Consider a straight crack subjected to mode I loading. Nominally, $K_{\mathrm{II}} = 0$, but due to the imperfections in the system and, consequently, in crack alignment, K_{II} will differ slightly from zero. The existence of such shear loading implies automatically that the crack deviates from a straight line [10,11]. Moreover, for this finite plate problem, there exists a "geometrical" shear loading that appears as soon as the crack deviates from the center of the strip. The combining of these two loadings leads to the appearance of the oscillatory instability [12]. The undulating crack may be due to an instability of the straight crack, which creates a mode II loading, and the geometry of the problem, which amplifies this shear loading and leads to the oscillatory instability of the crack tip.

These general considerations lead us to investigate the problem in a different manner than [6–8]. To define the wavy instability of the crack, we did not analyze it at the level of the criterion $K_{\mathrm{II}} = 0$, but before it. That is when a slight deviation from the straight crack will create a "physical" shear loading.

II. INSTABILITIES OF A CRACK IN A HEATED STRIP

The experimental configuration we will analyze is illustrated in Fig. 1. The coordinate system (x, y) is defined on an infinitely long strip of thin glass whose boundaries are located at $y = \pm b$. A semi-infinite crack, whose tip is taken as the origin of the coordinate system, is placed on the strip. In the following, we will take the half-width of the strip b as the unit length. Since we focus on quasistatic fractures, the advancing velocity V of the crack comes into the problem only through the temperature field, assumed to be constant in the cold bath $(x \leq -l)$ and independent of the transverse direction y. Here l denotes the crack tip position in the temperature gradient.

Under plane stress conditions, the strain tensor of a thin plate in the temperature field $T_l(x)$ is related to the stress tensor by [13]

$$\Sigma_{ij} = \frac{1}{1-\nu^2} \left\{ (1-\nu)E_{ij} + \nu E_{kk}\delta_{ij} - (1+\nu)T_l(x)\delta_{ij} \right\}, \qquad (2)$$

where $\overline{\overline{\Sigma}}(x,y)$ $[\overline{\overline{E}}(x,y)]$ is the two-dimensional stress (strain) tensor and ν the Poisson ratio. For convenience,

all the quantities in Eq. (2) are dimensionless: $\overline{\overline{\Sigma}}$ is scaled by $E\alpha_T\Delta T$, $\overline{\overline{E}}$ by $\alpha_T\Delta T$, and $T_l(x)$ by ΔT, where E is the Young modulus and α_T the coefficient of thermal expansion. Note that $\overline{\overline{\Sigma}}(x,y)$ $[\overline{\overline{E}}(x,y)$ and $T_l(x)]$ must be understood as the average of $\overline{\overline{\Sigma}}(x,y,z)$ $[\overline{\overline{E}}(x,y,z)$ and $T_l(x,z)]$ across the thickness of the strip [13]. Inversely, the strain tensor is given, in terms of the stress tensor, by

$$E_{ij} = \frac{1}{2}\left\{ \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right\}$$
$$= \{(1+\nu)\Sigma_{ij} - \nu\Sigma_{kk}\delta ij\} + T_l(x)\delta_{ij}, \qquad (3)$$

where $\vec{U}$ is the displacement vector. For this problem, the temperature field $T_l(x)$ is usually approximated by

$$T_l(x) = \left(1 - e^{-P(x+l)}\right)\theta(x+l), \qquad (4)$$

where $\theta(\ )$ is the Heaviside function. The parameter $P = bV/D$ is the ratio of the geometrical length b to the thermal diffusion length $d_{\mathrm{th}} = D/V$, where D is the diffusion constant. The temperature field given by Eq. (4) satisfies the stationary diffusion equation without a heat loss term $\Delta T_l + P\partial T_l/\partial x = 0$, but it does not take into account other effects corresponding mainly to the existence of other lengths in experiment [3]: the nonzero thickness e of the plate and the distance h between the cold bath and the heater. In fact, Eq. (4) assumes that the temperature is constant through the thickness (i.e., $eP \ll 1$) and that there is no heat exchange between the strip and its surroundings, which becomes important for low velocities [4].

The problem of a crack of unknown shape in a strip subjected to a temperature field consists in solving the equilibrium equations [7]

$$\frac{\partial \Sigma_{ij}}{\partial x_j} = 0, \quad \nabla^2\Sigma_{ii} = -\nabla^2 T_l(x), \qquad (5)$$

with the boundary conditions

$$\Sigma_{yy}(x,\pm 1) = \Sigma_{xy}(x,\pm 1) = 0, \qquad (6)$$
$$\Sigma_{ij}n_j = 0 \text{ on the crack,} \qquad (7)$$
$$\overline{\overline{\Sigma}} = 0 \text{ for } x = \pm\infty, \qquad (8)$$

where $\vec{n}$ is the unit vector normal to the crack edges. Under equilibrium conditions, the crack shape depends essentially on the stress, which is in this case related to the temperature field. The mathematics of this problem are, given the boundary conditions on a crack whose shape is a priori unknown, the solution of the whole problem will determine the correct shape of the crack [14].

Formally, there might exist more than one solution to the global problem and one has to select the crack shape that satisfies certain stability criteria. First, the solution must be in accordance with the criterion of local symmetry [10], which imposes that the crack path $s(x,y)$ is one for which $K_{\mathrm{II}} = 0$ at the tip. Also, the chosen crack path must satisfy another condition related to the energy criterion introduced by Griffith [2]. This criterion states

that the crack is at a *critical value of incipient growth* if the reduction in the stored elastic energy W_{el} associated with a small virtual crack advance ds from that state is equal to the fracture energy Γ

$$-\frac{\partial W_{el}}{\partial s} = \Gamma \quad \text{with} \quad -\frac{\partial^2 W_{el}}{\partial s^2} \leq 0, \qquad (9)$$

Γ being a material constant independent of the crack shape. The second condition in Eq. (9) means that the system must be stable in the sense of mechanical equilibrium.

The total thermoelastic free energy per unit thickness W_{el} of a thin plate is given by [14]

$$W_{el} = Eb^2 \alpha_T^2 (\Delta T)^2 \widetilde{W}, \qquad (10)$$

where $\widetilde{W}$ is the dimensionless free energy given by

$$\widetilde{W} = \frac{1}{2} \int_{\text{surface}} dS \left[\Sigma_{ij} E_{ij} - \Sigma_{ii}(x,y) T_l(x) \right.$$

$$\left. - \frac{3}{1-2\nu} T_l^2(x) \right]. \qquad (11)$$

Note that in this writing, an integration across the thickness of the strip has already been done. Equation (11) can be simplified without specifying the crack shape. The first term of this equation can be calculated by using successively the equilibrium equations (5), the divergence theorem, and the boundary conditions (6-8)

$$\int_{\text{surface}} dS \, \Sigma_{ij} E_{ij} = \int_{\text{surface}} dS \, \frac{\partial (\Sigma_{ij} U_i)}{\partial x_j}$$

$$= \oint_{\Omega} d\Omega \, \Sigma_{ij} U_i n_j' = 0, \qquad (12)$$

where $\vec{n}'$ is the unit vector perpendicular to the contour Ω limiting the strip, including the edges of the crack and the boundaries of the plate. This term always vanishes, regardless of the shape of the crack. The third term of Eq. (11) is infinite in the configuration of an infinite strip, which is evidently not the case experimentally. Nevertheless, since this term depends on the temperature distribution only and not on the crack location, one can omit it by a convenient choice of the zero free energy. Finally, $\widetilde{W}$ is simply given by

$$\widetilde{W} = -\frac{1}{2} \int_{-\infty}^{\infty} dx \, T_l(x) \int_{-1}^{+1} dy \, \Sigma_{ii}(x,y). \qquad (13)$$

Now we outline the analysis to be done and connect its relation to earlier works [6-8]. In the following, we will discuss first the problem of existence of a stably advancing straight crack. The transition between a non-cracked strip and a straight crack is studied. It consists in solving the problem of a centered straight crack and calculating the stress intensity factor $K_I(P, l)$. By using Eq. (9) and the well-known correspondence relation established by Irwin [15]

$$-\frac{\partial \widetilde{W}}{\partial l} = K_I^2, \qquad (14)$$

one defines the region of existence of stably extending straight cracks (see Fig. 2). This problem has been solved numerically [6-8] and the results are in agreement with the experimental measurements [3,4]. Nevertheless, in addition to the previous studies, we will introduce a simple analysis that yields the scaling laws governing the transition from a receding to a moving crack. That also leads to the confirmation [8] and to extension of the invalidity of the hypothesis, which consists in taking, in certain limits, the approximation of an infinite strip.

In Sec. IV we investigate the straight-oscillating crack transition. We perform a linear stability analysis of a straight crack submitted to a small perturbation of its shape. In the vicinity of the bifurcation, we introduce a small smooth deviation $y(x)$ to the shape of the centered straight crack in the form

$$y(x) \approx Af(x) + O(A^3) \approx A \sin \omega x \quad \text{with} \quad |A| \ll 1, \quad (15)$$

where A is a constant small amplitude and ω the wave vector of the oscillation. By small deviations from the centered straight crack, it must be understood that $|y(x)| \ll 1$ and $|y'(x)| \ll 1$ ($|A\omega| \ll 1$) because the length "difference" between the two paths must also be small. From this observation, one can already expect that the sought after transition will derive from a low wave vector (high wavelength $\lambda = 2\pi/\omega$) stability analysis. We develop the stress and deformation fields as

$$\Sigma_{ij} = \sigma_{ij} + As_{ij} + A^2 t_{ij} + O(A^3), \qquad (16)$$

$$U_i = u_i + Av_i + A^2 w_i + O(A^3). \qquad (17)$$

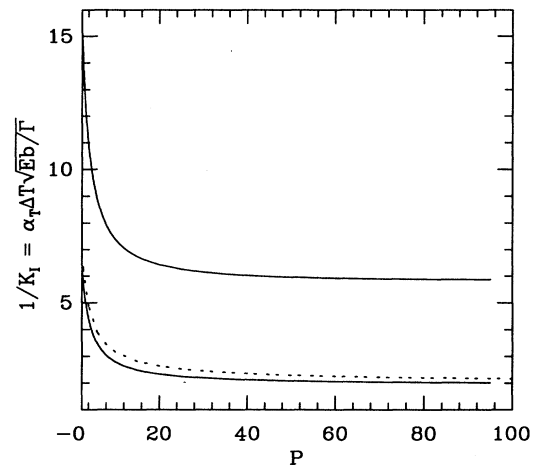Because of the symmetry $A \rightarrow -A$, the even perturbation

FIG. 2. Phase diagram in the P-K_I^{-1} plane. A crack can move in the region above the lower solid line. The transition to an oscillating propagation occurs at the upper solid curve. For comparison, the results obtained in [7] for the transition straight-oscillating morphology are also shown (dashed curve).

orders are of pure mode I type, while the odd ones are of pure mode II type. Therefore, using the tangential $U_t(x, y(x))$ and the normal $U_n(x, y(x))$ deformations [10]

$$U_t(x, y(x)) = \frac{1}{\sqrt{1 + y'^2(x)}}$$
$$\times \{U_x(x, y(x)) + y'(x) U_y(x, y(x))\}, \quad (18)$$

$$U_n(x, y(x)) = \frac{1}{\sqrt{1 + y'^2(x)}}$$
$$\times \{U_y(x, y(x)) - y'(x) U_x(x, y(x))\}, \quad (19)$$

one calculates $K_I^{tot}(P, l, \omega)$ and $K_{II}^{tot}(P, l, \omega)$, the stress intensity factors of mode I and mode II loadings, respectively. They are given by

$$K_I^{tot}(P, l, \omega) = \frac{1}{8} \lim_{x \to 0^-} \sqrt{\frac{2\pi}{-x}}$$
$$\times \{U_n(x, y^+(x)) - U_n(x, y^-(x))\}, \quad (20)$$

$$K_{II}^{tot}(P, l, \omega) = \frac{1}{8} \lim_{x \to 0^-} \sqrt{\frac{2\pi}{-x}}$$
$$\times \{U_t(x, y^+(x)) - U_t(x, y^-(x))\}, \quad (21)$$

where the superscripts $+$ and $-$ denote the upward and downward limits, respectively. At leading order, one finds that

$$K_I^{tot}(P, l, \omega) = K_I(P, l) + O(A^2), \quad (22)$$

$$\frac{1}{A} K_{II}^{tot}(P, l, \omega) = \frac{1}{A} K_{II}(P, l, \omega) + \frac{\omega}{2} K_I(P, l) + O(A^2). \quad (23)$$

It is shown that at leading order, K_I^{tot} is still given by $K_I(P, l)$, the stress intensity factor of the centered straight crack. The stress intensity factor $K_{II}(P, l, \omega)$ is the shear effect introduced by the first-order perturbation in loadings. It is given by the resolution of a pure mode II problem of a centered straight crack

$$K_{II}(P, l, \omega) = \frac{A}{8} \lim_{x \to 0^-} \sqrt{\frac{2\pi}{-x}} \{v_x(x, 0^+) - v_x(x, 0^-)\}. \quad (24)$$

Our linear stability analysis is based on the following physical arguments. If K_{II}^{tot}/A is found to be positive, this means that the stress intensity factor K_{II}^{tot} and the orientation of the crack tip $y'(0)$ have the same sign. Therefore, according to the criterion of local symmetry, the crack tip tends to follow a path that decreases $|y'(0)|$ and consequently the amplitude of the perturbation will decrease. On the other hand, when $K_{II}^{tot}/A < 0$, the slope $|y'(0)|$ will increase in order to restore a pure mode I local stress field at the tip. So, under a small perturbation of its shape, the straight crack will be stable if K_{II}^{tot}/A is found to be positive and unstable elsewhere. The oscillating crack configuration will then occur when $K_{II}^{tot}/A < 0$ is satisfied.
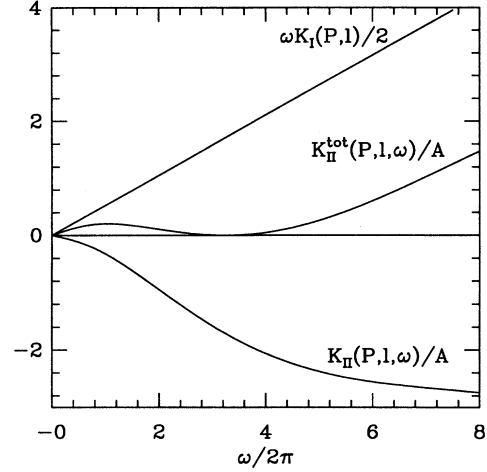

FIG. 3. Example of the variation of K_{II}/A, $\omega K_I/2$, and $K_{II}^{tot}/A = K_{II}/A + \omega K_I/2$ with respect to $\omega/2\pi$ for fixed values of l and P.

According to Eq. (23), Fig. 3 shows that K_{II}^{tot}/A is the sum of two competitive terms: the first one K_{II}/A, which is almost always negative, tends to amplify the instability of the straight crack. This destabilizing field effect is due to the variation of the stress field $\overline{\overline{\Sigma}}$ with respect to $\overline{\sigma}$. The second term of Eq. (23), $\omega K_I/2$, is a geometrical stabilizing effect. This quantity is always positive in the range of parameters where a straight crack can exist, so it tends to favor the straight configuration by damping the perturbation given by Eq. (15). It is foreseeable that the straight-undulating crack transition will occur when these two effects cancel each other. In Fig. 4 we plotted
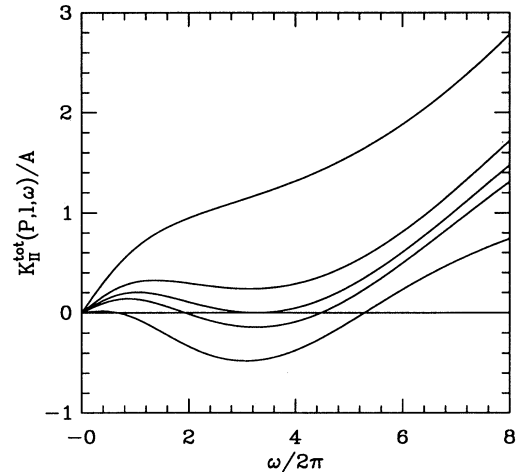

FIG. 4. K_{II}^{tot}/A versus $\omega/2\pi$ for fixed P ($P = 50$) and for different values of l (from the upper to the lower curve, $l = 0.08$, 0.11, 0.121, 0.13, and 0.16). By increasing l, K_{II}^{tot}/A decreases. When $K_{II}^{tot}/A > 0$, the problem of a wavy crack has no physical solutions. For a certain $l_u(P)$, the minimum of K_{II}^{tot}/A vanishes at $\omega = \omega_c$ and by increasing l further, any small perturbation of the straight crack in a well-defined range of wavelengths will cause a physical shear loading $K_{II}^{tot}/A < 0$.

K_{II}^{tot}/A with respect to ω at constant P and for different values of l. It is shown that there exist critical values of the parameters for which a small deviation from the centered straight crack begins to introduce a physical shear loading at the crack tip. At this point, the straight crack becomes unstable and an undulating crack path appears. From Fig. 4, one concludes that the straight-oscillating crack transition will occur when

$$K_{II}^{tot}(P,l,\omega) = 0, \quad \frac{\partial K_{II}^{tot}}{\partial \omega}(P,l,\omega) = 0. \qquad (25)$$

Equations (25) are satisfied for $l = l_u(P)$ and $\omega = \omega_u(P)$, functions that can be computed from (25). The value of $l = l_u(P)$ is the critical position of the crack tip in the temperature gradient where a straight and an oscillating crack coexist (see Fig. 5). If $l \leq l_u$ ($l \geq l_u$), the straight (oscillating) crack is the most stable configuration. The critical wavelength of the oscillation near the transition region (see Fig. 6) is simply determined by $\lambda(P) = 2\pi/\omega_u(P)$. In order to complete the phase diagram and to quantify the straight-oscillating crack transition, one has to calculate according to Eq. (22) the stress intensity factor of the straight crack $K_I^{(u)}$ at the critical points $l_u(P)$. This is done by using the results of Sec. III (see Fig. 2).

This stability analysis is not in contradiction with the criterion of local symmetry. Our process consists in searching for when a small perturbation of the linear crack can create a shear loading able to lead to an oscillatory instability. Of course, once this instability is reached, the undulating crack will choose a path satisfying $K_{II} = 0$. Clearly, this condition cannot be satisfied by the simple shape given by (15). The reason is that the chosen shape must be an exact solution of the problem to satisfy completely the criterion of local symmetry.

In order to see the effect of the crack oscillation on the free energy, we have calculated W_u, the elastic free energy of a weakly oscillating crack. This energy is expanded for
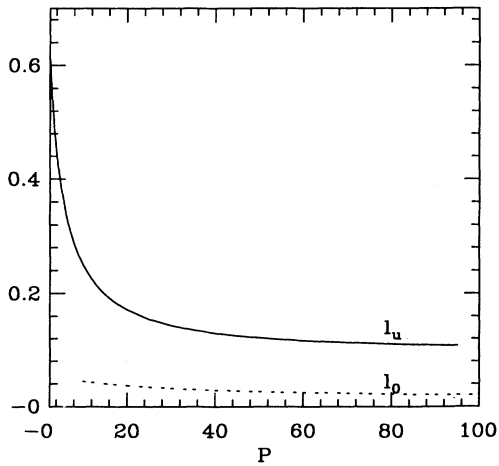
FIG. 5. $l_u(P)$ when $K_{II}^{tot}/A = 0$ (solid line) and the critical position $l_0(P)$ for which W_u becomes unstable at $\omega = 0$ (dashed line).
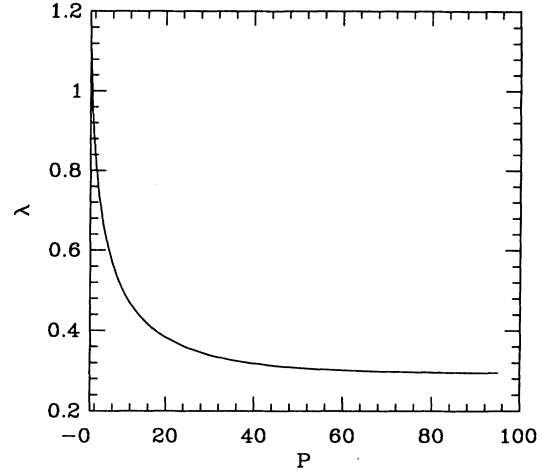
FIG. 6. Selected wavelength λ versus P at the transition from a straight to a wavy fracture.

A small under the form

$$W_u = W_s + A^2 \delta W + O(A^4) \qquad (26)$$

so that one has to solve the second-order perturbation problem to compute δW; W_s is the elastic free energy of a centered straight crack for the same tip position in the temperature gradient as the undulating crack. When the bifurcation to the wavy instability occurs, the deviation of the energy δW is always found to have an unstable maximum at $\omega = 0$. This serves as a consistency test for our approach (see Fig. 5).

III. THE STRAIGHT CRACK

This configuration has been studied previously [6–8] by numerical methods. Nevertheless, since we shall need some results for Sec. IV, we will discuss briefly the solution method (see [7] for details). The crack is assumed to be centered because of the constraint $K_{II} = 0$. The equilibrium equations (5) are rewritten as

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \nabla^2 \sigma_{ii} = -\nabla^2 T_l(x) \qquad (27)$$

and the boundary conditions (6) and (7) become

$$\sigma_{yy}(x,1) = \sigma_{xy}(x,1) = \sigma_{xy}(x,0) = 0, \qquad (28)$$

$$\sigma_{yy}(x,0) = 0 \text{ for } x \leq 0, \quad u_y(x,0) = 0 \text{ for } x \geq 0, \qquad (29)$$

where some boundary conditions have been added because of the symmetry of the problem. By working in the interval $0 \leq y \leq 1$ and in the Fourier space of the x direction, one obtains [7]

$$\hat{\sigma}_{yy}(k,0) = -F(k)\hat{u}_y(k,0) + D_l(k), \qquad (30)$$

$$\hat{\sigma}_{xx}(k,0) = H(k)\hat{\sigma}_{yy}(k,0) + S_l(k), \qquad (31)$$

with

$$F(k) = k\frac{\sinh^2 k - k^2}{\sinh 2k + 2k},$$

$$D_l(k) = 2\widehat{T}_l(k)\frac{(1 - \cosh k)(\sinh k - k)}{\sinh 2k + 2k},$$

$$H(k) = \frac{\sinh^2 k + k^2}{\sinh^2 k - k^2},$$

$$S_l(k) = \widehat{T}_l(k)\frac{\sinh k - k}{\sinh k + k}. \tag{32}$$

Once Eq. (30) is solved, the solution is complete. In order to satisfy the boundary conditions given by Eq. (29), one uses the Wiener-Hopf method [7,16], which consists of decomposing $F(k)$ as $F^-(k)/F^+(k)$, where $F^-(k)$ has neither poles nor zeros for $\mathrm{Im}(k) < 0$ and $F^+(k)$ has none for $\mathrm{Im}(k) > 0$. Then, one finds

$$\hat{u}_y(k, 0) = \frac{1}{F^-(k)}\int_{-\infty}^{0} dx\, g_{\mathrm{I}}(x + l)e^{ikx}, \tag{33}$$

where

$$g_{\mathrm{I}}(x) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}D_0(k)F^+(k)e^{-ikx}. \tag{34}$$

In our calculations, we chose the large k behavior of $F^+(k)$ to be $\sqrt{2/(\delta - ik)}$, with δ infinitesimal. The stress intensity factor of this mode I propagation is then $K_{\mathrm{I}} = g_{\mathrm{I}}(l)$.

The dimensionless free energy W_s for a straight crack is given by

$$W_s = \widetilde{W}_0(P) - \int_{-\infty}^{l} dx\, g_{\mathrm{I}}^2(x)$$

$$= \frac{1}{2}\widetilde{W}_0\left(\frac{P}{2}\right) + \int_{l}^{+\infty} dx\, g_{\mathrm{I}}^2(x), \tag{35}$$

with $\widetilde{W}_0(P)$ the free energy of a noncracked strip of width $2b$. It is explicitly given by

$$\widetilde{W}_0(P) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{1 - \frac{4\sinh^2[k]}{k(2k + \sinh[2k])}\right\}\frac{P^2}{k^2(k^2 + P^2)}. \tag{36}$$

Note that $\widetilde{W}_0$ is independent of l. Equations (35) and (36) can be obtained by two different, but equivalent, methods: either by putting directly in Eq. (13) the solution of the above problem or by using Eq. (14) and calculating the free energy $\widetilde{W}_0$ of a noncracked strip. This can be done easily since in this simple case, the mixed boundary condition (29) is replaced by $u_y(x, 0) = \hat{u}_y(k, 0) = 0$.

By using Eqs. (9) and (35), one concludes that the straight crack can extend when $l \geq l_s(P)$, where $l_s(P)$ corresponds to $K_{\mathrm{I}}^{(s)}(P)$, the maximum of $K_{\mathrm{I}}(P, l)$. Therefore the no crack–straight crack transition (lower solid curve in Fig. 2) obeys the law

$$\frac{1}{K_{\mathrm{I}}^{(s)}(P)} = \sqrt{\frac{Eb}{\Gamma}}\,\alpha_T\Delta T. \tag{37}$$

This law is in agreement with the experimental observations concerning this transition [6,8].

The stress intensity factor $K_{\mathrm{I}}(P, l)$ decays rapidly to zero when $l \to \pm\infty$ and varies only in a window of width of order L_c, which will be the characteristic length of the problem. Therefore, when the crack position is behind this region, the elastic free energy is not very different from a noncracked strip, while when it is in front of this region, the strip can be approximately treated as an infinitely cracked one. The latter case is equivalent to the problem of two strips of width b. To a first approximation, one can assume a linear energy variation between these two limits across the unknown characteristic length L_c. Thus the existence of an extending straight crack is constrained by the condition

$$Eb^2\alpha_T^2(\Delta T)^2\Delta\widetilde{W} \approx \Gamma L_c, \tag{38}$$

where $\Delta\widetilde{W} = \widetilde{W}_0(P) - \widetilde{W}_0(P/2)/2$. It can easily be shown that the asymptotic behavior of $\Delta\widetilde{W}$ with respect to P is simply given by

$$\Delta\widetilde{W} \propto P^2 \quad \text{for } P \ll 1,$$
$$\Delta\widetilde{W} \approx 1 \quad \text{for } P \gg 1. \tag{39}$$

Let us recall that from the beginning, we have scaled lengths by b and consequently k by $1/b$. So in these two limits, the relevant length scale is the width of the strip and the diffusion length d_{th} plays a secondary role compared to b. Therefore, in the asymptotic cases of Eqs. (39), the characteristic length is of the order of the width of the strip $L_c \propto b$. Extending straight cracks may occur when

$$b^3(\Delta T)^2 V^2 \approx Ct \quad \text{for } P \ll 1,$$
$$b(\Delta T)^2 \approx Ct \quad \text{for } P \gg 1. \tag{40}$$

Moreover, these scaling laws remain valid when one considers n equidistant straight cracks. This can be seen by considering the energy difference $W_{\mathrm{el}}(b) - (n + 1)W_{\mathrm{el}}(b/(n+1))$. The scaling laws of Eqs. (40) reproduce qualitatively the experimental behavior of the transition to an advancing straight crack. However, the most important conclusion that can be drawn from this analytical study is that the problem of a crack in a heated strip cannot be approximated by a problem of an infinite plate in both limiting cases $b \ll d_{\mathrm{th}}$ and $b \gg d_{\mathrm{th}}$. That could explain why the analysis of Refs. [7,8] assuming an infinite strip failed to explain the straight-oscillating crack transition. Let us now study the stability of crack path under mode I loading.

IV. THE OSCILLATING CRACK

In this section we lay down the calculations needed to compute numerically the conditions (25) and Eq. (26). To introduce perturbations to the symmetric straight crack, one takes the deviation from this configuration, as given by Eq. (15). The condition $f(0) = 0$ is not restrictive since the transition occurs between a centered straight crack and an oscillating one. It is therefore sufficient to compare these two configurations at the same location in the temperature gradient. In our approach, we must solve first for the straight crack and then for the first- and second-order perturbation in the amplitude A. The perturbation method does not differ too much from that followed in [10]. The following analysis is to be compared to the linearization performed in [10] for the study of slightly curved cracks. The components of the vector normal to the crack edges are $(n_x, n_y) \propto (Af'(x), -1)$. Expanding the equilibrium equations (5) and the boundary conditions (6) and (7) near $A = 0$, one has the following two problems to solve.

The first problem is a mode II loading given by the equilibrium equations

$$\frac{\partial s_{ij}}{\partial x_j} = 0, \quad \nabla^2 s_{ii} = 0 \tag{41}$$

with the boundary conditions

$$s_{yy}(x,1) = s_{xy}(x,1) = s_{yy}(x,0) = 0; \tag{42}$$

$$s_{xy}(x,0) = \frac{\partial}{\partial x}\left[f(x)\sigma_{xx}(x,0)\right] \quad \text{for} \quad x \leq 0,$$

$$v_x(x,0) = 0 \quad \text{for} \quad x \geq 0. \tag{43}$$

Note that $\lim_{x\to 0^-} \sigma_{xx}(x,0)$ is finite $[f_{xx}^I(\pm\pi) = 0$ in (1)]. This limit is equal to Σ [7], the stress in the transverse direction and near the tip, which remains once the square-root singularity has been subtracted out. By the same reasoning as in Sec. III, one finds that this problem satisfies the equation

$$\hat{s}_{xy}(k,0) = -P(k)\hat{v}_x(k,0), \tag{44}$$

where

$$P(k) = k\frac{\sinh^2 k - k^2}{\sinh 2k - 2k}. \tag{45}$$

To solve this equation, one uses again the Wiener-Hopf technique; one splits $\hat{s}_{xy}(k,0)$ as $\hat{s}_{xy}^-(k,0) + \hat{s}_{xy}^-(k,0)$ and $P(k)$ as $P^-(k)/P^+(k)$, where the signs $+$ and $-$ have the same meanings as in Sec. III. Then, using Eq. (43), one finds

$$\hat{v}_x(k,0) = \frac{1}{P^-(k)}\int_{-\infty}^0 dx\, g_{\mathrm{II}}(x)e^{ikx}, \tag{46}$$

where

$$g_{\mathrm{II}}(x) = -\int_{-\infty}^x dx'\frac{d}{dx'}\left[f(x')\sigma_{xx}(x',0)\right]p^+(x-x'), \tag{47}$$

with $p^+(x) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}P^+(k)e^{-ikx}$. The stress intensity factor of this mode II loading is then given by

$$\frac{1}{A}K_{\mathrm{II}}(P,l,\omega) \equiv g_{\mathrm{II}}(0)$$

$$= -\int_{-\infty}^0 dx\frac{d}{dx}\left[\sigma_{xx}(x,0)\sin\omega x\right]p^+(-x). \tag{48}$$

The second problem concerns the second order (in the amplitude A) of the perturbation analysis. It is a mode I loading given by the equilibrium equations

$$\frac{\partial t_{ij}}{\partial x_j} = 0, \quad \nabla^2 t_{ii} = 0 \tag{49}$$

with the boundary conditions

$$t_{yy}(x,1) = t_{xy}(x,1) = 0; \tag{50}$$

$$t_{xy}(x,0) = \theta(-x)\frac{\partial}{\partial x}\left(f(x)\frac{\partial}{\partial x}v_x(x,0)\right.$$

$$\left. -\frac{1}{2}f^2(x)\frac{\partial^2}{\partial x^2}u_y(x,0)\right); \tag{51}$$

$$t_{yy}(x,0) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[f^2(x)\sigma_{xx}(x,0)\right] \quad \text{for} \quad x \leq 0,$$

$$w_y(x,0) = 0 \quad \text{for} \quad x \geq 0. \tag{52}$$

It can be shown that this problem yields, after a Fourier transform,

$$\hat{t}_{yy}(k,0) = -F(k)\left[\hat{w}_y(k,0) - \nu\frac{\hat{t}_{xy}(k,0)}{-ik}\right]$$

$$-F(k)H(k)\frac{\hat{t}_{xy}(k,0)}{-ik}, \tag{53}$$

which has to be solved by using the boundary conditions (51) and (52). The functions $F(k)$ and $H(k)$ are given by Eqs. (32) and thus the Wiener-Hopf decomposition of $F(k)$ is already known.

The elastic energy W_u up to the second order in A is given by

$$W_u = -\frac{1}{2}\int_{-\infty}^{\infty}dx\,T_l(x)\left\{\int_0^1 dy\left[\Sigma_{ii}(x,y) + \Sigma_{ii}(x,-y)\right]\right.$$

$$-Af(x)\theta(-x)\left[\Sigma_{ii}(x,0^+) - \Sigma_{ii}(x,0^-)\right]$$

$$-\frac{A^2}{2}f^2(x)\theta(-x)$$

$$\left.\times\frac{\partial}{\partial y}\left[\Sigma_{ii}(x,y) + \Sigma_{ii}(x,-y)\right]\Big|_{y=0}\right\}. \tag{54}$$

Once the previous problems are solved analytically, the various quantities that appear in Eqs. (25) and (26) can be calculated numerically. The derivative of K_{II} with respect to ω is given by

$$\frac{1}{A}\frac{\partial K_{II}}{\partial \omega}(P, l, \omega)$$

$$= -\int_{-\infty}^{0} dx \frac{d}{dx}\left[x\sigma_{xx}(x, 0)\cos\omega x\right]p^+(-x). \quad (55)$$

After some algebraic manipulations on Eq. (54), the correction δW is found to be equal to

$$\delta W = \frac{1}{2}Pu_y(-l, 0)f^2(-l) - \int_{-\infty}^{0} dx \left\{ g_{II}^2(x) \right.$$

$$\left. -u_y(x, 0)\frac{d^2}{dx^2}\left[f^2(x)\sigma_{xx}(x, 0)\right]\right\}, \quad (56)$$

where the first term reflects the discontinuity of the derivative of $\sigma_{xx}(x, 0)$ at $x = -l$, when the temperature field is given by Eq. (4). Note that the variation δW is calculated using the first- and second-order terms of the perturbation and does not depend on g_{II} only.

The numerical analysis of the previous problems is straightforward. The Wiener-Hopf decomposition needed for $F(k)$ and $P(k)$ is done as described in [16]. The other quantities are computed using Fourier transforms, after treating the singular parts separately and analytically. The results of these numerical calculations are summarized in Figs. 2 and 6. Using Eqs. (34), (48), and (55) and according to (22) and (25), one calculates the position $l_u(P)$ where the straight crack becomes unstable against a small perturbation of its shape, the corresponding stress intensity factor $K_I(P, l_u)$, and the wavelength of the oscillation $\lambda(P)$. Using Eq. (56), the critical value $l_0(P)$ where δW starts to have a maximum instead of minimum at $\omega = 0$ has also been calculated (see Fig. 5).

The only hypothesis we made to introduce the conditions (22) and (25) is the smoothness of the fracture shapes. This assumption is not too drastic and is in agreement with the experimental observations [3]. We have also assumed, as in [10], the existence of a small random stress intensity factor K_{II} due to the imperfections in the loading system. Because of the criterion of local symmetry, the existence of such imperfections leads automatically to deviations from the original straight crack. Therefore, the stability analysis of a preexisting straight crack under a small fluctuation in its shape is necessary before studying the shape of its extension.

V. CONCLUSION

In this paper we treated completely the instabilities of an advancing crack in a strip subjected to thermal stresses. The first instability concerned the condition of existence of extending straight cracks. We quantified this transition (lower solid curve in Fig. 2) and proved the asymptotic scaling laws analytically. Since this bifurcation is well defined, its experimental study is a good way to fix the fracture energy value and to determine the relevant length scale in the imposed temperature field.

For the more subtle transition from a straight crack to a wavy crack (upper solid curve in Fig. 2), we introduced a notion that consists of defining this bifurcation by the existence of a physical stress intensity factor of mode II loading for a small instability of the straight crack. Any quantitative comparison between this treatment, the studies based on the criterion of local symmetry [8,6], and the experiment [3] is difficult to do because even the first bifurcation has not yet been studied quantitatively. Nevertheless, our results agree rather well with the numerical simulations of [6]. Moreover, a comparison with the experimental data in the region of the phase diagram where the diffusion length d_{th} is the dominant length scale in the temperature field ($V \approx 3$ mm/s, $\Delta T \approx 70$ K, and $b = 1.2$ cm) gives a value of the fracture energy $\Gamma \approx 7.3$ J/m^2, which is of the same order of magnitude as the value measured in [9].

The selected wavelength plotted in Fig. 6 also shows agreement with the results of [6] for $P \gg 1$. In both cases, it was found that $\lambda \approx 0.28$. For smaller P, the two wavelengths are different but still of the same order. In this limit, they have to be compared with the experimental value $\lambda \approx 0.56$ in [3]. However, since there is no information on V, one might attribute the plateau observed in this experiment to the effects of h, the spacing between the heater and the cold bath, or to three-dimensional effects [4].

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[12] Note that in [10], the crack is treated as a stable straight one if the asymptotic deviation $\lambda(x)$ from the original

path is $\lambda(x) \propto \sqrt{x}$. For an infinite geometry, this is a sufficient condition for a straight crack to be asymptotically stable. However, the existence of boundaries in our case leads to the creation of a "geometrical" shear stress. If the straight crack is not centered, it will be automatically unstable so that it will deviate towards the center. By the repetition of these processes, it is then possible to construct a wavy instability. Therefore, the condition that makes the crack stable in the infinite geometry can

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