Small Froude number asymptotics in two-dimensional two-phase flows

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A nonlinear wave equation is derived describing the behavior of gas- and liquid-fluidized beds in the small Froude number regime. It represents a two-dimensional perturbation of the Korteweg–de Vries equation and is shown to constitute a valid approximation of the original system. While greatly simplifying the analytical and numerical investigation of two-phase flow in fluidized beds, it also leads to the conclusion that the underlying model does not significantly discriminate between gas- and liquid-fluidized beds near the stability limit. An amplitude equation is derived governing the growth and stability of solitary plane waves. The results are linked to those obtained by previous nonapproximative analyses. It is expected that this analysis is applicable to other multiphase and traffic flow models due to the similarity in the governing equations and the completeness of the reduced wave equation.

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I. INTRODUCTION

As a particular two-phase flow, in which densely packed particles are suspended by an upward flowing fluid, fluidized beds have found a variety of engineering applications and, therefore, have been the subject of extensive experimental and theoretical investigations [1–7]. Typically, the desired state of uniform fluidization becomes unstable upon increasing the fluid flow rate and turns into a wavy flow in liquid-fluidized beds, or into a bubbly flow in gas-fluidized beds. Increased efforts have been made during the last few years to understand the origin of these nonlinear structures as well as the differences between gas- and liquid-fluidized beds [8–14].

To be more precise at this point, these differences seem to be less due to the differences in compressibility and viscosity of the fluids used for fluidization, usually air or water, and more due to the magnitude of the ratio δ of the specific densities of the fluid and the particulate phase [15]. This is reflected in common models of fluidized beds based on volume or ensemble averaged equations of motion, where both the fluid and the particulate phase are treated as incompressible. Assuming that the behavior of gas- and liquid-fluidized beds can be modeled in the same manner, the only parameters expressing an explicit difference between these two are δ and the ratio of the kinematic viscosities of the fluid and particle phase, ν. Usually, gas-fluidized beds are associated with very small values of δ and ν, and for simplification one often sets $\delta=\nu=0$. These values, however, depend on the properties of the fluidized particles as well. This reservation has to be kept in mind when we use the phrases gas- and/or liquid-fluidized bed below.

From the linear stability analysis of such models one knows that the primary instability occurs in the vertical direction, and the bifurcation analysis shows that this leads to plane periodic voidage waves traveling upwards through the bed [6,8,9]. Restricting attention to one-dimensional traveling waves, it turns out that the periodic waves either terminate in a homoclinic orbit, i.e., a solitary wave with a single hump and of infinite period, or return to the uniform state at a smaller wave speed. Although the other parameters play a role as well, it can be said that the second case can only take place for nonvanishing values of δ [9].

The basic fact behind this behavior is the occurrence of a codimension-2 degeneracy at the instability threshold, at which the interparticle collisional pressure balances the drag force exerted by the upward streaming fluid on the particles. It is essentially the difference between these two forces that determines the strength of the instability and the growth rate of the unstable modes; the latter is also proportional to the Froude number. At a certain wave speed, which is related to the drag force, a stationary solution bifurcates transcritically from the basic state—in special circumstances a pitchfork bifurcation is possible indicating the existence of heteroclinic connections, i.e., moving fronts—whereas a Hopf bifurcation takes place at a lower wave speed related to the value of the interparticle force [9]. As is well known, the interaction between these two solutions produces a homoclinic connection [16]. The temporal evolution of this solitary wave, which is a pure kinematic wave at the onset but can further be described by a perturbed Korteweg–de Vries (KdV) equation, has been intensively studied recently within a small Froude number approach for a one-dimensional model of gas-fluidized beds [14].

It is obvious, however, that higher-dimensional instabilities play the decisive role in the formation of bubbles or other, less distinctive multidimensional wave patterns. Again, bifurcation analysis reveals the presence of transverse instabilities allowing the bifurcation of two-dimensional periodic traveling waves off the state of uniform fluidization [8,10,17]. Their stability has not yet been studied analytically but known results for other systems and numerical analyses [18] suggest that, since they usually do not represent the first bifurcating branch, they are probably all unstable near their point of bifurcation. This holds except within a very narrow parameter regime where there exists no pure downstream instability, such

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that the first instability leads at once to a two-dimensional bubblelike structure (cf. Fig. 3 in [8]). This could explain the rapid formation of bubbles in gas-fluidized beds; however, this possibility is only valid for liquid-fluidized beds, which is contrary to experimental evidence. More precisely, for the case described, one needs to have $\nu \neq 0$.

Although these "mixed-mode" branches may play a prominent role in the nonlinear regime, as has been outlined in [17], they have not received much attention in the literature and are, therefore, not well understood. Instead, the search for the origin of bubbles has been directed to a secondary, transverse instability of the one-dimensional traveling wave solutions. Because such a behavior can be observed very clearly in liquid- but not in gas-fluidized beds, in the latter case the range of stability of the one-dimensional (1D) wave has to be very small in order to allow for the prompt onset of bubbling. A rudimentary approach has been presented recently and depends on the introduction of a diffusion term into the mass conservation equation for the particles [13]. In a more rigorous investigation, the present author has considered weakly stable perturbations of the uniform state with long transverse wavelength, and followed their development along the branch of vertically traveling periodic waves in a model with $\delta=\nu=0$ [19]. It could be shown that the interaction between the one-dimensional wave and a disturbance packet, consisting initially, i.e., at the primary bifurcation point, of two pure transverse perturbations and the above-mentioned mixed modes, leads to a secondary instability, at which the plane wave loses its stability to the wave packet. For transverse perturbations of wave number k, k "small," the secondary instability sets in and gives rise to two-dimensional traveling waves of bubble type (cf. also [18]), when the amplitude of the primary wave has grown up to the order of k^2. It has been found that a stationary as well as an oscillatory instability can occur (cf. also the discussion in [17] based on symmetry arguments). Because the standard modeling makes no difference in principle between the fluidization by a gas and that by a liquid, the same result will presumably hold for the latter case. However, since it is known from the pioneering work of Anderson and Jackson [20] that perturbations grow much faster in gas- than in liquid-fluidized beds, the 1D wave train can be expected to reach the critical amplitude much earlier, thus being more subject to a transverse instability in the first case. Of course, this supposition has to be confirmed by a detailed analysis, and it was one of the motivations of the present study to facilitate this endeavor.

A somewhat different approach based on a small parameter controlling the linear instability—without correlating it to the Froude number—has been applied quite recently to 1D [12] and simplified 2D models [11], and this has led to a 2D perturbation of the KdV equation. The authors of [11] have also carried out a 1D simulation and found that the initial solitary wave finally settles down onto a periodic wave train, which is no surprise in the light of the previous bifurcation results, as has been discussed in [17]. Finally, Hayakawa [21] has derived an amplitude equation describing the growth of pseudosolitons in one-dimensional gas-fluidized beds, and found that the one-soliton solution can blow up in finite time.

In this paper, we will derive a two-dimensionally perturbed KdV equation valid for both gas- and liquid-fluidized beds using the Froude number as small parameter. Reducing the complex two-fluid model to a scalar equation for the voidage greatly facilitates the further analytical and numerical treatment of the fluidized bed system. Therefore, we will first state the equations we deal with, then describe the procedure used to arrive at the simplified equation, state the latter, and show that it is a valid approximation of the original system. Then we transform it into canonical form and examine the coefficients of the various terms, thereby showing that they do not change their signs nor do they vanish in physically reasonable parameter domains. This indicates that gas- and liquid-fluidized beds behave very similarly within the considered approximation. Finally, we derive an amplitude equation governing the growth and stability of solitary plane waves and link the results to those of our previous bifurcation studies. The blow-up behavior of these soliton perturbations can be ascribed to the approximative nature of the reduced equation.

II. BASIC EQUATIONS AND SCALING

We start with the widely used two-phase flow model [4,8]

$$-\partial_t\phi+\operatorname{div}[(1-\phi)\mathbf{v}]=0\ , \tag{2.1}$$

$$\partial_t\phi+\operatorname{div}(\phi\mathbf{u})=0\ , \tag{2.2}$$

$$\begin{aligned}F(1-\phi)(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})\\ &=-(1-\phi)\mathbf{k}+B(\phi)(\mathbf{u}-\mathbf{v})-FG(\phi)\nabla\phi\\ &\quad-(1-\phi)\nabla p+\frac{F}{R}\Delta\mathbf{v}\ ,\end{aligned} \tag{2.3}$$

$$F\delta\phi(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})=-\delta\phi\mathbf{k}-B(\phi)(\mathbf{u}-\mathbf{v})-\phi\nabla p+\nu\frac{F}{R}\Delta\mathbf{u}\ . \tag{2.4}$$

Instead of (2.2) we will also use the following equation for the total volumetric flow, which is obtained by adding (2.1) and (2.2):

$$\operatorname{div}[(1-\phi)\mathbf{v}+\phi\mathbf{u}]=0\ . \tag{2.2$'$}$$

The variables are the fluid volume fraction or voidage ϕ, the effective fluid pressure p, and the particle and fluid phase velocities $\mathbf{v}$ and $\mathbf{u}$, respectively. The equations have been made dimensionless by scaling with an appropriate length scale L (cf. the discussion in [14] and below), fluidization velocity $\mathbf{u}_0=u_0\mathbf{k}$, pressure with $\rho_s gL$, and drag with $\rho_s g/u_0$, where g is the gravitational constant and $\mathbf{k}$ is the unit vector parallel to the x axis pointing against gravity; the transverse direction will be denoted by y.

Note that we have scaled the bulk modulus $G(\phi)$ (<0) appearing in the interparticle force differently from p, namely, by $\rho_s u_0^2$ which provides another natural pressure scale. However, the usual pressure term proportional to

∇p has to balance the weight of the particles as well as the drag force between the two phases, which are expressed by the terms $(1-\phi)\mathbf{k}$ and $B(\phi)(\mathbf{u}-\mathbf{v})$, respectively. Hence these forces should act on the same scale, whereas we assume that the interparticle force contributes on a scale $O(F)$ compared to ∇p; otherwise the kinematic wave occurring at the transition point would be damped [22].

The density ratio $\delta=\rho_f/\rho_s$ is small for gas-fluidized beds and often set to zero for simplicity, as is the ratio of the viscosity coefficients $\nu=\mu_f/\mu_s$; both may be taken of order $O(10^{-1})$. The $\mu_{s,f}$ may depend on the voidage but are taken to be constant here. Thus we end up with the nondimensional parameters Froude number $F=u_0^2/(gL)$ and a particle-based Reynolds number $R=\rho_s u_0 L/\mu_s$. Using a macroscopic length scale L yields $F \ll 1$, while $R=O(1)$ (see [14] and the discussion below).

Now the state of uniform fluidization is given by

$$\phi=\phi_0, \quad \mathbf{v}=0, \quad \mathbf{u}_0=\mathbf{k} , \qquad \nabla p_0=-[1-\phi_0(1-\delta)]\mathbf{k}\equiv p_0'\mathbf{k} , \tag{2.5}$$

while the drag coefficient has to satisfy the relation $B_0\equiv B(\phi_0)=(1-\delta)\phi_0(1-\phi_0)$. To match this, the drag coefficient is frequently assumed to be of the Stokes-like form

$$B(\phi)=(1-\delta)\frac{1-\phi}{\phi^n}\phi_0^{n+1} \tag{2.6}$$

with $n\approx 3.0$ (cf. [1]). Linearizing around the uniform flow leads to the equation

$$[A\partial_t^2+2C\partial_t\partial_x+C\partial_x^2+D\partial_x+E\partial_t -M\Delta-J\Delta\partial_t-H\Delta\partial_x]\phi=0 \tag{2.7}$$

for the voidage perturbation. Adopting the notation of [8], the positive constants are given by

$$A=\phi_0+C, \quad C=\delta(1-\phi_0), \quad E=\frac{B_0}{F\phi_0(1-\phi_0)}=\frac{1-\delta}{F} ,$$

$$D=\left[\frac{B_0}{\phi_0}-B_0'+(1-\delta)(1-2\phi_0)\right]/F =(n+2)(1-\phi_0)(1-\delta)/F , \tag{2.8}$$

$$M=-\phi_0 G_0, \quad H=\frac{\nu}{R}\frac{1-\phi_0}{\phi_0}, \quad J=\frac{\phi_0}{R(1-\phi_0)}+H .$$

In addition, it is convenient to introduce the following abbreviations:

$$m=M/A, \quad c=C/A ,$$
$$d=D/E\ [\doteq(n+2)(1-\phi_0)], \quad h=H/J ; \tag{2.9}$$
$$f(s)=m-c(1-c)-(s-c)^2 .$$

From (2.7) one derives easily the dispersion relation determining linear stability by setting $\phi\sim\exp(\sigma t+i\lambda x+iky)$. It can be shown (e.g., [8]) that the state of uniform fluidization is stable if $f(d)\geq 0$, otherwise an instability sets in at small wave numbers, where the growth rate is proportional to the Froude number. This is more clearly seen by expanding the critical eigenvalue for small wave numbers, where we set $\delta=\nu=0$ for simplicity:

$$\begin{aligned}\sigma^+=&-i\lambda d[1-FJ(\lambda^2+k^2)-F^2\{2\phi_0^2[(G_0+d^2)\lambda^2+G_0k^2]-J^2(\lambda^2+k^2)^2\}]\\&+F\phi_0[(G_0+d^2)\lambda^2+G_0k^2]-F^2\phi_0[J(2G_0+3d^2)+O(F)]\lambda^2k^2\\&-F^2\phi_0[J(G_0+3d^2)+O(F)]\lambda^4-F^2\phi_0G_0[J+O(F)]k^4+O(\lambda^6,\lambda^4k^2,\lambda^2k^4,k^6) .\end{aligned} \tag{2.10}$$

Up to terms of the order of F^3 this long wave expansion is virtually identical to the small Froude number expansion and shows how the various terms should be balanced in order to capture the behavior near the stability limit and for small Froude number: correlating the instability range to the Froude number via

$$0<-f(d)|_{\delta=0}=d^2-|G_0|\sim F^\alpha$$

leads to $\lambda^2F^{1+\alpha}\sim\lambda^4F^2$ and $k^2\sim\lambda^2F^\alpha$, because we want to include the regularizing fourth-order derivative in x and a contribution from the transverse direction; hence $\lambda^2\sim F^{\alpha-1}$, $k^2\sim F^{2\alpha-1}$. Choosing $\alpha=1$ eventually gives $d^2-|G_0|\sim F$, $\lambda\sim O(1)$, and $k\sim F^{1/2}$. Then, on the other hand, we cannot expect to keep track of the terms corresponding to F^2k^4 and $F^2\lambda^2k^2$, i.e., no derivatives ∂_Y^4, $\partial_X^2\partial_Y^2$ will show up in the simplified wave equation for the voidage to be derived in the next section.

Moreover, the imaginary part of (2.10) indicates that the analysis should be performed in a frame moving with velocity d against gravity, because the leading order approximation allows for an undistorted wave $\phi(x-dt)$ (cf. [12,14]). In [9] it has been shown that this corresponds to a transcritical bifurcation at $\omega_*=d$ in the system describing traveling plane waves $\phi(x-\omega t\pm ky/\lambda)$, where ω is the wave speed and bifurcation parameter, while a Hopf bifurcation to a branch of periodic solutions occurs at $\omega_k=c+\sqrt{m(k/\lambda)^2+f(c)}$, if $\omega_k<d$ holds. We mention that another Hopf bifurcation may take place at $\omega_k'=c-\sqrt{m(k/\lambda)^2+f(c)}$, if this expression is positive, which requires in particular $c\sim\delta>0$. However, this possibility—and thereby a possibly important difference between gas- and liquid-fluidized beds—is ruled out when only considering the weakly unstable case, because it has been shown in [9] that $\omega_k'<0$ if $\sqrt{m}$ is near d.

We note that Harris and Crighton [14] have employed the scaling $\alpha=\frac{1}{2}$, which would give $\lambda\sim F^{-1/4}$, $k\sim O(1)$; but they restrict their (one-dimensional) study to wave numbers $\lambda\sim O(1)$, which is not consistent within the above scaling, and so they miss the important fourth-order derivative that prevents the rapid growth of short waves (they are aware of this fact; see the Appendix of [14]). On the other hand, Komatsu and Hayakawa [11] used the value of $\epsilon^2=-f(d)$ as independent scaling parameter—leaving the Froude number finite and fixed—which suggests $\lambda\sim\epsilon$, $k\sim\epsilon^2$, and is certainly reasonable. However, we prefer to choose an intermediate scaling, such that $\lambda\sim O(1)$ and the fourth-order derivative is captured as well, and this can only be achieved by relating the strength of the instability to the Froude number.

The main reason behind this approach is our interest in the inclusion of one- and two-dimensional periodic solutions, which bifurcate at wave speeds $\omega_k=\omega_0+O(k^2)$, $\omega_0^2=|G_0|+O(\delta)$, that lie in the interval $\omega\in(h,d)$. The corresponding longitudinal and transverse wave numbers are given by [8,17]

$$\lambda^2=\frac{(1-\delta)m(d-\omega)}{FJ(\omega-h)[m-f(\omega)]},\quad k^2=-\lambda^2\frac{f(\omega)}{m}\,, \tag{2.11}$$

with the bifurcation condition $0\geq f(\omega)>f(d)$. As has been discussed above, the weakly unstable range is characterized by slightly negative values of $f(d)$, and marginally stable periodic solutions exist only for $\omega_0\leq\omega<d$. Hence the largest longitudinal wave number is given by that of the one-dimensional wave train, $\lambda_0^2\sim(d-\omega_0)/F$, while $\lambda\to0$ as $\omega\to d$. We see at once that λ_0^2 is of the order of $-f(d)/F$, so that a scaling of $f(d)\sim F^\alpha$ with an $\alpha<1$ would lead to an unphysical divergence of λ_0 as $F\to0$, while it would tend to zero for $\alpha>1$ or within the approach of [11]. Now, the *dimensional* wave number of the periodic plane wave at its bifurcation point indeed approaches zero with the shrinking range of the instability $[\omega_0\to d\Longrightarrow f(d)\to0]$. But the wavelengths have been nondimensionalized with $L\sim1/F$, and large L imply small F [note that u_0 must also become small in order to ensure $J^{-1}\sim R\sim u_0L\sim O(1)$]. Thus, if we use for L the physical wavelength of an emerging or nearby fully developed periodic wave, as has indeed been done in [14], we end up with longitudinal wave numbers λ, which are $O(1)$ on such a length scale. In the weakly unstable regime, this length scale is large, so the Froude number is small, and this induces the above scaling. The scaling of the transverse wave number can also be justified from (2.11), since $k^2\sim\lambda^2(\omega^2-m)+O(\delta)$, $\lambda\sim O(1)$, and $0<\omega^2-m<d^2-m\sim O(F)$.

In summary, our approach represents the unique and consistent way to unfold the degeneracy occurring at the stability limit $\omega_*=d$, which simultaneously marks the onset of the kinematic wave; the periodic ("dynamic") waves bifurcate within a distance of the order of the Froude number $[\omega=\omega_*-O(F)]$ and have finite longitudinal and small ($\sim F^{1/2}$) transverse wave numbers. Compared to the results in [11] we will obtain a more accurate (which means also more complicated) approximation of the system, which is better suited for the description of periodic solutions.

III. DERIVATION OF THE SIMPLIFIED WAVE EQUATION

Step 1: Setup

According to the above considerations we define $X=x-dt$, $Y=F^{1/2}y, T=Ft$, and propose that the voidage perturbation is of the order of the strength of the instability, namely, $O(F)$, whence the orders of the other perturbations follow from the equations

$$\begin{aligned}&\phi=\phi_0+F\tilde{\phi},\quad p=p_0X+F\tilde{p}\,,\\&v_x=Fv_X,\quad v_y=F^{3/2}v_Y\,,\\&u_x=1+Fu_X,\quad u_y=F^{3/2}u_Y\,.\end{aligned} \tag{3.1}$$

That is, assume a voidage perturbation of the form $\phi=\phi_0+F^a\tilde{\phi}$; then the perturbations of the vertical velocities and the pressure have to be of the same order to balance gravity with drag and pressure as mentioned at the beginning of Sec. II. Scaling the horizontal velocities by F^b we obtain for Eq. (2.1)

$$\begin{aligned}(1-\phi_0)&\partial_X\tilde{v}_X+d\partial_X\tilde{\phi}\\&=F^a\partial_X(\tilde{\phi}\tilde{v}_X)+F\partial_T\tilde{\phi}\\&\quad-F^{b-a+1/2}(1-\phi_0)\partial_Y\tilde{v}_Y+F^{b+1/2}\partial_Y(\tilde{\phi}\tilde{v}_Y)\,.\end{aligned}$$

This relation suggests the choices $a=1$ and $b=\frac{3}{2}$ above, which are also consistent with the perturbation form of the other equations. [For consistency $a>0$ is needed, otherwise the full nonlinearities from drag force and pressure would be included on the same scale as the zeroth-order linear terms that describe the kinematic wave. At the same time, the interparticle pressure as well as the viscous terms would act on a higher scale, which would be a highly unstable situation. In fact, $a=1$ is the proper choice, since it brings the lowest nonlinearities onto the same scale as the (slow) time derivative and the transverse perturbation terms.] Inserting this into the original Eqs. (2.1)–(2.4), rearranging terms, dividing by common factors of F, and dropping the tildes, we arrive at the following system:

$$(1-\phi_0)\partial_Xv_X+d\partial_X\phi=FR_{11}+F^2R_{12}\,, \tag{3.2}$$

$$\phi_0\partial_Xu_X+(1-d)\partial_X\phi=FR_{21}+F^2R_{22}\,, \tag{3.3}$$

$$\begin{aligned}(1+B_0'+p_0')\phi+B_0(u_X-v_X)&-(1-\phi_0)\partial_Xp\\&=F\hat{R}_{31}+F^2\hat{R}_{32}+O(F^3)\,,\end{aligned} \tag{3.4}$$

$$\begin{aligned}-(\delta+B_0'+p_0')\phi-B_0(u_X-v_X)&-\phi_0\partial_Xp\\&=F\hat{R}_{41}+F^2\hat{R}_{42}+O(F^3)\,,\end{aligned} \tag{3.5}$$

$$B_0(u_Y-v_Y)-(1-\phi_0)\partial_Yp=F\hat{R}_{51}+O(F^2)\,, \tag{3.6}$$

$$B_0(u_Y-v_Y)-\phi_0\partial_Yp=F\hat{R}_{61}+O(F^2)\,, \tag{3.7}$$

with

$$R_{11}=\partial_T\phi+\partial_X(\phi v_X)-(1-\phi_0)\partial_Yv_Y,\quad R_{12}=\partial_Y(\phi v_Y)\,, \tag{3.8}$$

$$R_{21}=-\partial_T\phi-\partial_X(\phi u_X)-\phi_0\partial_Yu_Y,\quad R_{22}=-\partial_Y(\phi u_Y)\,. \tag{3.9}$$

Furthermore, using (2.2′) instead of (2.2) gives rise to

$$\partial_X[(1-\phi_0)v_X+\phi_0 u_X+\phi] = -F\{\partial_X[\phi(u_X-v_X)]+\partial_Y[(1-\phi_0)v_Y+\phi_0 u_Y]\} - F^2\partial_Y[\phi(u_Y-v_Y)] . \tag{3.3'}$$

We will not state the $\hat{R}_{ij}$'s explicitly, because we are going to use the following combinations of the above equations: we add (3.4) and (3.5) and form $\phi_0\times$[Eq. (3.4)]$-(1-\phi_0)\times$[Eq. (3.5)], and do the same with (3.6) and (3.7). This gives

$$(1-\delta)\phi-\partial_X p=FR_{31}+F^2R_{32}+O(F^3) , \tag{3.10}$$

$$-a_0\phi+B_0(u_X-v_X)=FR_{41}+F^2R_{42}+O(F^3) , \tag{3.11}$$

$$\partial_Y p=FR_{51}+O(F^2) , \tag{3.12}$$

$$B_0(u_Y-v_Y)=FR_{61}+O(F^2) , \tag{3.13}$$

where

$$R_{31}=G_0\partial_X\phi-d(1-\phi_0)\partial_X v_X+\delta\phi_0(1-d)\partial_X u_X-\frac{1}{R}\partial_X^2 v_X-\frac{\nu}{R}\partial_X^2 u_X , \tag{3.14a}$$

$$R_{32}=G_0'\phi\partial_X\phi+(1-\phi_0)\partial_T v_X+\delta\phi_0\partial_T u_X+[(1-\phi_0)v_X+d\phi]\partial_X v_X +\delta[\phi_0 u_X+(1-d)\phi]\partial_X u_X-\frac{1}{R}\partial_Y^2 v_X-\frac{\nu}{R}\partial_Y^2 u_X , \tag{3.14b}$$

$$R_{41}=\phi_0 G_0\partial_X\phi-d\phi_0(1-\phi_0)\partial_X v_X-\delta\phi_0(1-\phi_0)(1-d)\partial_X u_X -\frac{\phi_0}{R}\partial_X^2 v_X+\frac{\nu}{R}(1-\phi_0)\partial_X^2 u_X-\frac{B_0''}{2}\phi^2-B_0'\phi(u_X-v_X)-\phi\partial_X p , \tag{3.15a}$$

$$R_{42}=\phi_0 G_0'\phi\partial_X\phi+\phi_0(1-\phi_0)(\partial_T v_X-\delta\partial_T u_X) +\phi_0[(1-\phi_0)v_X+d\phi]\partial_X v_X-\delta(1-\phi_0)[\phi_0 u_X+(1-d)\phi]\partial_X u_X -\frac{\phi_0}{R}\partial_Y^2 v_X+\frac{\nu}{R}(1-\phi_0)\partial_Y^2 u_X-\frac{B_0'''}{6}\phi^3-\frac{B_0''}{2}\phi^2(u_X-v_X) , \tag{3.15b}$$

$$R_{51}=G_0\partial_Y\phi-d(1-\phi_0)\partial_X v_Y+\delta\phi_0(1-d)\partial_X u_Y-\frac{1}{R}\partial_X^2 v_Y-\frac{\nu}{R}\partial_X^2 u_Y , \tag{3.16a}$$

$$R_{61}=\phi_0 G_0\partial_Y\phi-d\phi_0(1-\phi_0)\partial_X v_Y-\delta\phi_0(1-\phi_0)(1-d)\partial_X u_Y -\frac{\phi_0}{R}\partial_X^2 v_Y+\frac{\nu}{R}(1-\phi_0)\partial_X^2 u_Y-B_0'\phi(u_Y-v_Y) , \tag{3.16b}$$

and

$$-a_0=\phi_0(1+B_0'+p_0')+(1-\phi_0)(\delta+B_0'+p_0') =\frac{B_0}{\phi_0(1-\phi_0)}(1-\phi_0-d) . \tag{3.17}$$

The equations (3.13) and (3.16a) suggest that the transverse velocities are of a still higher order in F, but for the sake of greater generality we will first work with the present scaling and return to this point at the end of the section.

Now, in order to obtain a single equation for the voidage, we have to eliminate all the other variables by shifting them to terms of higher order in the Froude number. For instance, we can use (3.2) and (3.3) to express $\partial_X v_X$ and $\partial_X u_X$, respectively, by $\partial_X\phi+O(F)$; similarly, we use (3.11) and (3.13) to eliminate the relative velocity. This process may then be repeated in the $O(F)$ terms. Basically, forming

$$\phi_0(1-\phi_0)\times\partial_X[\text{Eq. } (3.11)]+B_0[\phi_0\times[\text{Eq. } (3.2)] -(1-\phi_0)\times[\text{Eq. } (3.3)]]$$

elminates the $O(1)$ terms due to (3.17) and leaves us with the $O(F)$ equation

$$\phi_0(1-\phi_0)\partial_X R_{41}+B_0[\phi_0 R_{11}-(1-\phi_0)R_{21}] =-F\{\phi_0(1-\phi_0)\partial_X R_{42} +B_0[\phi_0 R_{12}-(1-\phi_0)R_{22}]\}+O(F^2) . \tag{3.18}$$

Step 2: Treating the lower order terms

Let us first consider the $O(1)$ terms in (3.18). We use (3.2) and (3.3) up to $O(F)$ to eliminate $\partial_X v_X$ and $\partial_X u_X$ in R_{41}, and similarly (3.10) to remove $\partial_X p$ and (3.11) to remove u_X-v_X from R_{41}; thereby we obtain new contributions to the $O(F)$ terms on the right-hand side of (3.18). Defining the coefficients

$$a_2=\frac{1}{RB_0}[d\phi_0^2-\nu(1-\phi_0)^2(1-d)] , \qquad -a_3=\frac{a_0B_0'}{B_0}+\frac{B_0''}{2}+1-\delta , \tag{3.19}$$

this leads to

$$R_{41}=[\phi_0 G_0+\phi_0 d^2+\delta(1-\phi_0)(1-d)^2]\partial_X\phi+a_3\phi^2+\frac{B_0}{\phi_0(1-\phi_0)}a_2\partial_X^2\phi$$

$$+F\left\{-d\phi_0 R_{11}-\delta(1-\phi_0)(1-d)R_{21}-\frac{B_0'}{B_0}\phi R_{41}+\phi R_{31}\right.$$

$$\left.-\frac{1}{R\phi_0(1-\phi_0)}[\phi_0^2\partial_X R_{11}-\nu(1-\phi_0)^2\partial_X R_{21}]\right\}+O(F^2)\ , \qquad (3.20a)$$

while on using (3.11) and (3.13) one arrives at

$$\phi_0 R_{11}-(1-\phi_0)R_{21}=\partial_T\phi+\partial_X(\phi v_X)+\frac{a_0(1-\phi_0)}{B_0}\partial_X\phi^2$$

$$+F\left[\frac{1-\phi_0}{B_0}\partial_X(\phi R_{41})+\frac{\phi_0(1-\phi_0)}{B_0}\partial_Y R_{61}\right]+O(F^2)\ . \qquad (3.20b)$$

According to our scaling arguments we write [cf. (2.9)]

$$-A\,f(d)=\phi_0 G_0+\phi_0 d^2+\delta(1-\phi_0)(1-d)^2$$

$$=\frac{B_0}{\phi_0(1-\phi_0)}c_0 F\ , \qquad (3.21)$$

where c_0 is an $O(1)$ quantity, which is positive or negative depending on whether the system is on the unstable or stable side of the threshold, respectively. With the coefficient

$$a_1=\frac{1-\phi_0}{B_0}(a_0+a_3\phi_0)-\frac{d}{1-\phi_0}\ , \qquad (3.22)$$

we thus obtain as an intermediate result

$$\partial_T\phi+\partial_X(\phi v_X)+\left[a_1+\frac{d}{1-\phi_0}\right]\partial_X\phi^2+a_2\partial_X^3\phi=-F(c_0\partial_X^2\phi+\tilde{S})+O(F^2)\ , \qquad (3.23a)$$

$$\tilde{S}=\phi_0 R_{12}-(1-\phi_0)R_{22}+\frac{\phi_0(1-\phi_0)}{B_0}(\partial_Y R_{61}+\partial_X R_{42})+\frac{1-\phi_0}{B_0}\left[1-\frac{B_0'\phi_0}{B_0}\right]\partial_X(\phi R_{41})$$

$$+\frac{\phi_0(1-\phi_0)}{B_0}\partial_X[-d\phi_0 R_{11}-\delta(1-\phi_0)(1-d)R_{21}+\phi R_{31}]-\frac{1}{RB_0}\partial_X^2[\phi_0^2 R_{11}-\nu(1-\phi_0)^2 R_{21}]\ . \qquad (3.23b)$$

Step 3: Treating the higher order terms

To get rid of the unwanted variables in the $O(F)$ expressions, we use the $O(1)$ approximations of (3.2), (3.3), (3.11), and (3.13). Thus, to leading order,

$$\phi_0 R_{12}-(1-\phi_0)R_{22}=\partial_Y(\phi v_Y)\ ,$$

$$\partial_Y R_{61}=\phi_0 G_0\partial_Y^2\phi-\phi_0(1-\phi_0)[d+\delta(1-d)]\partial_X\partial_Y v_Y-\frac{1}{R}[\phi_0-\nu(1-\phi_0)]\partial_X^2\partial_Y v_Y\ ,$$

$$d\phi_0 R_{11}+\delta(1-\phi_0)(1-d)R_{21}=\frac{B_0}{\phi_0(1-\phi_0)}b_2[\partial_T\phi+\partial_X(\phi v_X)]$$

$$-\delta\frac{a_0}{B_0}(1-\phi_0)(1-d)\partial_X\phi^2-\phi_0(1-\phi_0)[d+\delta(1-d)]\partial_Y v_Y\ ,$$

$$\phi_0^2 R_{11}-\nu(1-\phi_0)^2 R_{21}=RB_0\,b_3[\partial_T\phi+\partial_X(\phi v_X)]+\nu\frac{a_0}{B_0}(1-\phi_0)^2\partial_X\phi^2-\phi_0(1-\phi_0)[\phi_0-\nu(1-\phi_0)]\partial_Y v_Y\ ,$$

$$R_{31}=[G_0+d^2-\delta(1-d)^2]\partial_X\phi+\frac{B_0}{\phi_0(1-\phi_0)}b_8\,\partial_X^2\phi\ ,$$

$$\partial_X R_{42} = \left[\frac{\phi_0 G_0'}{2} - \frac{B_0}{\phi_0(1-\phi_0)} \frac{db_2}{1-\phi_0} \right] \partial_X^2 \phi^2 + \frac{B_0}{\phi_0(1-\phi_0)} a_2 \partial_Y^2 \partial_X \phi$$
$$- \frac{B_0}{\phi_0(1-\phi_0)} b_2 \partial_X [\partial_T \phi + \partial_X(\phi v_X)] - \left[\frac{a_0 B_0''}{2B_0} + \frac{B_0'''}{6} \right] \partial_X \phi^3 ,$$

with the abbreviations

$$b_2 = \frac{\phi_0(1-\phi_0)}{B_0}[d\phi_0 - \delta(1-\phi_0)(1-d)] , \quad (3.24)$$

$$b_3 = \frac{1}{RB_0}[\phi_0^2 + \nu(1-\phi_0)^2] = \frac{1}{d}\left[a_2 + \frac{\nu(1-\phi_0)^2}{RB_0} \right] , \quad (3.25)$$

$$b_8 = \frac{1}{RB_0}[d\phi_0 + \nu(1-d)(1-\phi_0)] . \quad (3.26)$$

Upon inserting these expressions into (3.23b) the linear terms depending on v_Y cancel; after using the $O(1)$ part of (3.23a) to remove $\partial_T\phi + \partial_X(\phi v_X)$ from $\tilde{S}$ we are left with the relation

$$\tilde{S} = \partial_Y(\phi v_Y) - b_1 \partial_Y^2 \phi + a_2 \partial_Y^2 \partial_X \phi + 2a_2 b_2 \partial_X^4 \phi + a_2 b_3 \partial_X^5 \phi$$
$$+ \tilde{b}_4 \partial_X(\phi \partial_X^2 \phi) + b_5 \partial_X^2 \phi^2 + b_6 \partial_X^3 \phi^2 + \tilde{b}_7 \partial_X \phi^3 , \quad (3.27)$$

where

$$b_1 = \frac{\phi_0(1-\phi_0)}{B_0} \phi_0 |G_0|$$
$$= \frac{\phi_0(1-\phi_0)}{B_0}[\phi_0 d^2 + \delta(1-\phi_0)(1-d)^2] - c_0 F , \quad (3.28)$$

$$\tilde{b}_4 = b_8 + \frac{a_2}{\phi_0}\left[1 - \frac{B_0'\phi_0}{B_0} \right] ,$$
$$b_6 = b_3 \left[a_1 + \frac{d}{1-\phi_0} \right] - \frac{\nu a_0 (1-\phi_0)^2}{RB_0^2} , \quad (3.29)$$

$$b_5 = \frac{\phi_0(1-\phi_0)}{2B_0} \left[\phi_0 G_0' + G_0 + d^2 - \delta(1-d)^2 + 2\delta \frac{a_0}{B_0}(1-\phi_0)(1-d) \right] + \frac{db_2}{1-\phi_0} + 2a_1 b_2 , \quad (3.30)$$

$$\tilde{b}_7 = \frac{1-\phi_0}{B_0}\left[a_3 \left[1 - \frac{B_0'\phi_0}{B_0} \right] - \phi_0 \left[\frac{a_0 B_0''}{2B_0} + \frac{B_0'''}{6} \right] \right] . \quad (3.31)$$

Step 4: Finale

To express the quantity $\partial_X(\phi v_X) + F\partial_Y(\phi v_Y)$ as a function of ϕ, we return to Eq. (3.3′) and "solve" it by introducing the stream function W:

$$v_X + \phi_0(u_X - v_X) + \phi = F[\partial_Y W - \phi(u_X - v_X)] , \quad (3.32a)$$

$$v_Y + \phi_0(u_Y - v_Y) + \partial_X W = -F\phi(u_Y - v_Y) , \quad (3.32b)$$

with an obvious scaling of W [compare (3.33a) below with (3.2)]. Using the relations for the relative velocity components, we obtain

$$v_X + \frac{d}{1-\phi_0}\phi = F\left[\partial_Y W - \frac{a_0}{B_0}\phi^2 - \frac{\phi_0}{B_0} R_{41} \right] + O(F^2) , \quad (3.33a)$$

$$v_Y + \partial_X W = -F\frac{\phi_0}{B_0} R_{61} + O(F^2) . \quad (3.33b)$$

Hence,

$$\partial_X(\phi v_X) + F\partial_Y(\phi v_Y)$$
$$= -\frac{d}{1-\phi_0}\partial_X \phi^2 + F\left[\partial_X \phi\, \partial_Y W - \partial_Y \phi\, \partial_X W - \frac{a_0}{B_0}\partial_X \phi^3 - \frac{\phi_0}{B_0}\partial_X(\phi R_{41}) \right]$$
$$+ O(F^2) , \quad (3.34)$$

and this gives

$$\partial_T \phi + a_1 \partial_X \phi^2 + a_2 \partial_X^3 \phi$$
$$= -F(c_0 \partial_X^2 \phi + \partial_X \phi\, \partial_Y W - \partial_Y \phi\, \partial_X W + S) + O(F^2) , \quad (3.35a)$$

$$S = -b_1 \partial_Y^2 \phi + a_2 \partial_Y^2 \partial_X \phi + 2a_2 b_2 \partial_X^4 \phi + a_2 b_3 \partial_X^5 \phi$$
$$+ b_4 \partial_X(\phi \partial_X^2 \phi) + b_5 \partial_X^2 \phi^2 + b_6 \partial_X^3 \phi^2 + b_7 \partial_X \phi^3 , \quad (3.35b)$$

where the coefficients of the nonlinear terms can be written as

$$b_4 = \tilde{b}_4 - \frac{a_2}{1-\phi_0}$$
$$= b_8 + \frac{a_2}{\phi_0(1-\phi_0)}[1 - 2\phi_0 - B_0'\phi_0(1-\phi_0)/B_0]$$
$$= \frac{d^2\phi_0^2 + \nu(1-d)^2(1-\phi_0)^2}{RB_0\phi_0(1-\phi_0)} , \quad (3.36)$$

$$b_5 = \frac{\phi_0(1-\phi_0)}{2B_0}[\phi_0 G_0' + G_0 + d^2 - \delta(1-d)^2] + 2a_1 b_2$$
$$+ [d^2\phi_0^2 - \delta(1-d)^2(1-\phi_0)^2]/B_0 , \quad (3.37)$$

$$b_6=\frac{1}{RB_0\phi_0(1-\phi_0)}[d\phi_0^3+\nu(1-d)(1-\phi_0)^3]+a_1b_3\ ,\tag{3.38}$$

$$\begin{aligned}b_7&=\tilde{b}_7-(a_0+\phi_0a_3)/B_0\\&=\frac{a_3}{B_0}\left[1-2\phi_0-B_0'\frac{\phi_0(1-\phi_0)}{B_0}\right]\\&\quad-\frac{\phi_0(1-\phi_0)}{B_0}\left[\frac{a_0B_0''}{2B_0}+\frac{B_0'''}{6}\right]-\frac{a_0}{B_0}\ .\end{aligned}\tag{3.39}$$

Finally we argue that the contribution from the stream function can be neglected in the regarded order. Above all, $\partial_X\phi\partial_Y W-\partial_Y\phi\partial_X W=O(F)$ would hold, if $W=f(\phi)+O(F)$, where f is any differentiable function. However, this observation is of no help here. Instead, (3.13) suggests that not only is the transverse component of the relative velocity of the order of the Froude number but also the transverse velocity components of both phases themselves, which means that no transverse velocity (which would otherwise be the same for the two phases) develops to leading approximation. This is in agreement with $\partial_Y\phi=O(F)$ from (3.10) and (3.12), and is reflected in the structure of the perturbed KdV equation (3.35). Assuming therefore

$$v_Y=FV_Y,\quad u_Y=FU_Y\ ,\tag{3.40}$$

(3.12) and (3.13) become

$$\partial_Y p=FG_0\partial_Y\phi+O(F^2)\ ,\tag{3.41}$$

$$B_0(U_Y-V_Y)=\phi_0G_0\partial_Y\phi+O(F)\ .\tag{3.42}$$

Fortunately, this additional scaling does not affect the final result, (3.35a) and (3.35b), except that the terms containing the stream function W are shifted to the next higher order, since now $\partial_X W=O(F)$; hence

$$W=O(F)\ .\tag{3.43}$$

The latter follows because the contributions from the transverse velocities in (3.2), (3.3), and (3.3′) are shifted to higher order, too, resulting in $\partial_Y W=O(F)$.

IV. VERIFICATION BY COMPARISON WITH KNOWN RESULTS

Using results from previous analyses, the validity of the reduced equation (3.35)—taking account of (3.43)—can be checked easily: the linear terms should agree with the stability and bifurcation results, while the nonlinear terms should correspond to those obtained from an ordinary differential equation describing plane voidage waves.

First of all, it is immediately seen that the linear stability of the null solution of (3.35), evaluated for $\delta=\nu=0$, represents the $O(F)$ approximation to that of the uniform solution (2.5) of the original system (2.1)–(2.4); see (2.10) and take account of the transformation $X=x-dt$ as well as of the scaling $k=F^{1/2}K$ and that expressed in (3.21). In passing, we note the following relations to the original coefficients defined in (2.8) and (2.9):

$$a_2=b_3(d-h),\quad b_3=\frac{\phi_0(1-\phi_0)}{B_0}J=\frac{J}{1-\delta}\ ,\tag{4.1}$$

$$\begin{aligned}b_2&=\frac{\phi_0(1-\phi_0)}{B_0}\left[Ad-C-\frac{Af(d)}{2(d-h)}\right]\\&=\frac{Ad-C}{1-\delta}+O(F)\ .\end{aligned}\tag{4.2}$$

To prove for nonvanishing δ and ν that the bifurcation behavior in the vicinity of the trivial solution has also been attained correctly, we have to search for traveling wave solutions $\phi(X-\tilde{\omega}T,Y)$ which are periodic in both variables. Such solutions bifurcate at values of the propagation velocity and longitudinal and transverse wave numbers λ and K, respectively, satisfying the two relations

$$\begin{aligned}&\tilde{\omega}=a_2\lambda^2+Fa_2(K^2-b_3\lambda^4)+O(F^2)\ ,\\&-c_0\lambda^2+b_1K^2+2b_2a_2\lambda^4=O(F)\ .\end{aligned}\tag{4.3}$$

Indeed, the same result is obtained by replacing $\omega=d-F\tilde{\omega}$ $(\tilde{\omega}>0)$ and $k^2=FK^2$ in the full relations (2.11) for the bifurcation points [8,17] and expanding them with respect to the Froude number.

Most importantly, the appearance of the various nonlinear terms and the correctness of their coefficients can be tested against the equation describing traveling plane waves. Proposing that all variables are functions of $z=x-\omega t\pm ky$ only reduces (2.1)–(2.4) to a system of ordinary differential equations. Upon elimination of the other variables, this system is further reduced to a single equation for the voidage [9]:

$$f_1\phi''=f_2(\phi')^2+f_3\phi'+f_4\ ,\tag{4.4}$$

where the prime denotes the derivative with respect to z, and the f_i are functions of ϕ, ω, F, k^2, and the other parameters. Applying the same procedure as in the preceding section (see Appendix A) leads to a reduced equation, which has to be compared with the one following from (3.35). There, the corresponding ansatz for traveling plane waves gives, after dropping one common derivative (and the tilde over ω),

$$\begin{aligned}&\omega\phi+a_1\phi^2+a_2\phi''\\&\quad=-F[(c_0-b_1K^2)\phi'+a_2K^2\phi''\\&\qquad+2a_2b_2\phi'''+a_2b_3\phi^{(4)}+b_4\phi\phi''\\&\qquad+b_5(\phi^2)'+b_6(\phi^2)''+b_7\phi^3]\ .\end{aligned}\tag{4.5}$$

The results of Appendix A show that the structure of the two equations is exactly the same, and a lengthy but straightforward calculation reveals that also the coefficients are in concordance within the proposed approximation. We conclude, therefore, that the two-dimensional perturbed KdV equation (3.35) is a valid approximation of the original two-fluid system and can be used for simplified investigations of fluidized beds in the small Froude number regime.

V. AMPLITUDE EQUATION FOR SOLITARY PLANE WAVES

A. Rescaling and examination of coefficients

We transform (3.35) into canonical form by changing variables to

$$\phi=\alpha u,\quad \tau=\beta T,\quad x=\gamma X,\quad y=\rho Y,\quad F=\eta\tilde{F} \tag{5.1}$$

with

$$\gamma=\left[\frac{|c_0|}{2a_2b_2}\right]^{1/2},\quad \beta=a_2\gamma^3,\quad \alpha=\frac{a_2\gamma^2}{a_1},\quad \rho=\left[\frac{2a_2b_2}{b_1}\right]^{1/2}\gamma^2,\quad \eta=\frac{1}{2b_2\gamma}. \tag{5.2}$$

This gives our main result

$$\begin{aligned}u_\tau+(u^2)_x+u_{xxx} &=-\tilde{F}[\epsilon u_{xx}-u_{yy}+c_1u_{yyx}+u_{xxxx}+c_2u_{xxxxx}\\ &\quad+c_3(uu_{xx})_x+c_4(u^2)_{xx}+c_5(u^2)_{xxx}\\ &\quad+c_6(u^3)_x]+O(\tilde{F}^2),\end{aligned} \tag{5.3}$$

where

$$\epsilon=\operatorname{sgn}c_0,\quad c_1=\frac{a_2\gamma}{b_1},\quad c_2=\frac{b_3\gamma}{2b_2},\quad c_3=\frac{b_4\gamma}{2a_1b_2},\quad c_4=\frac{b_5}{2a_1b_2},\quad c_5=\frac{b_6\gamma}{2a_1b_2},\quad c_6=\frac{a_2b_7\gamma}{2a_1^2b_2}. \tag{5.4}$$

The above scaling requires $a_2b_2>0$. But it is easily seen that for the drag coefficient (2.6), yielding $d=(n+2)(1-\phi_0)$ with any positive n, $a_2\sim d-h>0$ for all $\nu\in[0,1]$ and $b_2\sim d-c>0$ for all $\delta\in[0,1]$ (see also [8]), such that this condition is always satisfied. Then the scaling parameters (5.2) are all positive as they should be and, furthermore,

$$c_i>0,\quad i=1,2,3,5,6;\quad \operatorname{sgn}c_4=\operatorname{sgn}b_5, \tag{5.5}$$

due to (4.1), (3.36), and explicit calculations revealing that

$$2\phi_0a_1=(n+2)[n+1-(n+3)\phi_0]>0 \quad \text{for } \phi_0<\frac{n+1}{n+3}\approx\frac{2}{3}, \tag{5.6}$$

$$\begin{aligned}RB_0b_6&=\phi_0^2\left[\frac{d}{1-\phi_0}+a_1\right]+\nu(1-\phi_0)^2\left[\frac{1-d}{\phi_0}+a_1\right]\\ &=\frac{(n+1)(1-\phi_0)}{2\phi_0}\\ &\quad\times\{(n+2)\phi_0^2+\nu(1-\phi_0)[n-(n+2)\phi_0]\}\\ &\quad> \text{ for } n>\sqrt{2}-1,\end{aligned} \tag{5.7}$$

$$6\phi_0^2b_7=(n+1)[n(11n+10)-\phi_0(11n^2-4n+6)]>0 \quad \text{for } n>\tfrac{3}{7}. \tag{5.8}$$

We see that the dependence of these coefficients on δ and ν is weak; in particular, a_1 and b_7 do not depend on δ or ν at all, while a_2,b_3,b_4, and b_6 are all inversely proportional to $1-\delta$ (assuming $B\sim1-\delta$ only). Since b_5 does not depend on ν either, we conclude that the fluid viscosity has no essential influence on the small Froude number behavior of both gas- and liquid-fluidized beds. At first glance,

$$\begin{aligned}(1-\delta)b_5&=\frac{\phi_0G_0'}{2}+\frac{d^2\phi_0}{1-\phi_0}+2a_1[d\phi_0-\delta(1-\phi_0)(1-d)]-\frac{\delta}{2\phi_0}(1-d)^2(3-2\phi_0)\\ &=\tfrac{1}{2}\phi_0G_0'+\frac{d-1}{\phi_0}[(n+2)\phi_0d+\delta\{\tfrac{1}{2}+(1-\phi_0)[n(n+\tfrac{3}{2})-(n+2)^2\phi_0]\}]\end{aligned} \tag{5.9}$$

could change its sign depending on the values of δ, ϕ_0, and G_0'. Assuming, however, a monotonic interparticle force, $G_0'\geq0$, over the range of physically accessible voidage values, then b_5 is positive if $d>1$, i.e., $\phi_0<(n+1)/(n+2)$; hence $c_4>0$, if $\phi_0<(n+1)/(n+3)$. These conditions are usually met, since fluidized beds are normally operated at uniform voidage values between 0.4 and 0.6, while the Richardson-Zaki exponent n lies in the range of 3 to 4 [1].

B. The soliton perturbation

Based on (5.3) we can derive an amplitude equation governing the growth and stability of a developing solitary wave in two-dimensional fluidized beds. To leading order, (5.3) has the single-soliton solution $u=6A^2\text{sech}^2[A(x-4A^2\tau-x_0)]$, where A and x_0 are arbitrary constants. However, the $O(\tilde{F})$ part of (5.3) determines A and x_0 as functions of a slow time $\tilde{\tau}$ and the transverse direction. In order to avoid the introduction of spurious resonant terms like $\int\partial_yA^2d\tilde{\tau}/\tilde{F}$, we have to assume that the amplitude A is a function of time only; this is explained in more detail in Appendix B. As will be shown below, this determines x_0 as the sum of a time-dependent part and a time-independent part linear in y, such that this ansatz is capable of yielding plane wave solutions only.

Thus we assume $A=A(\tilde{\tau})$ with $\tilde{\tau}=\tilde{F}\tau$ and introduce

$$\theta=A(\tilde{\tau})[x-\Omega-x_0(\tilde{\tau},y)], \tag{5.10}$$

with $\partial_\tau\Omega=4A^2/\tilde{F}$, $\partial_y\Omega=0$, such that (5.3) becomes

$$\partial_{\tilde{\tau}}u-4A^3\partial_\theta u+A\partial_\theta u^2+A^3\partial_\theta^3u=-\tilde{F}N(u)+O(\tilde{F}^2), \tag{5.11a}$$

$$N(u)=\partial_{\tilde{\tau}}u+\frac{A_{\tilde{\tau}}}{A}\theta\partial_\theta u-A\dot{x}_0\partial_\theta u+\epsilon A^2\partial_\theta^2 u+Ax_{0,yy}\partial_\theta u-A^2x_{0,y}^2\partial_\theta^2 u-c_1A^2x_{0,yy}\partial_\theta^2 u+c_1A^3x_{0,y}^2\partial_\theta^3 u+A^4\partial_\theta^4 u+A^5c_2\partial_\theta^5 u+A^3c_3\partial_\theta(u\partial_\theta^2 u)+A^2c_4\partial_\theta^2 u^2+A^3c_5\partial_\theta^3 u^2+Ac_6\partial_\theta u^3 \ . \tag{5.11b}$$

Expanding the sought solution with respect to the Froude number,

$$u=u_0+\tilde{F}u_1+O(\tilde{F}^2)$$

$$\text{with } u_0=6A^2(\tilde{\tau})v,\quad v=\text{sech}^2\theta \ , \tag{5.12}$$

gives to the next order of approximation

$$\partial_{\tilde{\tau}}u_1+Lu_1=-N(u_0) \ ,\qquad Lu_1=-4A^3\partial_\theta u_1+2A\partial_\theta(u_0u_1)+A^3\partial_\theta^3 u_1 \ . \tag{5.13}$$

The solvability condition for (5.13) reads

$$\int_{-\infty}^{\infty}N(u_0)v\,d\theta=0 \ , \tag{5.14}$$

since $v=\text{sech}^2\theta$ is the integrable solution of $L^*u^*=0$, with the adjoint operator $L^*=4A^3\partial_\theta-2Au_0\partial_\theta-A^3\partial_\theta^3$. The evaluation of (5.14) yields the following amplitude equation:

$$\partial_{\tilde{\tau}}(A^2)=\tfrac{16}{15}A^4[\epsilon+\tfrac{4}{7}(12c_4-5)A^2-(x_{0,y}^2+c_1x_{0,yy})] \ . \tag{5.15}$$

It shows that $x_{0,y}^2+c_1x_{0,yy}$ has to be a function of $\tilde{\tau}$ only, suggesting already that x_0 is linear in y. In order to obtain more information about x_0, we must actually solve (5.13). For that purpose we note that $N(u_0)$ may be written in the form

$$N(u_0)=av+(b+c\theta)v'+dv^2+evv'+fv^3+gv^2v' \ , \tag{5.16a}$$

with the coefficients

$$\begin{aligned} a&=6[(\partial_{\tilde{\tau}}A^2)+4A^4(\epsilon+4A^2-x_{0,y}^2-c_1x_{0,yy})] \ ,\\ c&=3(\partial_{\tilde{\tau}}A^2) \ ,\\ b&=6A^3(16c_2A^4-\dot{x}_0+x_{0,yy}+4c_1A^2x_{0,y}^2) \ ,\\ f&=720A^6(1-c_4) \ ,\\ d&=36A^4[-\epsilon+4A^2(4c_4-5)+x_{0,y}^2+c_1x_{0,yy}] \ ,\\ e&=72A^5[4A^2(4c_5+c_3-5c_2)-c_1x_{0,y}^2] \ ,\\ g&=216A^7(10c_2-3c_3-10c_5+3c_6) \ . \end{aligned} \tag{5.16b}$$

Now, the general solution of the differential equation

$$p'''-4p'+12(pv)'=r$$

is determined by

$$p=wv',\quad w'=\frac{k_1+v\int r\,d\theta-\int vr\,d\theta}{4v^2(1-v)} \ ,$$

with an arbitrary integration constant k_1. Hence the stationary solution of (5.13) is given by

$$-A^3u_1=-\frac{5k_1}{16}-\frac{k_2}{4}+\left[\frac{c}{8}+\frac{d}{30}+\frac{4f}{105}\right]\tanh\theta+\frac{f}{130}v'\ln v+\left[\frac{15}{16}k_1+\frac{3}{4}k_2+\frac{b}{4}+\frac{e}{12}+\frac{g}{12}+\frac{c}{4}\theta\right]v-\frac{g}{24}v^2+\left[k_3+\left[\frac{15}{32}k_1+\frac{3}{8}k_2+\frac{b}{8}\right]\theta+\frac{c}{16}\theta^2\right]v'-\frac{k_1}{8}\frac{1}{v}+\frac{1}{8}\left[\frac{2a}{3}-\frac{c}{3}+\frac{8d}{15}+\frac{16f}{35}\right]\frac{\tanh\theta}{v} \ , \tag{5.17}$$

where $k_{1,2,3}$ are constant with respect to θ. The requirement that u_1 should be bounded at $\pm\infty$ yields the conditions $k_1=0$ and that the coefficient of the last term vanishes, which is nothing else but the amplitude equation (5.15). Imposing the condition $u_1\to 0$ as $\theta\to+\infty$, so that the bed is in its unperturbed state before the passage of the solitary wave, leads to the relation

$$k_2=4\left[\frac{c}{8}+\frac{d}{30}+\frac{4f}{105}\right] \ . \tag{5.18}$$

The coefficient k_3 of v' is arbitrary, because v' is always a solution of the linearized equation. This and the coefficient of the $\theta v'$ term can be set to zero, as they can be incorporated into A and x_0 by changing them by an $O(\tilde{F})$ quantity (the same argument as in [14]). Thus, $3k_2+b=0$, and upon combination with (5.18) this becomes the additional constraint

$$b+\frac{3}{2}c+\frac{2}{5}d+\frac{16}{15}f=0 \ . \tag{5.19}$$

Now we could use (5.15) to eliminate $\dot{A}$ from (5.19) and get an evolution equation for x_0. It is more practical, however, to eliminate the quadratic term $x_{0,y}^2$, so that we are left with an equation of the type

$$\dot{x}_0=(1-4c_1^2A^2)x_{0,yy}+h(A,\dot{A}) \ . \tag{5.20}$$

Because $x_{0,y}^2+c_1x_{0,yy}$ has to be a function of time only due to (5.15) and the requirement $\partial_yA=0$, the only possible solution of (5.20) is given by

$$\begin{aligned} &x_0(\tilde{\tau})=x_{01}(\tilde{\tau})\pm\tilde{k}y,\quad \tilde{k}\equiv\text{const} \ ,\\ &\dot{x}_{01}=[-\tfrac{8}{5}\epsilon+16c_2A^3+\tfrac{32}{35}(5-12c_4)A^2+4(\tfrac{2}{5}+c_1A)\tilde{k}^2]A \ . \end{aligned} \tag{5.21}$$

Thus the only two-dimensional perturbations that can be reached by the perturbation ansatz based on the one-dimensional KdV soliton are obliquely traveling plane waves, parametrized by their transverse wave number $\tilde{k}$. We note that Barcilon and Lovera [23] have also unsuccessfully attempted to find nearby two-dimensional solutions in the form of solitary wave perturbations (cf. Appendix B). These failures suggest that genuine 2D solutions are of a different structure and have to be approached by other methods.

It is nevertheless possible to draw some conclusions about the existence and stability of solitary plane waves. To achieve this, we finally investigate the amplitude equation, which now reads

$$\dot{A}=\tfrac{8}{15}[\epsilon-\tilde{k}^2+\tfrac{4}{7}(12c_4-5)A^2]A^3 . \tag{5.22}$$

In general, the coefficients entering the amplitude equation can be read off from the energy equation following from (5.3):

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}u^2dx\,dy &= \tilde{F}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx\,dy[\epsilon u_x^2-u_y^2-u_{xx}^2+2c_4uu_x^2\\ &\quad+(c_3-2c_5)uu_xu_{xx}] ,\end{aligned} \tag{5.23}$$

where the last term vanishes if u is an even function of x (which is the case here). The first term arises from the antidiffusion in the downstream direction and acts as an energy source in the unstable case, the second term is due to the diffusion into the transverse direction and has a stabilizing effect as well as the third term, which describes the effect of dissipation. The remaining terms represent nonlinear corrections and are either stabilizing or destabilizing (cf. a similar discussion in [24]). It is therefore not surprising that the only coefficient out of the c_i that has survived the above procedure is c_4. Moreover, with a little algebra it can be shown that $12c_4-5>0$ holds in the same range in which a_1 stays positive. This is important for stationary solutions of (5.22), for which

$$\tfrac{4}{7}(12c_4-5)A_0^2=\tilde{k}^2-\epsilon,\quad \dot{x}_{01,0}=4A_0^2(c_1\tilde{k}^2+4c_2A_0^2) . \tag{5.24}$$

Therefore, as long as the base state is stable ($\epsilon=-1$), steady state solutions to (5.22) exist for all transverse wave numbers $\tilde{k}^2\geq 0$, but they are all unstable and perturbations develop either to the base state or to other one- or two-dimensional patterns, e.g., periodic waves. On the other hand, if the base state is unstable ($\epsilon=+1$), (5.23) allows finite amplitude solutions for $\tilde{k}^2>1$ only, but these are again unstable. In addition, the amplitudes of those plane solitary waves whose propagation directions do not deviate too much from the vertical axis ($\tilde{k}^2<1$) grow without bound, such that the above approximation breaks down and one has to return to the full equations. Fortunately, the fully developed quasisteady plane waves are described by an ordinary differential equation, namely, (4.4), and can thus be studied without any approximation (at least qualitatively). This has been done in [9], where it has been shown that the interaction of periodic and stationary solutions produces solitary waves in the form of homoclinic connections. Indeed, the traveling directions of these periodic waves have been found to be confined around the vertical axis, corresponding to the (now scaled) range $\tilde{k}^2<1$ in which the solitary waves become singular according to (5.22) [25].

For the sake of completeness we notice that for $c_4<\frac{5}{12}$ solitary waves would exist only if the base state is unstable and $\tilde{k}^2<1$, and that this band of waves would be stable (in the frame of one-soliton perturbations).

VI. DISCUSSION

For a commonly used two-fluid model of fluidized beds a nonlinear voidage wave equation has been derived by relating the strength of the instability to a small Froude number perturbation. The equation covers the full range of the density and viscosity ratios of the two phases, but depends only weakly on these parameters; it can easily be extended to include voidage-dependent viscosity coefficients. From this reduced equation, an amplitude equation for the perturbation of a Korteweg–de Vries soliton to solitary plane waves has been derived, whose solutions either are unstable or become singular in finite time, signaling the breakdown of the approximation. That the singularity is artificial is obvious from a comparison to the equations describing fully developed traveling plane waves [9]. The range of critical transverse wave numbers coincides with that allowing for Hopf bifurcations from the uniform state, which can be concluded in a straightforward manner from the scaled equation (5.3). The scaled version of (4.3) relates the wave speed to the longitudinal and transverse wave numbers via

$$\begin{aligned}\tilde{\omega}&=\lambda^2+\tilde{F}(c_1k^2-c_2\lambda^4)+O(\tilde{F}^2) ,\\ -\epsilon\lambda^2&+k^2+\lambda^4=O(\tilde{F}) ,\end{aligned} \tag{6.1}$$

so that bifurcations to periodic solutions are possible for positive ϵ only (i.e., $\epsilon=+1$), with the transverse wave number restricted to $k^2<\frac{1}{4}$ (at $k^2=\frac{1}{4}$ a degeneracy occurs). Simultaneously, the longitudinal wave number is confined to $\lambda^2\leq 1$, whereby at $\lambda=0$ a stationary and at $\lambda^2=1$ a Hopf bifurcation to a branch of one-dimensional vertically traveling waves takes place. Note that the distance of these two limiting points has been scaled to $|\epsilon|=1$. Eliminating λ, the expression

$$\tilde{\omega}=\frac{1-c_2\tilde{F}}{2}(1\pm\sqrt{1-4k^2})+\tilde{F}(c_1+c_2)k^2 \tag{6.2}$$

shows again (see [8]) that the two limiting points are connected by a continuous branch of Hopf bifurcation points to two-dimensional traveling waves. By symmetry, three (probably unstable) solution branches emanate from each of these points, one vertically traveling wave $\phi(x-\omega t,y)$ with a nontrivial transverse structure [26] and a pair of obliquely countermoving plane waves $\phi(x-\omega t\pm\tilde{k}y)$. Due to (6.1) and the relation $k=\tilde{k}\lambda$, such plane waves exist in the range $\tilde{k}^2<1$ only, highlighting the intimate connection between plane periodic and solitary waves. We recall that in the full system the branch of periodic waves (for a given $\tilde{k}$) must either terminate in an infinite period bifurcation, i.e., develop to a solitary wave, or return to the uniform state via another Hopf bifurcation; the latter case is excluded here. A similar behavior might be conjectured for the two-dimensional vertically traveling waves.

It should be mentioned that Komatsu and Hayakawa [11] have derived a similar reduced wave equation for the case $\delta=\nu=0$, and with the assumption of a constant, vertically directed, volumetric mean flow [$W\equiv W_0 Y$ in our notation, cf. (3.32)]. Because they do not weigh the strength of the instability with the Froude number, which occurs explicitly in the equations, they obtain a simpler perturbation of the Korteweg–de Vries equation; namely, all except one of the terms with coefficients c_i appearing in (5.3) are absent in their formulation. The exception regards the nonlinear diffusion term $c_4(u^2)_{xx}$, which has been shown above to govern the soliton perturbation. Therefore the amplitude equation is the same in both cases; the 1D version has been derived by Hayakawa [21], but he gives no value for c_4. However, the terms missing in the approximation in [11] are important for the description of periodic waves as is already seen from the relations (6.1) and (6.2) following from the linear part of (5.3). We believe, therefore, that our equation gives a better approximation of the fluidized bed system near the stability limit, although the reduced equation presented in [11] has the advantage that it is not restricted to small Froude numbers.

Our results are in agreement with other findings in one space dimension [11,12,21,27]. In a numerical simulation of the *original* 1D equations with periodic boundary conditions it has been observed in [11] that an initial periodic disturbance of the uniform state first evolves like solitary waves but then settles down to a steadily moving wave train meaning that the latter finite-wavelength pattern represents the stable solution for the considered parameter values. Another simulation in the same paper shows the characteristic evolution of an initial two-soliton. Unfortunately, no time scale is provided for this process, so that it remains unclear whether a finite-time singularity could develop or not. A blow-up seems indeed to appear in a third simulation far away from the stability limit, in which the voidage locally becomes ~ 1.

The same equation as in [12] has been investigated in [27] in the context of Rayleigh-Bénard convection, leading of course to the same amplitude equation (5.22) without the $\tilde{k}^2$ contribution (see also [21]). In addition, the authors of [27] have considered the stability of the 1D periodic solutions and the numerical behavior for small and large box sizes admitting single or multiple pulses, respectively. If the system is small, such that there is only one unstable wave number, namely, $\lambda=1$ (scaled), and the bifurcation is subcritical, they did not find any stable finite-amplitude solution. In the case of a large system they observed the saturation of the amplitude of the fastest growing pulse and the evolution towards a train of pulses for a given value of the perturbation parameter ($\hat{=}\tilde{F}$) and values $c_4<c_{\tilde{F}}$. For values c_4 larger than the threshold value $c_{\tilde{F}}$, which approaches $\frac{5}{12}$ as $\tilde{F}$ is decreased, the size of the pulse grew unbounded; this process could involve the collision between two pulses.

Although our focus has been on fluidized beds, we are not restricted to this particular application. Similar equations are used to describe other multiphase flows, e.g., polymer-solvent mixtures [28], roll waves down an open inclined channel [29], and traffic jams [30]. All these models fall into the general class of wave-hierarchy problems with dissipation, a mathematical treatment of which has been presented in [31]. Due to this common feature of all of the above-mentioned models, and because the reduced wave equation derived in this paper is fairly complete in the low-order terms, it should describe, in one or the other form, the behavior near the stability limit in all the models. Finding exact or approximate solutions of these reduced wave equations would give valuable insight into the formation of two-dimensional patterns in two-phase flow systems.

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APPENDIX A: VERIFICATION OF THE NONLINEAR TERMS

To check the nonlinear terms of the reduced wave equation (3.35) we apply the same but much simpler asymptotic analysis to the ordinary differential equation (4.4) describing plane voidage waves:

$$f_1\phi''=f_2(\phi')^2+f_3\phi'+f_4\ , \tag{A1}$$

where

$$f_1=(1+k^2)\phi(1-\phi)\times[\omega(1-\phi_0)\phi^3-\nu(1-\omega)\phi_0(1-\phi)^3]/R\ , \tag{A2}$$

$$f_2=-2(1+k^2)[\omega(1-\phi_0)\phi^4+\nu(1-\omega)\phi_0(1-\phi)^4]/R\ , \tag{A3}$$

$$f_3=-\phi(1-\phi)[(1+k^2)\phi^3(1-\phi)^2G(\phi)+\omega^2(1-\phi_0)^2\phi^3+\delta\phi_0^2(1-\omega)^2(1-\phi)^3]\ , \tag{A4}$$

$$f_4=\phi^2(1-\phi)^2\{B(\phi)[\phi_0(1-\omega)+\phi(\omega-\phi_0)]-(1-\delta)\phi^2(1-\phi)^2\}/F\ . \tag{A5}$$

Of course, the uniform state is a solution for all ω, i.e.,

$$f_4(\phi_0,\omega)=0\quad[\Longrightarrow B_0=(1-\delta)\phi_0(1-\phi_0)]\ . \tag{A6}$$

Inserting the double expansion of

$$f_i(\phi,\omega)=f_i(\phi_0+F\tilde{\phi},d-F\tilde{\omega})=f_i^0+F(f_i'^0\tilde{\phi}-f_{i,\omega}^0\tilde{\omega})+\cdots \tag{A7}$$

into (A1) yields in $O(F^{-1})$ and $O(1)$

$$f_4^0\equiv f_4(\phi_0,d)=0,\quad f_{4,\omega}^0\equiv\partial_\omega f_4(\phi_0,d)=0\ ,\qquad f_4'^0\equiv\partial_\phi f_4(\phi_0,d)=0\ , \tag{A8}$$

respectively; the first two relations are in accordance with (A6), while the last determines the value of d. Omitting the tildes, we get in $O(F)$

$$f_1^0\phi''-f_3^0\phi'+f'^0_{4,\omega}\phi\omega-f''^0_4\phi^2/2$$
$$=-F(-f^0_{1,\omega}\omega\phi''+f'^0_1\phi\phi''-f^0_2(\phi')^2+f^0_{3,\omega}\omega\phi'$$
$$-f'^0_3\phi\phi'+f''^0_{4,\omega}\omega\phi^2/2-f'''^0_4\phi^3/6)$$
$$+O(F^2)\ . \tag{A9}$$

Solving the $O(1)$ part of (A9) for $\omega\phi$ and using this expression to eliminate ω from the $O(F)$ part leads to

$$f'^0_{4,\omega}\omega\phi-f''^0_4\phi^2/2+f^0_1\phi''=f^0_3\phi'-F\hat{S}\ , \tag{A10}$$

with

$$f'^0_{4,\omega}\hat{S}=f^0_1f^0_{1,\omega}\phi^{(4)}-(f^0_1f^0_{3,\omega}+f^0_{1,\omega}f^0_3)\phi'''-f^0_{1,\omega}f''^0_4(\phi^2)''/2+(f'^0_1f^0_{4,\omega}-f^0_1f''^0_{4,\omega}/2)\phi\phi''$$
$$+(f''^0_4f''^0_{4,\omega}/4-f'^0_{4,\omega}f'''^0_4/6)\phi^3-f^0_2f'^0_{4,\omega}(\phi')^2+f^0_{3,\omega}f^0_3\phi''+(f^0_{3,\omega}f''^0_4-f'^0_3f^0_{4,\omega}+f^0_3f''^0_{4,\omega}/2)(\phi^2)'/2\ . \tag{A11}$$

Stating some of the coefficients gives more insight:

$$f'^0_{4,\omega}=\phi_0^2(1-\phi_0)^2B_0,\quad f^0_1=(1+k^2)a_2f'^0_{4,\omega}\ ,$$
$$f^0_{1,\omega}=(1+k^2)b_3f'^0_{4,\omega}\ ,$$
$$f^0_2=-2(1+k^2)(b_6-a_1b_3)f'^0_{4,\omega}\ , \tag{A12}$$
$$f^0_{3,\omega}=-2b_2f'^0_{4,\omega}\ ,$$
$$f^0_3=-\phi_0^3(1-\phi_0)^3[(1+k^2)\phi_0G_0+\phi_0d^2+\delta(1-\phi_0)(1-d)^2]$$
$$=F(K^2b_1-c_0)f'^0_{4,\omega}\ .$$

According to our scaling, k^2 and f^0_3 are $O(F)$ quantities, so that their contributions to $\hat{S}$ vanish. The remaining equation has to be compared with (4.5), which is the version of (3.35) for vertically and oblique traveling plane waves. Replacing ϕ'^2 by $(\phi^2)''/2-\phi\phi''$ in (A11) shows that the structure of the two equations is exactly the same, as claimed in Sec. IV.

APPENDIX B: TRANSVERSE SOLITON PERTURBATIONS

Here we argue that the amplitude of the one-soliton perturbation must not depend on the transverse variable. In order to transform the wave equation (5.3) into a system moving with the anticipated solitary wave, we generalize the usual one-dimensional ansatz (see, e.g., [14,21], and the references therein) and introduce

$$\theta=A(\tilde{\tau},y)[x-\Omega(\tilde{\tau},y)-x_0(\tilde{\tau},y)]\ ,$$

with $\tilde{\tau}=\tilde{F}\tau$ and $\partial_{\tau}\Omega=4A^2/\tilde{F}$ in order to match the 1D soliton. Shifting a pure function of y into x_0, we may write $\Omega=4\int A^2d\tilde{\tau}/\tilde{F}$. Hence the operators become

$$\partial_x\to A\partial_\theta,\quad \partial_\tau\to\partial_\tau-4A^3\partial_\theta+\tilde{F}\partial_{\tilde{\tau}}+\tilde{F}\left[\frac{\dot{A}}{A}\theta-A\dot{x}_0\right]\partial_\theta\ ,$$
$$\partial_y\to\partial_y+\left[\frac{A_y}{A}\theta-Ax_{0,y}\right]\partial_\theta-A\Omega_y\partial_\theta\ , \tag{B1}$$

where we denote the derivative with respect to $\tilde{\tau}$ with an overdot. Then (5.3) assumes the form

$$\partial_\tau u-4A^3\partial_\theta u+A\partial_\theta u^2+A^3\partial_\theta^3u$$
$$=-\tilde{F}\left[\dot{u}+\frac{\dot{A}}{A}\theta\partial_\theta u-A\dot{x}_0\partial_\theta u+N(u)\right]\ , \tag{B2}$$

where $N(u)$ represents the transformed operator on the right-hand side of (5.3). It is decisive that N contains terms proportional to Ω_y, Ω_{yy}, and Ω_y^2 stemming from the second-order y derivatives. However, Ω_y and Ω_{yy} are of the order of $\tilde{F}^{-1}$, while $\Omega_y^2\sim O(\tilde{F}^{-2})$. Performing the expansion of u with respect to the Froude number, $u=u_0+\tilde{F}u_1+\tilde{F}^2u_2+\cdots$, leads in $O(\tilde{F}^{-1})$ to

$$\left[4A\int\partial_yA^2d\tilde{\tau}\right]^2\partial_\theta^2u_0=0\ . \tag{B3}$$

Assuming $\int\partial_yA^2d\tilde{\tau}\neq0$ yields $u_0=u_0(\tau,\tilde{\tau},y)$, since u should be bounded for $\theta\to\pm\infty$. Similarly, one obtains in the next order

$$\left[4A\int\partial_yA^2d\tilde{\tau}\right]^2u_1=f(\tau,\tilde{\tau},y)\quad\text{and}\quad\partial_\tau u_0=0\ . \tag{B4}$$

One could go on and find further restrictions on u_0 and u_1, or claim $u\to0$ as $\theta\to\infty$ (cf. Sec. V B), upon which u_0,u_1, and also u_2 would vanish. But it is already obvious that one can never get the KdV soliton in leading order, unless the above assumption $\partial_y\int A^2d\tilde{\tau}\neq0$ is abandoned.

We notice that Barcilon and Lovera [23] tried a similar perturbation of a one-dimensional solitary wave solution of a somewhat different wave equation. To leading order they obtained two solvability conditions for a phase shift perturbation $\theta^{(0)}$ (corresponding to our x_0), which they viewed as incompatible. Taking them literally, however, leads to $\theta^{(0)}_{yy}=\theta^{(0)}_{\tilde{\tau}\tilde{\tau}}=0$, showing that a perturbation to nearby oblique traveling plane waves is still possible (the dependency on the slow time scale is more restrictive than in our case). Anyway, their investigation indicated that the 1D solitary wave is unstable to two-dimensional perturbations. On the other hand, they showed that their equation possesses isotropic solutions, which depend on $(x-\omega t)^2+y^2$ only (this remains doubtful in our case). Obviously, such solutions cannot be reached by a perturbation of one-dimensional single solitary waves.

[1] *Fluidization,* edited by J. F. Davidson, R. Clift, and D. Harrison (Academic, London, 1985). See in particular the contributions by J. F. Richardson and R. Jackson.

[2] G. M. Homsy, R. Jackson, and J. R. Grace, J. Fluid Mech. **236**, 477 (1992).

[3] A. K. Didwania and G. M. Homsy, Int. J. Multiphase Flow **7**, 563 (1981); J. Fluid Mech. **122**, 433 (1982).

[4] S. K. Garg and J. W. Pritchett, J. Appl. Phys. **46**, 4493 (1975).

[5] A. Kluwick, Z. Angew. Math. Mech. **63**, 161 (1983); D. G. Crighton, in *Nonlinear Waves in Real Fluids,* edited by A. Kluwick (Springer, Berlin, 1991), p. 83.

[6] D. J. Needham and J. H. Merkin, Z. Angew. Math. Phys. **37**, 322 (1986).

[7] G. H. Ganser and D. A. Drew, SIAM J. Appl. Math. **47**, 726 (1987); Int. J. Multiphase Flow **16**, 447 (1990).

[8] M. F. Göz, J. Fluid Mech. **240**, 379 (1992).

[9] M. F. Göz, Physica D **65**, 319 (1993).

[10] M. F. Göz, in *Instabilities in Multiphase Flows,* edited by G. Gouesbet and A. Berlemont (Plenum, New York, 1993), p. 251.

[11] T. S. Komatsu and H. Hayakawa, Phys. Lett. A **183**, 56 (1993).

[12] S. Sasa and H. Hayakawa, Europhys. Lett. **17**, 685 (1992); H. Hayakawa, T. S. Komatsu, and T. Tsuzuki, Physica A **204**, 277 (1994).

[13] G. K. Batchelor, J. Fluid Mech. **157**, 359 (1993).

[14] S. E. Harris and D. G. Crighton, J. Fluid Mech. **266**, 243 (1994).

[15] Fluidizing lead shot by water keeps δ small and gives rise to bubbly structures, see D. Harrison, J. F. Davidson, and J. W. de Kock, Trans. Inst. Chem. Eng. **39**, 202 (1961).

[16] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields* (Springer, Heidelberg, 1991).

[17] M. F. Göz, Phys. Lett. A **200**, 355 (1995).

[18] B. Glasser, I. G. Kevrekidis, and S. Sundaresan (unpublished).

[19] M. F. Göz, Z. Angew. Math. Mech. **75SI**, S381 (1995); J. Fluid Mech. (to be published).

[20] T. B. Anderson and R. Jackson, Ind. Eng. Chem. Fundam. **7**, 12 (1968).

[21] H. Hayakawa, Teor. Mat. Fiz. **99**, 309 (1994).

[22] This scaling also suggests that the interparticle force depends on the relative velocity of the two phases as well, as is indeed claimed in more sophisticated multiphase flow models. Here we will not take account of this additional complication, which is needed only at high Reynolds numbers.

[23] V. Barcilon and O. M. Lovera, J. Fluid Mech. **204**, 121 (1989).

[24] A. A. Nepomnyashuchy and M. G. Velarde, Phys. Fluids **6**, 187 (1994).

[25] As has been mentioned in the Introduction, in a small parameter region it may happen that the propagation directions are bounded away from the vertical axis, but this case is not covered within the weakly unstable limit; cf. also the remark at the end of Sec. II.

[26] This solution further splits up into a symmetric and an antisymmetric one with respect to y; but since they are related by a phase shift by $\pi/2$, they are counted as one single branch.

[27] B. Janiaud *et al.*, Physica D **55**, 269 (1992).

[28] M. Doi and A. Onuki, J. Phys. (France) II **2**, 1631 (1992).

[29] D. J. Needham and J. H. Merkin, Proc. R. Soc. London Ser. A **394**, 258 (1984).

[30] B. S. Kerner and P. Konhäuser, Phys. Rev. E **48**, R2335 (1993); **50**, 54 (1994).

[31] M. F. Göz, Math. Methods Appl. Sci. **17**, 787 (1994).