PHYSICAL REVIEW E VOLUME 52, NUMBER 4 OCTOBER 1995

Block-analyzing method in cellular automata

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(Received 13 April 1995)

In this paper we introduce the block-analyzing method to describe the evolution of cellular automata. A certain kind of one-dimensional cellular automaton is decomposed into a sequence of several particular kinds of blocks standing consecutively. We can then view the evolution of the whole automaton in terms of the evolution of these blocks. We show that this method is useful in analyzing one-dimensional cellular automata.

PACS number(s): 05.90.+m, 02.90.+p, 01.70.+w

The subject of cellular automata (CA) has attracted much attention in the past decade [1–10]. Theoretically, cellular automata may be used as simple models for a wide variety of physical, biological, and computational systems. Analysis of general features of their behaviors may therefore yield general results on the behaviors of many complex systems and may perhaps ultimately suggest generalizations of the laws of thermodynamics appropriate for systems with irreversible dynamics. Wolfram had used cellular automata to discuss the undecidability and intractability in theoretical physics [2]. Cellular automata have also found a great diversity of application. Some examples are tapestry designs, simple models for self-organizing phenomena in physical and chemical systems, patterns of flow in turbulent fluids, and biological systems [4]. Apparently, the evolution of the cellular automaton is a key problem in this field of study and much work had been done [1–10]. Wolfram first examined the behavior of one-dimensional cellular automata extensively mainly by computer simulation [1]. In another paper [3] Wolfram discussed the self-organizing behavior in cellular automata as a computational process and he used formal language theory to extend the dynamical systems theory description of cellular automata. Martin, Odlyzko, and Wolfram treated this problem in a rather mathematical way. In their paper [5], algebraic techniques are used to give an extensive analysis of the global properties of a class of finite cellular automata. These cellular automata exhibit the simplifying feature of "additivity." The configurations of such cellular automata satisfy an additive superpositon principle that allows a natural representation of the configurations by characteristic polynomials. The time evolution of the configurations is represented by iterated multiplication of characteristic polynomials (generating function) [6,7] by fixed polynomials. The complete structure of state transition diagram is derived in terms of algebraic and number theoretical quantities. Urías discussed the cellular automata in arithmetic representation. One- and two-dimensional cellular automata were described in terms of arithmetic relations and were interpreted as finite state machines [8,9]. Boccara discussed this problem by using translation-invariant local subjective mappings [10].

In this paper we introduce a method to the treatment of one-dimensional cellular automata in the case of $r = 1$ and $k = 2$ [1]. A one-dimensional cellular automaton is decomposed into a sequence of certain kinds of blocks standing consecutively. We can view the evolution of the automata in terms of the evolution of these blocks. It is shown that this method can help us understand the evolution of this kind of cellular automaton.

First, let us begin with the general condition. A one-dimensional cellular automaton consists of a line of sites, with each site taking on a finite set of possible values, updated in discrete time steps according to a deterministic rule involving a local neighborhood of sites around it. The value of sites i at time step t is denoted as S_i and is an integer chosen from the set

$$S = \{0, 1, 2, \ldots, k-1\}. \tag{1}$$

At each time step each site's value is updated according to the values of a neighborhood of $2r + 1$ sites around it by a local rule:

$$\tilde{S}_i = f(S_{i-r}, S_{i-r+1}, \ldots, S_{i+r}), \tag{2}$$

where $\tilde{S}_i$ represents the value of site i at time step $t + 1$ and $S_j, j = i-r, i-r+1, \ldots, i+r$ represents the value of site j at time step t. There are total $N(k, 2r+1) = k^{k^{2r+1}}$ kinds of different local rules according to Eq. (1) and Eq. (2).

Define a $(2r + 1)$-dimensional vector that takes the form:

$$\vec{m} = (m_1, m_2, \ldots, m_{2r+1}),$$

where $m_j, j = 1, 2, \ldots, 2r + 1$ is an arbitrary integer chosen from the set S. There are k^{2r+1} total such vectors and we denote the set of such vectors as M.

First, we point out that every local rule defined by Eq. (2) can be substituted by a polynomial which takes the following form:

$$\tilde{S}_i = \sum_{j=1}^{q} d_j P_j \quad (\mathrm{mod} k), \tag{3}$$

where $q = k^{2r+1}$ and d_j is an arbitrary integer chosen from set S. P_j has the form

$$P_j = S_{i-r}^{m_{j1}} S_{i-r+1}^{m_{j2}} \cdots S_{i+r}^{m_{j2r+1}}, \tag{4}$$

where $\vec{m_j} = (m_{j1}, m_{j2}, \ldots, m_{j2r+1}) \in M, j = 1, 2, \ldots, q$ and different j correspond to different $\vec{m_j}$. In Eq. (4) $m_{j1}, m_{j2}, \ldots, m_{j2r+1}$ represents the power of $S_{i-r}, S_{i-r+1}, \ldots, S_{i+r}$, respectively. According to Eqs. (3) and (4) there are $k^{k^{2r+1}}$ total different independent polynomials which take the form of Eq. (3). They correspond to the $k^{k^{2r+1}}$ kinds of local rules defined in Eq. (2). We call $P_j, j = 1, 2, \ldots, q$ the independent factors of the automaton and we can see that every local rule is nothing but the linear combination of these factors. In the case of $r = 1, k = 2$ there are eight total independent factors as defined above. They are

$$P_1 = 1,\ P_2 = S_{i-1},\ P_3 = S_i,\ P_4 = S_{i+1},\ \ P_5 = S_{i-1}S_i,$$
$$P_6 = S_iS_{i+1},\ P_7 = S_{i-1}S_{i+1},\ P_8 = S_{i-1}S_iS_{i+1}. \qquad (5)$$

For the case of $r = 1, k = 2$, we consider a line with N sites. We apply the periodic boundary condition: $S_{i+N} = S_i$, which makes the line form a loop. We first decompose the structure of the loop into a sequence comprised of six different kinds of "building blocks" structures, each of which consists of several consecutive sites. Let $\mathbb{Z}^+$ be the set of the positive integers, and we define

$$\begin{aligned} A_l &: \ 00\cdots0, \ \ l \geq 2, \\ B_l &: \ 11\cdots1, \ \ l \geq 2, \\ C_l &: \ 0101\cdots01, \ \ l = 2n, \ \ n \in \mathbb{Z}^+, \\ D_l &: \ 0101\cdots010, \ \ l = 2n-1, \ n \in \mathbb{Z}^+, \\ E_l &: \ 1010\cdots10, \ \ l = 2n, \ \ n \in \mathbb{Z}^+, \\ F_l &: \ 1010\cdots101, \ \ l = 2n-1, \ \ n \in \mathbb{Z}^+, \end{aligned} \qquad (6)$$

where the subscript l denotes the length of the block, i.e., the number of sites contained in the block. Many blocks of the same kind with different length l can exist in a loop. Then the loop comprised of N sites can be viewed as a loop of many consecutive blocks which take the forms of Eq. (6). We must point out that consecutive blocks like $A_{l_1}A_{l_2}$ have no meaning because it can be coalesced into a single block $A_{l_1+l_2}$. According to the definition of the block structures in Eq. (6) we obtain Table I.

For instance, the block sequence $BCAEBDBCAFAE$ is a possible loop. Note that the block E on the right end of the line is connected to the block B on the left end of the line according to the periodic boundary condition and we can see that this connection satisfies the rules described in Table I. One may notice that the block assignment of a certain configuration may not be unique, but we can make the assignment unique by the following rule of decomposition. One should first assign as many A_l and B_l, $l \geq 2$, as possible. That means that as long as there exists in a loop consecutive 0's or 1's whose length ≥ 2 one should assign them as A_l or B_l. Then the remaining part of the loop is some sequences comprised of alternating 0's and 1's and these sequences can be assigned according to Eq. (6). Thus the block assignment of any configuration is unique.

Now we apply the eight independent factors in Eq. (5) to these six kinds of building blocks. The results are shown in Table II.

In this way we can simplify the evolution analysis by only examining the evolution of the blocks. In the following text we use this method to analyze three certain kinds of automata. In Wolfram's formulation their codes are 232, 2, and 20, respectively [1].

The first rule's code 232 is also known as "voting" and the rule can be written in the the form of Eq. (3):

$$\tilde{S}_i = S_{i-1}S_i + S_iS_{i+1} + S_{i-1}S_{i+1} \pmod 2.$$

Applying Table II we get Table III.

In order to go on with the next step's evolution, we must now redecompose the new loop obtained from the first step's evolution. According to Table I, only B blocks can appear immediately to the left side of C_l and only A blocks can appear immediately to the right side of C_l. We consider the following block sequence's evolution:

$$B_{l_1}C_{l_2}A_{l_3} \to B_{l_1}E_{l_2}A_{l_3}.$$

Because E_l has the structure $1010\cdots10$, we can coalesce the first number 1 in block E_{l_2} into the left block B_{l_1} and coalesce the last number 0 in block E_{l_2} into the right block A_{l_3}; then we get the sequence in the new loop: $B_{l_1+1}C_{l_2-2}A_{l_3+1}$. We can get exactly the same results for blocks D_l, E_l, and F_l. This means that in every time step all the blocks of type C, D, E, and F reduce their length by 2. The automaton's final structure evolving from an arbitrary initial state will be the consecutive $ABAB\ldots$ sequence.

For the second rule, which has the code number 2, Wolfram said that it will evolve into a "chaotic" state [1]. The rule can be written as follows:

TABLE I. Rule of possible block sequences.

Block	Possible kinds of blocks immediately to the left side of the block	Possible kinds of blocks immediately to the right side of the block
A	$B, C, F,$	B, E, F
B	A, D, E	A, D, C
C	B	A
D	B	B
E	A	B
F	A	A

TABLE II. First step results of the six building blocks by the eight independent factors in Eq. (5).

Factor/structure	A_l	B_l	C_l	D_l	E_l	F_l
$P_1 = 1$	A_l	B_l	C_l	D_l	E_l	F_l
$P_2 = S_{i-1}$	$1A_{l-1}$	$0B_{l-1}$	E_l	F_l	C_l	D_l
$P_3 = S_i$	A_l	B_l	C_l	D_l	E_l	F_l
$P_4 = S_{i+1}$	$A_{l-1}1$	$B_{l-1}0$	E_l	F_l	C_l	D_l
$P_5 = S_{i-1}S_i$	A_l	$0B_{l-1}$	A_l	A_l	A_l	A_l
$P_6 = S_iS_{i+1}$	A_l	$B_{l-1}0$	A_l	A_l	A_l	A_l
$P_7 = S_{i-1}S_{i+1}$	A_l	$0B_{l-2}0$	E_l	F_l	C_l	D_l
$P_8 = S_{i-1}S_iS_{i+1}$	A_l	$0B_{l-2}0$	A_l	A_l	A_l	A_l

TABLE III. First step results of the six building blocks by rule 232.

	A_l	B_l	C_l	D_l	E_l	F_l
$S_{i-1}S_i$	A_l	$0B_{l-1}$	A_l	A_l	A_l	A_l
S_iS_{i+1}	A_l	$B_{l-1}0$	A_l	A_l	A_l	A_l
$S_{i-1}S_{i+1}$	A_l	$0B_{l-2}0$	E_l	F_l	C_l	D_l
+(mod2)	A_l	B_l	E_l	F_l	C_l	D_l

$$\tilde{S_i} = (1 - S_{i-1})(1 - S_i)S_{i+1} \pmod 2$$
$$= S_{i+1} + S_{i-1}S_{i+1} + S_iS_{i+1}$$
$$+S_{i-1}S_iS_{i+1} \pmod 2.$$

Similar to the treatment of the rule "voting" we can get the following table according to Table II.

	A_l	B_l	C_l	D_l	E_l	F_l
$\tilde{S_i}$	$A_{l-1}1$	A_l	A_l	A_l	A_l	A_l

We can easily see that after the first step's evolution the system will evolve into the structure $AF_1AF_1\ldots$. Then after the first step the evolution of the automata circle can be viewed as rotating one site to the left at each consecutive time step. Because of the arbitrary choice of the initial length of block A the evolution of the automata may appear chaotic. But from the analysis above we can see that the evolution of the automata can be clearly predicted only after the first step. We could say that rule 2 is relatively simple.

For the third example we examine the rule 20, which Wolfram considered "more complicated." The rule can be written as follows:

$$\tilde{S_i} = S_{i-1}(1 - S_i)(1 - S_{i+1})$$
$$+(1 - S_{i-1})S_i(1 - S_{i+1}) \pmod 2$$
$$= S_{i-1} + S_i + S_iS_{i+1} + S_{i-1}S_{i+1} \pmod 2.$$

Similarly, we can get the following table according to Table II.

	A_l	B_l	C_l	D_l	E_l	F_l
$\tilde{S_i}$	$1A_{l-1}$	A_l	C_l	D_l	E_l	F_l

For the convenience of further discussion we list the distribution of blocks after the fist time step in Table IV.

From the table we can get some characteristics of the system after a sufficient long period of time:

(1) B_l can exist only as B_2.

(2) Let us look at the evolution of E_l. It has three different evolutions: (a): $E_l \to F_{l-1} \to E_{l-2} \to \cdots$; (b): $E_l \to F_{l+1} \to \cdots$; (c): $E_l \to C_{l-1} \to E_{l-2} \to \cdots$. $E_l \to F_{l+1}$ happens in the structure $(B_2)(A_2)E_l(B)$: $B_2A_2E_lB_2 \to A_{l'}F_{l+1}A_{l''}(l', l'' \geq 3) \to A_{l'-1}E_lB_2$. So the length of E_l has two trends: remaining constant or decreasing. After a long period of time, if block E_l does not reduce to F_1 it must oscillate like $E_l \to F_{l+1} \to E_l$ or $E_l \to F_{l-1} \to E_l$.

(3) Similar to (2), block $F_l(l \geq 3)$ can only oscillate like $F_l \to E_{l+1} \to F_l$ or $F_l \to E_{l-1} \to F_l$.

(4) According to (2) and (3), after a long period of time, if blocks E, F do not vanish, they must oscillate between each other. The blocks E, F are stable and this determines that block A immediate to the left side of blocks E, F can only take the forms of A_2 or A_3. This further confirms that the final forms of block A can only take the forms of A_2 or A_3 because if the block immediate to the right side of A is B the B block will be soon "swallowed" by the A block and the A block will sooner or later encounter block E, F. It is worth remembering that in this case block C, D does not exist. If blocks $E, F(F_l, l \geq 3)$ do vanish after a long period of time the final structure of the system can only take the form of $F_1A_3B_2$ and this structure evolves as $F_1 \to B_2 \to F_1 \to B_2$. At a time step the typical configuration is as follows: $\cdots AF_1AF_1AB_2AF_1AB_2AB_2\cdots$.

In the following text we concentrate our discussion in the case that blocks E, F do not vanish after a long period of time (assuming $F_l, l \geq 3$):

(i) Because structure A_3B_2 generates A_4 and struc-

TABLE IV. The redistribution of blocks after the first step by rule 20.

Block	Structure before evolution	Structure after first step	Resulting structure
C_l	$\cdots(B)C_l(A)\cdots$	$\cdots(A)E_{l-2}(B_2)\cdots$	$E_{l-2}B_2$ for $l \leq 2$ E_l vanishes
D_l	$\cdots(B)C_l(B)\cdots$	$\cdots(A)F_{l-2}(A)\cdots$	F_{l-2} for $l < 2$ F_l vanishes
	$\cdots(A_{l'})E_l(B)\cdots l' \geq 3$	$\cdots(A_{l'-1})F_{l-1}(A)\cdots$	F_{l-1}
E_l	$\cdots(B)(A_2)E_l(B)\cdots$	$\cdots(A)F_{l+1}(A)\cdots$	F_{l+1}
	$\cdots(C,F)(A_2)E_l(B)\cdots$	$\cdots(B_2)C_l(A)\cdots$	B_2C_l
	$\cdots(A_{l'})F_l(A)\cdots l' \geq 3$	$\cdots(A_{l'-1})E_{l-1}(B_2)\cdots$	$E_{l-1}B_2$
F_l	$\cdots(B)(A_2)F_l(A)\cdots$	$\cdots(A)E_{l+1}(B_2)\cdots$	$E_{l+1}B_2$
	$\cdots(C,F)(A_2)F_l(B)\cdots$	$\cdots(B_2)D_l(B_2)\cdots$	F_{l-1}
	$\cdots(A_{l'})B_l(A)\cdots$	$\cdots A_{l'+l-1}\cdots$	$A_{l'+l-1}$
B_l	$\cdots(A_{l'})B_l(D,C)\cdots$	$\cdots A_{l'+l}\cdots$	$A_{l'+l}$
	$\cdots(D,E)B_l(D,C)\cdots$	$\cdots A_{l+2}\cdots$	A_{l+2}
	$\cdots(B)A_l(B)\cdots$	$\cdots(A)F_1(A)\cdots$	$F_1A_{l'}(l' > l)$
A_l	$\cdots(C,F)A_l(B)\cdots$	$\cdots(B_2)A_{l'}\cdots$	$B_2A_{l'}(l' > l)$
$l \geq 3$	$\cdots(C,F)A_l(E,F)\cdots$	$\cdots(B_2)A_{l-1}\cdots$	B_2A_{l-1}
A_2	$\cdots(B)A_2(B)\cdots$	$\cdots(A)F_1(A)\cdots$	F_1
	$\cdots(B)(C,F)A_2(E,F)\cdots$ referring to E_l, F_l		

ture $F_1A_2(E,F)$ generates C,D, structures $A_3B_2, F_1A_2(E,F)$ are forbidden.

(ii) Because structure $B_2A_3(E,F)$ generates $F_1A_2(E,F)$, structure $B_2A_3(E,F)$ is forbidden [referring to (i)]. Besides the several forbidden structures shown above, at any one time step the typical configuration of the system is the E,F sequence interpolated with A_2, A_3, B_2, F_1 according to the rules of Table II. After every two time steps the configuration of the system rotates exactly two sites to the left.

In summary, we have introduced a block-analyzing method to the treatment of a typical one-dimensional cellular automata, i.e., in the case of $r = 1$ and $k = 2$. A one-dimensional cellular automaton is decomposed into a sequence of certain kinds of blocks standing consecutively. We then view the evolution of the automaton in terms of the evolution of these blocks. From the examples that we examined above, we can see that the block-analyzing method is useful. Sometimes after the first step's evolution we can predict the final configuration of the cellular automata. Although our study is mainly concentrated on a typical kind of one-dimensional cellular automata, we hope that this method may give some insight to the treatment of higher-dimensional problems and we think that more work should be done to apply this method to some wider situations.

We thank Dr. H. F. Chau for helpful discussion and critical reading of the manuscript.

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