Lack of self-averaging in critical disordered systems

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We consider the sample to sample fluctuations that occur in the value of a thermodynamic quantity P in an ensemble of finite systems with quenched disorder, at equilibrium. The variance of P, V_P, which characterizes these fluctuations is calculated as a function of the systems' linear size l, focusing on the behavior at the critical point. The specific model considered is the bond-disordered Ashkin-Teller model on a square lattice [Phys. Rev. **64**, 178 (1943)]. Using extensive Monte Carlo simulations, several bond-disordered Ashkin-Teller models were examined, including the bond-disordered Ising model and the bond-disordered four-state Potts model. It was found that far from criticality all thermodynamic quantities which were examined (energy, magnetization, specific heat, susceptibility) are strongly self-averaging, that is $V_P \sim l^{-d}$ (where $d = 2$ is the dimension). At criticality though, the results indicate that the magnetization M and the susceptibility χ are non-self-averaging, i.e., $\frac{V_\chi}{\chi^2}, \frac{V_M}{M^2} \not\to 0$. The energy E at criticality is clearly weakly self-averaging, that is $V_E \sim l^{-y_v}$ with $0 < y_v < d$. Less conclusively, and possibly only as a transient behavior, the specific heat too is found to be weakly self-averaging. A phenomenological theory of finite size scaling for disordered systems is developed, based on physical considerations similar to those leading to the Harris criterion. Its main prediction is that when the specific heat exponent $\alpha < 0$ (α of the disordered model) then, for a quantity P which scales as l^ρ at criticality, its variance V_P will scale asymptotically as $l^{2\rho+\frac{\alpha}{\nu}}$. The theory is not applicable in the asymptotic limit ($l \to \infty$) to the bond-disordered Ashkin-Teller model where $\frac{\alpha}{\nu} = 0_+$. Nonetheless in the accessible range of lattice sizes we found very good agreement between the theory and the data for V_χ and V_E. The theory may also be compatible with the data for the variance of the magnetization V_M and the variance of the specific heat V_C, but evidence for this is less convincing.

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I. INTRODUCTION

How is the critical behavior affected by the introduction of disorder (usually dilution or bond-randomness) into a model? This question has been extensively studied [1] experimentally, analytically [2], and numerically [3] for quite some time now. Many studies concentrate on finding out to which universality class certain disordered models belong, e.g., calculating critical exponents. In this work we consider a different aspect of the same question. The measurement of any *density* of an extensive thermodynamic property P (e.g., $P = E, M, C_h$, or χ) in a disordered system may hypothetically be done in the following way. An ensemble of macroscopic disordered samples of size l is prepared; denote by x a sample with a particular random realization of the quenched disorder. Now in each sample x, $P_x(t)$ is measured over a long time interval, and $\overline{P_x}$, the average over time t is calculated. Close to the critical point the measurement of $\overline{P_x}$ will require long times due to large thermal fluctuations which will occur. In addition, since in every sample a different configuration of the quenched disorder is present, a different value for $\overline{P_x}$ will be measured. Next, the average of $\overline{P_x}$ over the ensemble $[\overline{P_x}]$ ($[\cdots]$ stands for an ensemble average over the different samples) is calculated and so is its variance

$$V_P = [\{\overline{P_x} - [\overline{P_x}]\}^2] \ . \tag{1}$$

Assume that the time interval of the measurement was long enough so that thermal fluctuations in $P_x(t)$ were averaged out perfectly and $\overline{P_x}$ may be considered to be exact. The question then rises: How will the variance V_P change as the critical temperature is approached or as the correlation length ξ [4] is increased? This question, which concerns the way in which disorder affects the behavior of systems near their critical point, is approached in this work using the framework of finite size scaling.

A common practice in Monte Carlo (MC) simulations is to examine the critical behavior by simulating a system at its critical temperature T_c^∞ and changing the lattice size l. According to the theory of finite size scaling [5] the lattice size l sets the scale of the correlation length in such a finite system. Thus the dependence of P on ξ in an infinite system close to criticality is substituted by dependence on l in a finite system at criticality. When a disordered system is considered, then many samples need to be simulated in order to obtain estimates of P which are averaged over the disorder. In this case, the question, which is the main theme of this work, would be: how does the effect of disorder on the sample to sample fluctuations in $\overline{P_x}$ change, as the lattice size l is increased at the critical temperature? Or how does V_P scale with l? This question is not only of theoretical interest in its own right, but also of practical interest for MC studies of critical disordered systems. If the relative variance $V_P/[P]^2$ decreases with increasing l, then the number of

samples needed to obtain [P] to a given accuracy goes down with increasing l. If, on the other hand, $V_P/[P]^2$ is independent of l, then the number of samples which need to be simulated is independent of l and the total amount of work rises very strongly with l.

The issue which we study in this work should not be confused with two closely related issues. The first is usually referred to as the property of self-averaging of additive (extensive) quantities in disordered systems [6]. Consider again the ensemble of macroscopic disordered samples of size l. The question is then whether

$$V_P/[P]^2 \to 0 \quad \text{as } l \to \infty \ . \tag{2}$$

If so, then the measurement of $\overline{P_x}$ in one very large sample x which occurs with reasonable probability will provide a good estimate of the ensemble average. This is very important for the comparison of theoretical work, where the configurational average is taken, with experiments, where only a large single sample is examined. As first argued by Brout [7], we may divide the sample x into n large subsamples (much larger than the correlation length ξ). If we assume that the coupling between neighboring subsystems is negligible, then the value of any density of an extensive quantity over the whole sample is equal to the average of the (independent) values of this quantity over the subsamples. Provided the probability distribution of the P's of the subsamples has a finite variance, then according to the central limit theorem the value of $\overline{P_x}$ is distributed with a Gaussian probability distribution around its mean $[\overline{P_x}]$. The square of the width of the Gaussian, V_P, is proportional to $\frac{1}{n} \sim l^{-d}$. In this case (2) is fulfilled, and P is called self-averaging.

The quantities which are studied here are all densities of extensive self-averaging quantities (far from criticality). Nonetheless, note that our question, as it was formulated for macroscopic samples ($l \gg \xi$), concerned the dependence of V on the correlation length ξ and not on the sample size l. On the other hand, as we will examine finite samples of size l at criticality where $\xi \sim l$, the Brout argument does not hold, since the average of P over neighboring subsamples may not be considered as independent. Thus at criticality there is no reason to expect that $V_P \sim l^{-d}$. An example for a phase transition, where sample to sample fluctuations result in non-self-averaging of certain quantities, is the percolation transition. It has been shown [8] that the resistive susceptibility and the conductivity are non-self-averaging at the percolation threshold.

A second related issue is that of self-averaging in homogeneous systems. This question concerns the thermal fluctuations in the value of a density P in a homogeneous system of size l. Define the thermal variance as $\sigma_T^2 = \langle (P - \langle P \rangle)^2 \rangle$, where $\langle \cdots \rangle$ denotes thermal or time averaging. The following notions (slightly modified) have been introduced by Milchev, Binder, and Heermann [9,10]: If $\sigma_T^2/\langle P \rangle^2 \to 0$ as $l \to \infty$, then P is self-averaging, otherwise it is said to exhibit lack of self-averaging. If

$$\sigma_T^2/\langle P \rangle^2 \sim l^{-d} \ , \tag{3}$$

then P is strongly self-averaging. If

$$\sigma_T^2/\langle P \rangle^2 \sim l^{-x_1} \quad \text{and} \quad 0 < x_1 < d \ , \tag{4}$$

then P is weakly self-averaging. When $l \gg \xi$ it was found [9,10] that averages of simple densities such as E, M are strongly self-averaging while quantities obtained from the fluctuations of these densities such as the specific heat C and susceptibility χ are non-self-averaging. At criticality the singular part of the energy E is weakly self-averaging while C, M, and χ exhibit lack of self-averaging. For example, $\langle M \rangle^2 \sim l^{-2\beta/\nu}$ and $\sigma_{T,M}^2 \sim \chi/l^d \sim l^{\gamma/\nu - d} = l^{-2\beta/\nu}$, so that M is non-self-averaging.

The issues of self-averaging in disordered systems and homogeneous systems concern the asymptotic behavior of the fluctuations due to disorder and the thermal fluctuations, respectively, as the system size is increased. While self-averaging in homogeneous systems at criticality has been addressed previously [9,10], this study involves the question of self-averaging in disordered systems at criticality. With the increase in the available computational power, a numerical investigation of the sample to sample fluctuations of thermodynamic quantities is nowadays feasible (whereas previously only calculation of the ensemble average, which is less demanding computationally, was feasible).

The particular model which is used here to study the question of the self-averaging of fluctuations due to disorder at criticality is the bond-disordered Ashkin-Teller model on a square lattice. Actually this work is based on further analysis of results which were obtained in a previous MC study [11] which aimed to determine the universality class of the model. The random-bond Ashkin-Teller model is particularly suitable for studying the effects of disorder on critical behavior. This is because the pure model possesses a line of critical points along which critical exponents vary continuously. In particular, the scaling exponent corresponding to randomness $\phi = (\alpha/\nu)_{\text{pure}}$ varies continuously and is positive. Thus, according to the Harris criterion [12], randomness is a relevant operator of varying strength, and the critical behaviour of the disordered model was indeed found to differ from that of the pure system. Our conclusion in the present work is that the effective exponent ratio α/ν of a disordered model plays a central role in determining the self-averaging of the fluctuations due to disorder at criticality. For the susceptibility, for instance, our results agree very well with a finite size scaling theory which we develop, according to which the relative variance of the susceptibility, $V_\chi/[\chi]^2$, scales as $l^{\alpha/\nu}$ at the critical temperature. This implies lack of self-averaging when $\alpha = 0$ (as is found for the random-bond Ashkin-Teller model) and only weak self-averaging for negative α. Our theory is successful also in describing, for models with weak disorder, the effect of crossover on the variance.

Our finite size scaling theory is very similar to the physical arguments that led to the Harris criterion [12], which was derived near the pure system fixed point. The difference is that we are assuming that similar considerations are valid near the disordered fixed point as well.

This work is organized as follows. In Sec. II we define

the random-bond Ashkin-Teller model (RBAT) and summarize its critical properties as found in a previous study [11]. In Sec. III we define various variances of thermodynamic quantities in disordered systems and explain their meaning. We explain how the "sample to sample variance" can be estimated from MC results. In Sec. IV we display our results for several bond disordered Ashkin-Teller models, including the four-state Potts and Ising models. We have measured the "sample to sample variance" at criticality for different lattice sizes and also for different degrees of disorder. We discuss some qualitative features of these results, such as the apparent lack of self-averaging and the dependence on the amount of disorder and on the specific heat exponent α. In Sec. V we develop a phenomenological finite size scaling theory for the "sample to sample variance." In Sec. VI we compare the predictions of the theory with the numerical results. We find good agreement in the case of the susceptibility and the energy, while the agreement in the case of the specific heat and magnetization is more questionable.

II. THE RANDOM-BOND ASHKIN-TELLER MODEL

The model we study is the random-bond Ashin-Teller model (RBAT) on a square lattice. On every site of the lattice two Ising spin variables, σ_i and τ_i, are placed. Denoting by $\langle ij \rangle$ a pair of nearest-neighbor sites, the Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j \rangle} [K_{i,j}\sigma_i\sigma_j + K_{i,j}\tau_i\tau_j + \Lambda_{i,j}\sigma_i\tau_i\sigma_j\tau_j] \ . \quad (5)$$

The positive coupling constants $K_{i,j}$ and $\Lambda_{i,j}$ are chosen according to

$$(K_{i,j}, \Lambda_{i,j}) = \begin{cases} (K^1, \Lambda^1) & \text{with probability } \frac{1}{2} \\ (K^2, \Lambda^2) & \text{with probability } \frac{1}{2} \ . \end{cases} \quad (6)$$

The homogeneous model [13] [$(K^1, \Lambda^1) = (K^2, \Lambda^2)$] possesses a line of critical points, along which critical exponents vary continuously. This critical line interpolates between the Ising and four-state Potts models. Even though the scaling exponent corresponding to randomness, $\phi = (\alpha/\nu)$, also varies continuously along this line, it takes positive values, $(1 \geq \phi \geq 0)$, so that randomness is relevant. Indeed the critical behavior of the disordered model was found to be different from that of the pure one [11]. In [11] a duality transformation was used to locate a critical plane of the disordered model; the random model is critical when (K^2, Λ^2) are the dual couplings of (K^1, Λ^1) [14–17]. This critical plane corresponds to the line of critical points of the pure model, along which critical exponents vary continuously. A finite size scaling study was performed for several critical models, extrapolating between the critical bond-disordered Ising and four-state Potts models. The critical behavior of each disordered model was compared with the critical behavior of an anisotropic Ashkin-Teller model which was used as a reference pure model [18]. Whereas we found no essential change induced by randomness in the order parameters' critical exponents, the divergence of the specific heat C did change dramatically. Our results favor a logarithmic type divergence at T_c, $C \sim \log l$ for the entire critical manifold of the random-bond Ashkin-Teller model, including the four-state Potts model, but excluding the random-bond Ising model, for which $C \sim \log\log l$ was obtained.

Here we give some of the details of the simulations and our main numerical results for the critical behavior. These are necessary for understanding and analyzing our variance results. All the results listed here were presented in detail in [11]; some essential points are reviewed here for the sake of completeness.

Two series of critical RBAT models were studied in order to monitor two effects. The first series of measurements were performed at five models (or points in the couplings space), $\{(K^1, \Lambda^1), (K^2, \Lambda^2)\}_i$, $i = 0, \ldots, 4$, which we label as C_i, $i = 0, \ldots, 4$. These were chosen so as to interpolate between C_0, which is a random-bond Ising critical point $(\Lambda^1 = \Lambda^2 = 0)$, and C_4, which is a random-bond four-state Potts critical point $(\Lambda^1 = K^1, \Lambda^2 = K^2)$. The points C_i interpolate in a similar manner to the way in which the critical line of the pure AT connects the pure Ising critical point with the pure four-state Potts critical point. The extent of deviation from pure behavior is determined by the difference between the two sets of couplings. For the series C_i, $i = 0, \ldots, 4$ the ratio of $\frac{1}{10}$ was chosen, i.e.,

$$K^2 \approx \frac{1}{10} K^1 \ , \quad (7)$$

so that randomness will be pronounced [19,20].

Two additional measurement points (or models) were intended to monitor the effect of varying the amount of randomness on the critical behavior. The points A_2, B_2, C_2 represent three RBAT models with coupling ratios $\frac{\Lambda^1}{K^1} \approx \frac{1}{2}$ and $\frac{K^2}{K^1} \approx \frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{10}$, respectively. Thus the model A_2 possesses the lowest degree of randomness, while the model C_2 possesses the highest degree of randomness. The usual definitions for energy E, specific heat C, magnetization M [21], susceptibility χ, polarization $p = \langle \sigma\tau \rangle$, and susceptibility of the polarization $\chi^{(p)}$ were used. Since the specific heat seems to play a dominant role in the behavior of the variance, we elaborate on the specific heat results, and even reproduce one graph. For the specific heat we found excellent agreement with the finite size scaling form

$$C = a_0 + b_0 \ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)] \ , \quad (8)$$

where $(\alpha/\nu)_{\text{pure}}$ is the critical exponent ratio of the corresponding anisotropic (pure [18]) model. Equation (8) reproduces expected scaling forms in various limits as we now show. The constant c_0 can be expressed as

$$c_0 = (l_c^{(\alpha/\nu)_{\text{pure}}} - 1)^{-1} \ , \quad (9)$$

where l_c is a crossover length, at which crossover from the pure model's power law behaviour to the random logarithmic behavior occurs. Thus for $l \ll l_c$, Eq. (8)

reduces to the pure model behavior,

$$C = a_0 + b_1 l^{(\alpha/\nu)_{\rm pure}} , \tag{10}$$

while for $l \gg l_c$ and $l^{(\alpha/\nu)_{\rm pure}} \gg 1$ a logarithmic behavior is attained,

$$C = a + b \ln l . \tag{11}$$

Apart from crossing over to the correct pure result (10) when $c_0 \to 0$, in the Ising model limit, $(\alpha/\nu)_{\rm pure} \to 0$, Eq. (8) becomes

$$C = a + b \ln(1 + g \ln l) , \tag{12}$$

with $g = c_0(\alpha/\nu)_{\rm pure}$. This is the finite size scaling form which was predicted analytically [22] and confirmed numerically [19] for the random-bond Ising model.

In Fig. 1 the specific heat of the critical RBAT models is plotted on a log-log scale, with fits to (8) using the full lattice size range $4 \le l \le 256$. The fitting parameters a_0, b_0, and c_0 together with $(\alpha/\nu)_{\rm pure}$ and the crossover lengths l_c are listed in Table I. Note that $(\alpha/\nu)_{\rm pure}$ was not a fitting parameter, and was taken for each RBAT model from results of independent simulations of the corresponding anisotropic AT model. For the models $C_{0\ldots4}$ (with large randomness, $\frac{K^2}{K^1} \approx \frac{1}{10}$), the crossover lengths l_c were found to be 1. Nonetheless these models differ by exhibiting some crossover with different values of $(\alpha/\nu)_{\rm pure}$ (see Table I). On the other hand, $(\alpha/\nu)_{\rm pure}$ of the three models A_2, B_2, C_2 is very similar (0.40, 0.37, 0.37, respectively) but they differ in their amount of randomness, $\frac{K^2}{K^1} \approx \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$, respectively. Consequently, as one would expect, we found that their crossover lengths decrease as randomness increases: $l_c = 51 \pm 7$, $l_c = 4.0 \pm 0.4$, $l_c = 1$, respectively.

We found that the magnetization M, susceptibility χ, and the susceptibility of the polarization $\chi^{(p)}$ are well described at criticality by the following scaling laws:

$$M = A_M l^{-\frac{\beta}{\nu}} , \tag{13a}$$

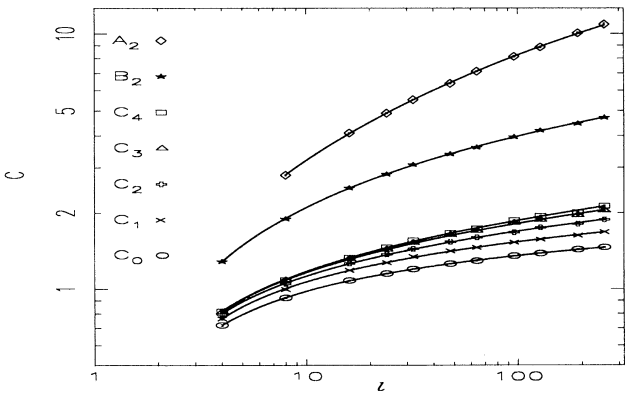

FIG. 1. Specific heat, C, as a function of l on a log-log scale for seven critical RBAT models. C_0 is a random-bond Ising model and C_4 is a random-bond four-state Potts model. The curves are fits to the form (8), yielding estimates for the coefficients of (8) which are listed in Table I.

TABLE I. The fitting parameters of the critical specific heat of the random-bond Ashkin-Teller model. a_0, b_0, and c_0 were obtained by fitting the specific heat results of the seven critical RBAT models $C_{0\ldots4}$ and A_2, B_2 to Eq. (8) using lattice sizes $4 \le l \le 256$. $(\frac{\alpha}{\nu})_{\rm pure}$ is the specific heat exponent of the corresponding anisotropic (pure) models. l_c is the crossover length defined in (9). Errors are given in parentheses only when the error is smaller than or of the same order as the number itself.

	a_0	b_0	c_0	l_c	$(\frac{\alpha}{\nu})_{\rm pure}$
C_0	-0.37(12)	0.58(1)	5.2E4(1.5E4)	1.0	0 .0001(150)
(Ising)					
C_1	-4.6	0.51(2)	1.5E6	1.0	0.171(5)
C_2	-4.1	0.46(127)	5.5E4	1.0	0.375(5)
C_3	-3.9	0.43(4)	5.5E4	1.0	0.549(8)
C_4	-4.1	0.42(1)	1.0E5	1.0	0.630(8)
(Potts)					
B_2	-0.09(5)	2.00(4)	1.47(10)	4.0(4)	0.371(5)
A_2	-0.07(6)	9.35(33)	0.26(2)	51(7)	0.40(1)

$$\chi = A_\chi l^{\frac{\gamma}{\nu}} , \tag{13b}$$

$$\chi^{(p)} = A_{\chi^{(p)}} l^{\frac{\gamma^{(p)}}{\nu}} . \tag{13c}$$

The estimates for the exponents $\frac{\beta}{\nu}$, $\frac{\gamma}{\nu}$, and $\frac{\gamma^{(p)}}{\nu}$, which were obtained using lattice sizes $l \ge 24$, are listed in Table II. Even though one observes slight variation of $\frac{\beta}{\nu}$ and $\frac{\gamma}{\nu}$ from model to model, the results are consistent also with fixed, nonvarying exponents $\frac{\beta}{\nu} = \frac{1}{8}$ and $\frac{\gamma}{\nu} = \frac{7}{4}$, modified by a logarithmic correction. So $\frac{\gamma}{\nu}$ shows very little variation or does not vary at all. This is nearly the same behavior as was found for the corresponding anisotropic models where $\frac{\gamma}{\nu}$ is predicted analytically [23,24] to be constant for all models $\frac{\gamma}{\nu} = \frac{7}{4}$. The exponent ratio $\frac{\gamma^{(p)}}{\nu}$ connected with the susceptibility of the polarization which varies continuously for the pure Ashkin-Teller model seems to do so also for the random models (see Table II).

TABLE II. Critical exponents ratios for seven critical RBAT models $C_{0\ldots4}$ and A_2, B_2. These exponent ratios for the magnetization M, the susceptibility χ, and the susceptibility of the polarization $\chi^{(p)}$ were obtained by fitting results for lattice sizes $l \ge 24$ to Eq. (13).

	$\frac{\gamma}{\nu}$	$\frac{\beta}{\nu}$	$\frac{\gamma^{(p)}}{\nu}$
C_0 (Ising)	1.751(5)	0.125(3)	1.549(9)
C_1	1.751(6)	0.124(3)	1.575(8)
C_2	1.743(5)	0.129(3)	1.597(9)
C_3	1.736(3)	0.133(2)	1.638(5)
C_4 (Potts)	1.714(5)	0.145(3)	1.714(5)
B_2	1.738(4)	0.132(3)	1.586(6)
A_2	1.739(5)	0.132(3)	1.590(8)

III. VARIANCES: DEFINITIONS AND ESTIMATORS

In this section we define two types of variances of thermodynamic quantities in disordered systems and explain their relation to error estimates. We explain how the "sample to sample variance" can be estimated from MC data.

First consider some sample x which is simulated at some temperature T. Because of the thermal fluctuations and the finite simulation time, we obtain for this sample an estimate $\overline{P_x}$ of the exact P_x, with an error

$$(\delta\overline{P_x})^2 = \frac{\sigma^2_{T,x}}{T_{\rm MC}/\tau_x}, \quad T_{\rm MC} \text{ large} . \tag{14}$$

$T_{\rm MC}$ is the length of the MC runs and τ is the autocorrelation time of the MC dynamics. $\sigma^2_{T,x}$ is the variance of P within the sample x due to thermal fluctuations. In practice, in order to avoid the estimation of τ which requires a long simulation time, we estimate $(\delta\overline{P_x})^2$ by binning the MC sequence into ~ 10 subsequences and treating each subsequence as independent (the Jack-knife procedure).

The estimate for the error in the estimation of $[\overline{P_x}]$, the average of P over all samples, is given by

$$(\delta[\overline{P_x}])^2 = \frac{1}{(n-1)n}\sum_{x=1}^{n}(\overline{P_x} - [\overline{P_x}])^2, \quad n \text{ large} , \tag{15}$$

where n is the number of random-bond samples. In contrast to (14) this total error has two contributions, namely the sample to sample fluctuations of the exact P_x around $[P_x]$ and the thermal fluctuations of $\overline{P_x}$ around P_x within each sample, that is,

$$(\delta[\overline{P_x}])^2 = \frac{V}{n} + \left[\frac{\sigma^2_{T,x}}{nT_{MC}/\tau}\right], \quad n, T_{\rm MC} \text{ large}. \tag{16}$$

Thus by estimating $(\delta\overline{P_x})^2$ for all x and $(\delta[\overline{P_x}])^2$ with (15) we obtain V through (16); it is an unbiased estimate of the variance of the exact P_x due to sample to sample fluctuations (see Ref. [25] for a basic statistical explanation). In order to minimize the error of $[\overline{P_x}]$ for a given amount of computer time, one needs to adjust $T_{\rm MC}$ so that the two terms in (16) are equal. However, if one is interested in obtaining a reasonable estimate of V, $T_{\rm MC}$ needs to be chosen larger, so as to obtain accurate estimates of the $\overline{P_x}$'s and minimize the second term on the left-hand-side of (16).

As explained in the Introduction, the dependence of thermal variance σ^2_T on the lattice size l has been examined (for homogeneous models) in Refs. [9,10]. Thus from here on the term variance will refer to the variance due to disorder. Here it is our aim to examine the dependence of the variance V on l at criticality, one reason being that for MC simulations of disordered systems, it has the bigger influence on their accuracy. This is in addition to the theoretical motivation given in the Introduction. In the next section we display our results for the variance V of the random-bond Ashkin-Teller model.

IV. VARIANCE RESULTS OF THE RANDOM-BOND ASHKIN-TELLER MODELS

A. Far from criticality

Far from criticality the correlation length is finite and one would expect the system to behave similarly to a collection of independent smaller systems. Thus one would expect the Brout argument to hold with the variance scaling as l^{-d}. Nonetheless this is not obvious: Note that the thermal fluctuations of the specific heat C and the susceptibility χ are non-self-averaging even away from criticality [9,10]. Thus the RBAT model C_2 (the choice of model was arbitrary) was simulated at the reduced temperature $t = 1$. In Fig. 2 we show the relative variances $V_P/[P]^2$, where V_P is defined in (16) and $P = E, M, C, \chi$, as a function of $\log_{10} l$. The linear curves are fits to the form $V_P/P^2 = Al^{-\rho}$. We find $\rho = 2.06(7), 2.13(7), 2.04(6), 2.12(7)$ for χ, C, E, M, respectively. Thus the Brout argument is confirmed and far from criticality strong self-averaging holds.

B. The variance at criticality

1. Distributions

In order to visualize how large the sample to sample fluctuations are, at the critical temperature, several histograms of the number of occurrences of samples according to their susceptibility $\overline{\chi_x}$ or according to their specific heat $\overline{C_x}$, are shown in Figs. 3–7. The abscissa is scaled by the average susceptibility $[\chi]$ (or specific heat $[C]$) of all samples. The histogram of the susceptibility for lattice size $l = 192$ is shown in Fig. 3 for the Ising model C_0 and in Fig. 4 for the four-state Potts model C_4. The y axis of both figures is scaled so that the area of both histograms is the same. Even though the lattice size is

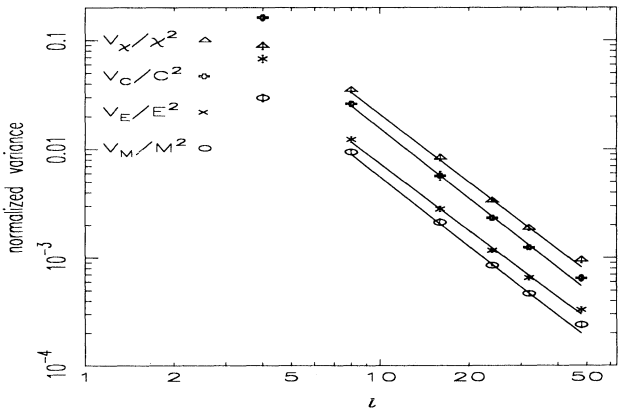

FIG. 2. The relative variances $V_P/[P]^2$, $P = E, M, C, \chi$ as a function of $\log_{10} l$ for the RBAT model C_2 at the reduced temperature $t = 1$. The linear curves are fits to the form $V_P/P^2 = Al^{-\rho}$, yielding $\rho = 2.06(7), 2.13(7), 2.04(6), 2.12(7)$ for χ, C, E, M, respectively.

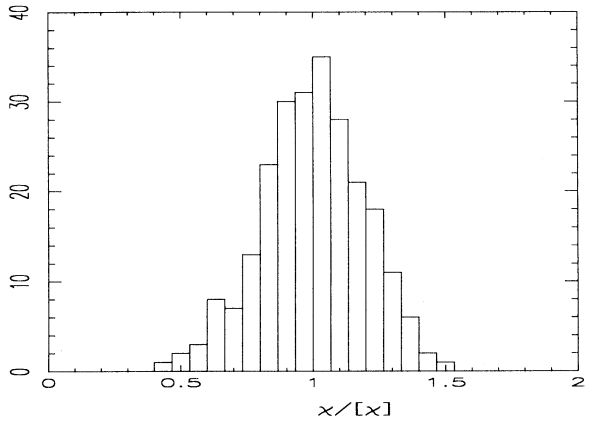

FIG. 3. Histogram of the number of occurrences of samples according to their susceptibility scaled by the average susceptibility, for the Ising model C_0 and lattice size $l = 192$; with 240 samples.

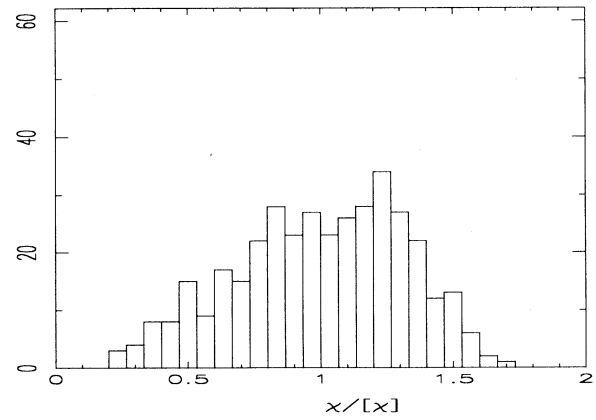

FIG. 4. Histogram of the number of occurrences of samples according to their susceptibility scaled by the average susceptibility, for the four-state Potts model C_4 and lattice size $l = 192$; with 370 samples.

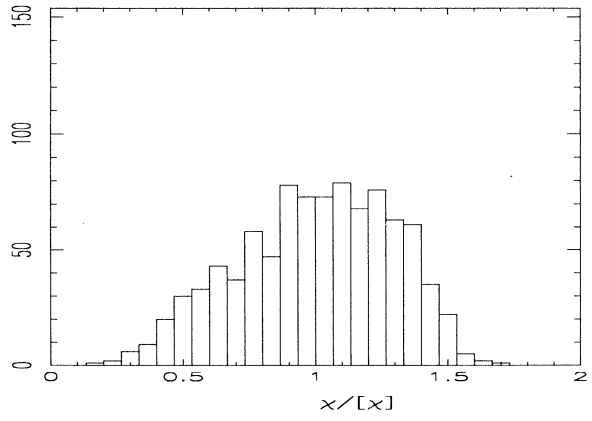

FIG. 5. Histogram of the number of occurrences of samples according to their susceptibility scaled by the average susceptibility, for the four-state Potts model C_4 and lattice size $l = 24$; with 920 samples.

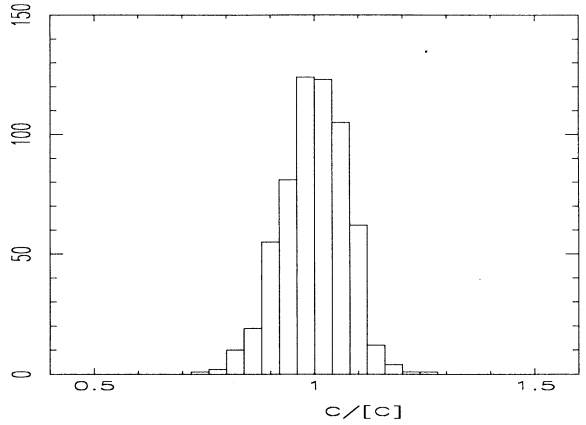

FIG. 6. Histogram of the number of occurrences of samples according to their specific heat scaled by the average specific heat, for the Ising model C_0 and lattice size $l = 48$ with 600 samples.

rather large, the distributions are very wide; a measurement of a value of χ at 40% above the mean $[\chi]$ has a non-negligible probability for the four-state Potts model.

There is a marked difference between the width of the distribution of the Ising model ($\sqrt{V_\chi}/[\chi] \approx 0.2$) and the much wider distribution of the four-state Potts model ($\sqrt{V_\chi}/[\chi] \approx 0.32$). The histogram of the susceptibility for the four-state Potts model with lattice size $l = 24$ is shown in Fig. 5. Note that the width of the distribution here is slightly narrower (≈ 0.29) than that of Fig. 4. This very small difference (and even slight increase) of the width as l increases hints at a lack of self-averaging of the susceptibility of the four-state Potts model. An additional striking difference between the susceptibility distributions of the four-state Potts model, Figs. 4 and 5, and the Ising model, Fig. 3, is that the former are strongly asymmetric (this asymmetry was measured by measuring

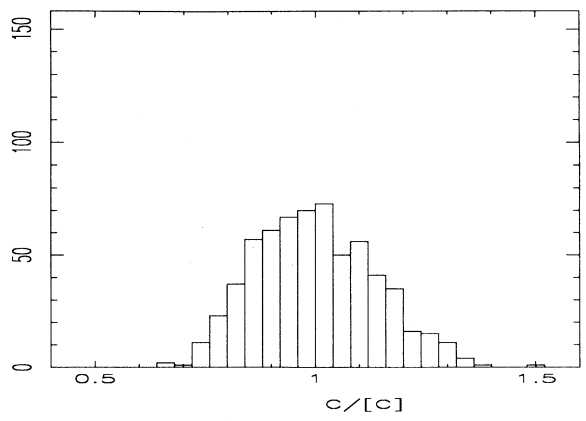

FIG. 7. Histogram of the number of occurrences of samples according to their specific heat scaled by the average specific heat, for the four-state Potts model C_4 and lattice size $l = 48$ with 630 samples.

the third moment of the distribution). A possible explanation for this asymmetry, which exists to some degree in all the models, is given in Sec. V and in [26] . The average errors in the estimation of the susceptibility of a single sample x, divided by the average susceptibility, $[\delta\overline{\chi_x}]/[\chi]$, are 0.01, 0.017, and 0.012 in Figs. 3, 4, and 5, respectively. Since these errors are negligible as compared to the widths, the histograms are highly reliable.

The histograms of the specific heat for lattice size $l = 48$ are shown in Fig. 6 for the Ising model C_0 and in Fig. 7 for the four-state Potts model C_4. Note that the distributions of the specific heat are much narrower than those of the susceptibility. The width of the distribution for the four-state Potts model (≈ 0.126) is about twice wider than the width of the distribution for the Ising model (≈ 0.062). The asymmetry of the distribution for the four-state Potts model is almost unnoticeable and is of the opposite sign than the asymmetry of the susceptibility. The average error in the estimation of the specific heat of a single sample x divided by the average specific heat $[\delta\overline{C_x}]/[C]$ is 0.038 for the Ising model and 0.048 for the four-state Potts model, so that these histograms are much less accurate than those of the susceptibility. For the larger lattices the ratio between the width and the error becomes smaller, mostly because the width becomes smaller, and histograms become even less accurate. Thus in order to obtain more accurate histograms and also better estimates of the variance (which is the square of the width of the histograms), longer simulation times would be needed, in order to obtain more accurate estimates of the $\overline{C_x}$'s. This may be done in a future study.

2. *The variance*

In Fig. 8 we show the variance of χ, V_χ, of the seven critical RBAT models. For the sake of clarity (so that the data do not fall on top of each other) V_χ of the model C_i was multiplied by 2^{i+1}. The lines are fits according to a theory which we develop in the next section. Here we just note that V_χ is measured with high precision, so that it may be faithfully tested against theory.

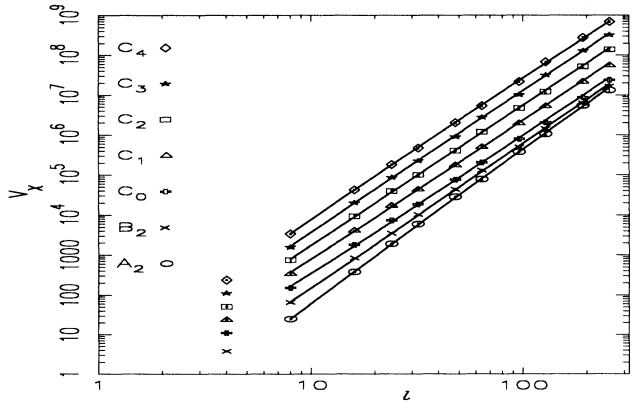

FIG. 8. The variance of χ, V_χ, as a function of $\log_{10} l$ for all critical models, $C_{0\ldots4}$, and A_2, B_2 of the RBAT model. For the sake of clarity V_χ of the model C_i was multiplied by 2^{i+1}. The solid lines are fits to the form (41), yielding estimates for fitting parameters which are listed in Table IV.

The relative variance $V_\chi/[\chi]^2$ is plotted in Fig. 9. Since it is the ratio of two fluctuating quantities, the errors are quite large. Nonetheless the main trends can be seen. First note that apparently for all models (except for the weakly random model A_2) $V_\chi/[\chi]^2 \to$ const, so that the susceptibility is non-self-averaging. It is also possible that $V_\chi/[\chi]^2$ is slightly increasing with l for some models (e.g., the four-state Potts model C_4) or slightly decreasing for the Ising model C_0. Upon comparison of the models C_i $i = 0, \ldots, 4$ we make the following observations. The higher the specific heat of a model is, the larger is its relative variance (see Fig. 1). The higher the exponent $(\frac{\alpha}{\nu})_{\text{pure}}$ of the pure model (see Table I) is, the larger is the initial slope of the relative variance of the corresponding random model. Thus the relative variance of the Ising model C_0 is the smallest and the increase with l, for small l, is the smallest. The relative variance of the four-state Potts model C_4 is the largest and the increase with l, for small l, is the largest. The relative variances of the RBAT models $C_{1,2,3}$ fall in between. The relative variance of the weakly random model A_2 shows a steady increase with l, in contrast with the highly random model C_2, in which a shorter increase is followed by a plateau. This is reminiscent of the specific heat of the A_2 model which exhibits very slow crossover from the power-law behavior (10) to the asymptotic logarithmic behavior (11) with a crossover length of $l_c \approx 50$. Thus for small lattice sizes the model A_2 exhibits effective exponents (of the specific heat) of the pure model, and also exhibits a small variance due to its small degree of randomness. But as the lattice size increases, this effect diminishes, the effective exponents approach the random value, and the variance approaches that of the highly random models.

A very similar picture is obtained for the relative variance of the magnetization $V_M/[M]^2$, as seen in Fig. 10. The qualitative picture of the magnetization results, Fig. 10, is very similar to that of the susceptibility results, Fig. 9, showing the same trends as outlined above. Yet

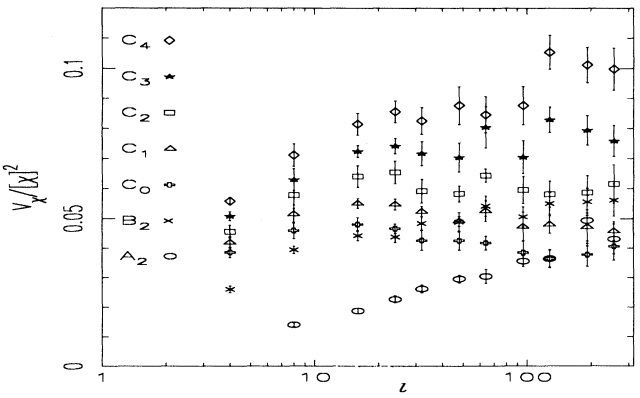

FIG. 9. The scaled variance of the susceptibility, $V_\chi/[\chi]^2$, as a function of $\log_{10} l$ for all critical models, $C_{0\ldots4}$, and A_2, B_2 of the RBAT model.

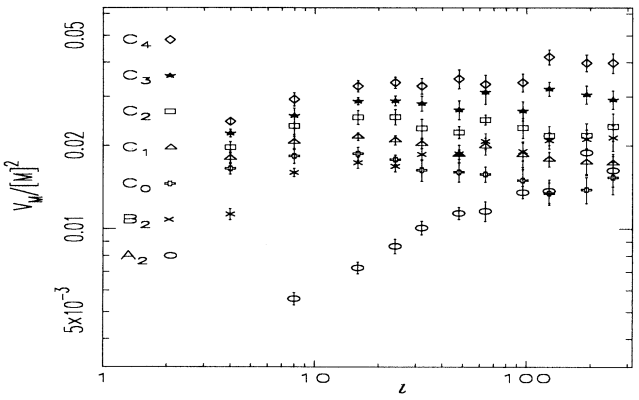

FIG. 10. The scaled variance of M, $V_M/[M]^2$ as a function of $\log_{10} l$ for all critical models, $C_{0\ldots4}$, and A_2, B_2 of the RBAT model.

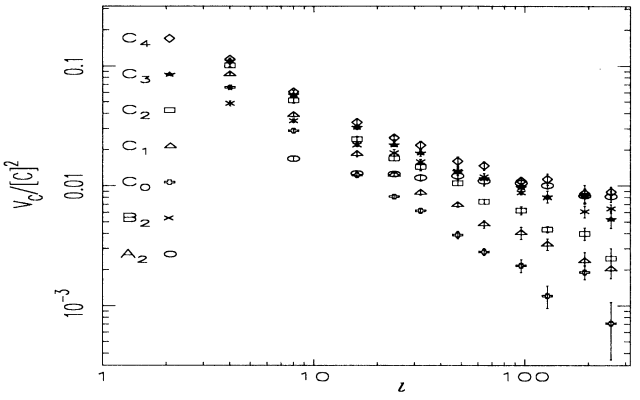

FIG. 12. The scaled variance of C, $V_C/[C]^2$, as a function of $\log_{10} l$ for all critical models, $C_{0\ldots4}$, and A_2, B_2 of the RBAT model.

we emphasize that even though the magnetization is an intensive quantity, it does not seem that $V_M/[M]^2 \to 0$ as l increases so that the magnetization is *not* self-averaging at criticality.

In Fig. 11 the variance of the energy V_E [27] is plotted on a log-log scale. For the sake of clarity (so that the data do not fall on top of each other) V_E of the model C_i was multiplied by 2^{i+1}. We fit the data to the form $V_E \sim l^{-\theta}$ for lattice sizes $l \geq 16$ (but the fitting curves shown in Fig. 11 are not made with this form but with a more complicated one which is due to a theory which we develop in the next section). The highest value of θ, $\theta = 1.855(13)$, was obtained for the Ising model C_0. For the four-state Potts model, C_4, we obtained $\theta = 1.72(2)$, while for the models $C_{1,2,3}$ the values of θ fell between these two values. Thus in contrast with the susceptibility and the magnetization, the variance of the energy V_E is weakly self-averaging. But similar to the susceptibility, models with a higher specific heat or with a higher effective $\frac{\alpha}{\nu}$ have a smaller θ, and thus their V_E decreases more slowly with l. For the weakly random model A_2, $\theta = 1.30(3)$ so that again the slope of the variance is correlated with the high slope of the specific heat of this model.

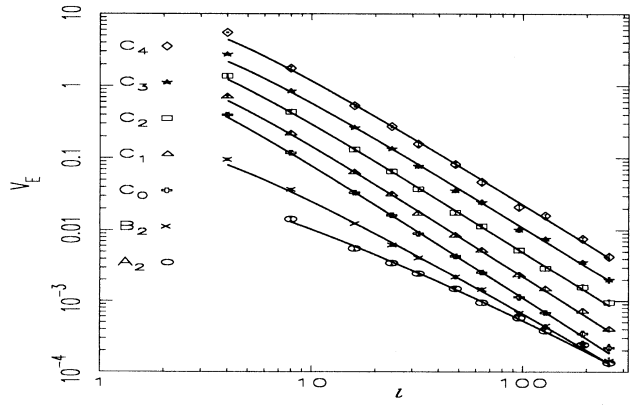

FIG. 11. The variance of the energy, V_E, as a function of $\log_{10} l$ for all critical models, $C_{0\ldots4}$, and A_2, B_2 of the RBAT model. For the sake of clarity V_E of the model C_i was multiplied by 2^{i+1}. The solid lines are fits to the form (42), yielding estimates for the fitting parameters a_v, b_v which are listed in Table IV.

The results of the relative variance V_C/C^2, plotted in Fig. 12, seem to indicate that the specific heat is weakly self-averaging. Nonetheless the effective slopes increase with l (or the absolute values of the slopes decrease with l, this trend being strongest for the four-state Potts model C_4) so that it is possible that self-averaging does not hold for very large l. It also seems possible that the Ising model is self-averaging while the other models are not. Clearly more accurate data and data from larger systems would be useful. As in other variances, we observe qualitatively that the relative variance of the moderately random models, A_2 and B_2, approaches that of the highly random ones as l increases and even exceeds it. The findings of this section are partly summarized in Table III, where the self-averaging properties of the highly random critical models are displayed.

In the next section we develop a phenomenological finite size scaling theory for the variance. This theory explains the apparent connection between the variance and

TABLE III. Summary of the self-averaging properties of the critical random-bond four-state Potts C_4, Ashkin-Teller $C_{1..3}$, and Ising C_0 models. The letter "n" stands for non-self-averaging, "w" for weakly self-averaging, and "?" for inconclusive results. This summary is according to a subjective examination of the numerical results as displayed in Figs. 9, 10, 11, 12, and 13. According to our theory, only the energy E is weakly self-averaging while all other quantities are non-self-averaging in all of these models. If the theory is correct, then any numerical indication to the contrary is due to a "transient" finite size effect.

Model	χ	$\chi^{(p)}$	M	C	E
C_0 (Ising)	?	?	?	w	w
$C_{1\ldots3}$ (Ashkin-Teller)	n	n	n	w	w
C_4 (Four-state Potts)	n	n	n	w	w

the specific heat behavior of the random models. In the final section we explain how this theory was applied to the results we have displayed here and discuss the comparison between our scaling theory and the numerical results.

V. FINITE SIZE SCALING OF SAMPLE TO SAMPLE FLUCTUATIONS AT CRITICALITY

As our numerical results show, we have obtained quite accurate estimates of the variance V of the thermodynamic functions at the critical temperature for different lattice sizes l. In order to understand these results, a phenomenological theory of finite size scaling of disordered systems, which will take into account sample to sample fluctuations, needs to be developed.

The main result of our theory will be the scaling of the variance V with l at criticality. To be precise, we will calculate the variance V of P (e.g., $P = C$, M, E, or χ, where all these quantities are normalized per volume; i.e., they are densities)

$$V(T,l) = [\{P_x(T,l) - [P_x(T,l)]\}^2] \ . \tag{17}$$

$P_x(T,l)$ is the exact value of P (that is, after the thermal fluctuations have been averaged over) of a specific sample x (with some specific realization of randomly distributed bonds) of linear size l at temperature T. Again the square brackets denote averaging over the different samples x.

Our conclusion will be that when the specific heat exponent α is negative, the leading behavior of V at T_c is

$$V(T_c,l) \sim K_r^2 l^{2\rho+\frac{\alpha}{\nu}} \quad \text{implying} \quad \frac{V(T_c,l)}{[P_x(T_c,l)]^2} \sim l^{\frac{\alpha}{\nu}} \ . \tag{18}$$

Where K_r is a measure of the amount of randomness or disorder and ρ is the critical exponent of the quantity P, e.g., if $P = \chi$, then $\rho = \frac{\gamma}{\nu}$. Equation (18) implies that disordered systems at criticality are only weakly self-averaging when $\frac{\alpha}{\nu} < 0$. For $\frac{\alpha}{\nu} = 0_+(\log_{10})$, as was found [11] for the random bond Ashkin-Teller model, our derivation is strictly not valid for $l \gg 1$. Nonetheless for the range of lattice sizes considered, we found good agreement between the numerical results for the variance of χ, χ_p, and E and theoretical fits according to (18) together with next to leading terms (see Figs. 8, 13, and 11 and the discussion in the next section). If no dramatic change occurs at larger sizes, then the sample to sample fluctuations of the random-bond Ashkin-Teller model are non-self-averaging.

The result (18) indicates that the sample to sample fluctuations at the critical temperature T_c depend strongly on the specific heat exponent $\frac{\alpha}{\nu}$. This strong dependence can be made plausible based on heuristic arguments. These heuristic arguments will serve to define some basic ingredients of our approach and will be followed by a more quantitative treatment.

We start by characterizing every specific sample x of size l by a pseudocritical temperature $T_c(x,l)$. This pseudocritical temperature can, for instance, be the temperature at which a maximum in the specific heat of the sample occurs. We denote the average pseudocritical temperature as $T_c(l) = [T_c(x,l)]$. We assume that, as is the case in homogeneous systems,

$$T_c(l) - T_c = a l^{-y_t} \ , \tag{19}$$

where a is a constant, $y_t = 1/\nu$, and $T_c = \lim_{l\to\infty} T_c(l)$. T_c is the average critical temperature of the ensemble of infinite samples. Equation (19) is supported by a numerical study [28] of the three-dimensional dilute Ising model.

Different realizations of bonds lead to (slightly) different pseudocritical temperatures $T_c(x,l)$. We assume that $T_c(x,l)$ fluctuates around $T_c(l)$ with width

$$\delta T_c(l) \sim l^{-d/2} \ . \tag{20}$$

This assumption is in the spirit of the assumption leading to the Harris criterion [12,1], that *weak* bond disorder in subsystems of volume ξ^d (where ξ is the correlation length) leads to fluctuations of the local subsystem effective critical temperature, which are of width $\delta T_c \sim \xi^{-d/2}$. Thus (20) is probably true for small disorder and small l, or close to the pure system fixed point. Without proof, we assume (20) to be valid for large disorder as well, though for large disorder (or close to the random fixed point) the possibility that $\delta T_c(l) \sim l^{-y_t}$ has been raised [29].

Next define reduced temperatures

$$t_c(x,l) = \frac{T_c(x,l) - T_c}{T_c} \ , \tag{21a}$$

$$t_c(l) = \frac{T_c(l) - T_c}{T_c} \ , \tag{21b}$$

and the reduced width

$$\delta t_c(l) = \frac{\delta T_c(l)}{T_c} \ . \tag{21c}$$

We make (20) more specific by assuming for $t_c(x,l)$ a Gaussian probability distribution $q(t_c(x,l))$

$$q(t_c(x,l)) = \frac{l^{d/2}}{\sqrt{2\pi}K_r} \exp\{-[t_c(x,l) - t_c(l)]^2 l^d / 2K_r^2\} \ . \tag{22}$$

The width of the distribution is controlled by the lattice size l and by K_r which is a measure of the amount of randomness or disorder.

The scaling relations (19) and (20) already make the result (18) plausible. The main idea is that the sample to sample fluctuations at T_c are governed by the relative magnitude of two temperature differences. The first is the difference between the average pseudocritical temperature $T_c(l)$ and the critical temperature of the infinite system T_c. The second is the difference between $T_c(l)$ and $T_c(x,l)$, the pseudocritical temperature of the sample x, which is governed by $\delta t_c(l)$. If $\delta t_c(l) \gg |t_c(l)|$, then fluctuations in $t_c(x,l)$ are so large that for some samples one will find $T_c > T_c(x,l)$ while for other sam-

ples $T_c < T_c(x,l)$. In this case, even though we are simulating all samples at T_c, some samples are in their high-temperature phase while others are in their low-temperature phase. This will obviously increase the sample to sample fluctuations in any observable. If, on the other hand, $\delta t_c(l) \ll |t_c(l)|$, then $T_c - T_c(x,l)$ will always have the same sign and fluctuations will be smaller. The condition $\delta t_c(l) \ll |t_c(l)|$ will be fulfilled for large l if $y_t - \frac{d}{2} < 0$ or, using the hyperscaling relation $\frac{\alpha}{\nu} = 2y_t - d$, if $\frac{\alpha}{\nu} < 0$. For disordered systems, the bound $y_t \leq d/2$ has been proven by Chayes *et al.* [30], so that asymptotically one always finds $\frac{\alpha}{\nu} \leq 0$. However, for small l and small disorder, the system may be governed by a positive $(\frac{\alpha}{\nu})_{\text{pure}}$. In this case sample to sample fluctuations can increase with lattice size, as is indeed seen in our numerical results for the weakly disordered model A_2. Thus on the basis of these considerations one can conclude that the sign and magnitude of the specific heat exponent α of the disordered model have a strong influence on the sample to sample fluctuations [26], and will determine whether they are self-averaging. The discussion above is analogous to the physical arguments leading to the Harris criterion [12], but in a finite size scaling formulation. The difference is that the Harris criterion was derived near the pure system fixed point, while we are assuming that similar conditions apply also next to the disordered critical fixed point.

In order to put these general considerations on more quantitative grounds, we proceed to derive the finite size scaling expression (18) for the variance of various thermodynamic quantities. Start by introducing the reduced temperature of each sample x,

$$\dot{t}_x = \frac{T - T_c(x,l)}{T_c} \ . \tag{23}$$

We assume [for samples with T close to $T_c(x,l)$] a finite size scaling form for the singular part of P_x,

$$P_x^{\text{sing}}(T,l) = l^{\rho} \tilde{Q}_x(\dot{t}_x l^{y_t}) \ . \tag{24}$$

The form of the function $\tilde{Q}_x(Z)$ (or its coefficients) is assumed to be sample dependent but the critical exponents ρ, y_t are assumed to be universal or sample independent.

Equation (24) embodies the usual [31] finite size scaling assumption that in the vicinity of the critical temperature the behavior of a large finite system is governed by the scaled variable ξ_x/l. We use this assumption, even though in the present context it implies that a single correlation length ξ_x is sufficient to describe the state of a disordered sample, which is not obvious at all. This "thermal" l dependence is compounded by the fact that if we increase l, we must generate additional random bonds, and hence increasing l necessitates, effectively, changing x (that represents a particular realization of the random-bond variables). Since x affects P_x^{sing} through the nonuniversal coefficients of $\tilde{Q}_x$, a nonthermal dependence of P_x^{sing} on l is induced. The main task of our analysis is to separate the thermal l dependence from the nonthermal component.

At this stage it is possible to draw some more conclusions based on (24), without making strong assumptions about the coefficients of $\tilde{Q}_x$. We leave such derivations for the Appendix. Here we proceed in a more straightforward manner by using a simplifying ansatz. Our ansatz states that the coefficients of $\tilde{Q}_x$ depend only on $\Delta t_c(x,l)$, the deviation of the pseudocritical temperature of the sample from the average pseudocritical temperature, defined as

$$\Delta t_c(x,l) = t_c(x,l) - t_c(l) \ . \tag{25}$$

It is convenient to proceed by rewriting $\dot{t}_x$ as $\dot{t}_x = t - \Delta t_c(x,l) - t_c(l)$ with $t = \frac{T - T_c}{T_c}$. Using the scaling of $t_c(l)$ [see (19) and (21b)], we substitute $\tilde{Q}_x$ by a different scaling function Q_x and rewrite Eq. (24) as

$$\begin{aligned} P_x^{\text{sing}}(T,l) &= l^{\rho} \tilde{Q}_x\{(t - \Delta t_c(x,l) - t_c(l))l^{y_t}\} \\ &= l^{\rho} Q_x\{(t - \Delta t_c(x,l))l^{y_t}\} \ . \end{aligned} \tag{26}$$

For completeness of the treatment which will later prove to be necessary we do not neglect the analytic dependence of $P_x(T,l)$ on $(t - \Delta t_c(x,l))$ [32], and write

$$\begin{aligned} P_x(T,l) = A_x + B_x(t - \Delta t_c(x,l)) + C_x(t - \Delta t_c(x,l))^2 \\ + \cdots + l^{\rho} Q_x\{(t - \Delta t_c(x,l))l^{y_t}\} \ . \end{aligned} \tag{27}$$

The coefficients A_x, B_x, C_x are assumed to be sample dependent in the same way that the coefficients of Q_x are, namely, they depend only on $\Delta t_c(x,l)$ [33]. Next, assume the dependence of the coefficients on $\Delta t_c(x,l)$ is analytic. Since according to (22) and (25) $\Delta t_c(x,l)$ is distributed around zero with width that scales as $l^{-d/2}$, we can expand

$$A_x = A_0 + A_1 \Delta t_c(x,l) + A_2(\Delta t_c(x,l))^2 + \cdots \ , \tag{28}$$

where A_0, A_1, A_2 are sample independent. The same type of expansion is assumed for B_x, C_x, etc.

We are interested in knowing what happens at $T = T_c$, the average critical temperature of the ensemble of infinite samples. Thus we set $T = T_c$ which implies $t = 0$. For the analytic part of (27) we get

$$\begin{aligned} P_x^{\text{analytic}}(T_c,l) &= A_x - B_x \Delta t_c(x,l) + C_x \Delta t_c(x,l)^2 + \cdots \\ &= A_0 + (A_1 - B_0)\Delta t_c(x,l) + (A_2 - B_1 + C_0)(\Delta t_c(x,l))^2 + \cdots \\ &\equiv a + b\Delta t_c(x,l) + c(\Delta t_c(x,l))^2 \ , \end{aligned} \tag{29}$$

where the second equality is reached by use of (28) and the same expansions for other coefficients. The last equality is a redefinition of constants. In a similar way we expand Q_x,

$$Q_x(Z) = D_x + E_x Z + F_x Z^2 + \cdots \ , \tag{30}$$

where D_x, E_x, F_x are again expanded as in (28). Again setting $t = 0$, we obtain for the singular part of (27)

$$P_x^{\text{singular}}(T_c, l) = l^{\rho}\{D_0 + (D_1 - E_0 l^{y_t})\Delta t_c(x,l) + (D_2 - E_1 l^{y_t} + F_0 l^{2y_t})(\Delta t_c(x,l))^2\} + \cdots . \tag{31}$$

We stress that since we set $Z = -\Delta t_c(x,l) l^{y_t}$, Z is fluctuating around zero with width that scales as $\sim l^{\frac{\alpha}{2\nu}}$. Thus the expansion (30) is justified asymptotically only for $\alpha < 0$. Putting together (29) and (31) we have

$$\begin{aligned} P_x(T_c, l) &= (a + D_0 l^{\rho}) + (b + D_1 l^{\rho} - E_0 l^{\rho + y_t})\Delta t_c(x,l) \\ &\quad + (c + D_2 l^{\rho} - E_1 l^{\rho + y_t} + F_0 l^{\rho + 2y_t})(\Delta t_c(x,l))^2 + \cdots \equiv d + e\Delta t_c(x,l) + f(\Delta t_c(x,l))^2 . \end{aligned} \tag{32}$$

Notice that here the only dependence on the specific sample x is through explicit dependence on $\Delta t_c(x,l)$, the deviation of its reduced pseudocritical temperature from the average pseudocritical temperature. Taking the quenched sample average [ ] with the probability distribution (22), using $[\Delta t_c(x,l)] = 0$, we get

$$[P_x(T_c, l)] = d + f[(\Delta t_c(x,l))^2] , \tag{33}$$

and using $[(\Delta t_c(x,l))^3] = 0$ we further obtain

$$[(P_x(T_c,l))^2] = d^2 + e^2[(\Delta t_c(x,l))^2] + f^2[(\Delta t_c(x,l))^4] + 2df[(\Delta t_c(x,l))^2] . \tag{34}$$

The variance is then given by

$$V(T_c, l) = e^2[(\Delta t_c(x,l))^2] + f^2\{[(\Delta t_c(x,l))^4] - [(\Delta t_c(x,l))^2]^2\} = e^2[(\Delta t_c(x,l))^2] + 2f^2[(\Delta t_c(x,l))^2]^2 , \tag{35}$$

where the last equality is a property of the Gaussian distribution. Lastly we use $[(\Delta t_c(x,l))^2] = K_r^2 l^{-d}$ and obtain to the leading orders in l

$$\begin{aligned} V(T_c, l) &= (b^2 + D_1^2 l^{2\rho} + E_0^2 l^{2\rho + 2y_t} + 2bD_1 l^{\rho} - 2bE_0 l^{\rho + y_t} - 2D_1 E_0 l^{2\rho + y_t}) K_r^2 l^{-d} \\ &\quad + (F_0^2 l^{2\rho + 4y_t}) 2K_r^4 l^{-2d} + \cdots . \end{aligned} \tag{36}$$

Since $y_t > 0$, and usually $\rho + y_t > 0$, the leading term in (36) is

$$V(T_c, l) \sim E_0^2 K_r^2 l^{2\rho + 2y_t - d} = E_0^2 K_r^2 l^{2\rho + \frac{\alpha}{\nu}} . \tag{37}$$

The last term in (36) is proportional to $K_r^4 l^{2\rho + 2\frac{\alpha}{\nu}}$, and may be neglected with respect to (37) only for $\frac{\alpha}{\nu} < 0$, or if $K_r^2 \ll 1$ and l is not too large. (37) is our main result for the variance, where all exponents ρ, α, ν are exponents that characterize the disordered system. It means that disordered systems at criticality are only weakly self-averaging when $\frac{\alpha}{\nu} < 0$. Though our derivation is not valid when $\frac{\alpha}{\nu} = 0_+$ ($C \sim \log_{10} l$) it seems that in this case there is no self-averaging of the sample to sample fluctuations. This point is further discussed in the next section.

We note that for $\alpha < 0$ and in the large l limit considered at the end of the Appendix [Eq. (A4)], where

$$Q_x(-\Delta t_c(x,l) l^{y_t}) \to Q(-\Delta t_c(x,l) l^{y_t}) , \tag{38}$$

the coefficients of Q_x are independent of x so that $D_x = D_0, E_x = E_0$, etc. Neglecting the analytic part of P, this limit corresponds to our derivation with only $D_0, E_0, F_0 \neq 0$ and all other coefficients (D_1, D_2, E_1, etc.) equal to zero. Thus in this limit the main result (37) is unchanged, though less assumptions are needed.

From (33) corrections to the scaling of P are obtained,

$$\begin{aligned} [P_x(T_c, l)] &= a + D_0 l^{\rho} + (c + D_2 l^{\rho} - E_1 l^{\rho + y_t} \\ &\quad + F_0 l^{\rho + 2y_t}) K_r^2 l^{-d} . \end{aligned} \tag{39}$$

So that the leading behavior of P is

$$\begin{aligned} [P_x(T_c, l)] &= a + D_0 l^{\rho} + F_0 K_r^2 l^{\rho + 2y_t - d} \\ &= a + D_0 l^{\rho} + F_0 K_r^2 l^{\rho + \frac{\alpha}{\nu}} . \end{aligned} \tag{40}$$

Thus for negative α, the third term in (40) is a correction to scaling due to sample to sample fluctuations. It follows that for $\frac{\alpha}{\nu} < 0$, (40) and (37) are consistent with $V/[P]^2 \sim l^{\frac{\alpha}{\nu}}$. A special case is when $\alpha_{\text{pure}} < 0$ and randomness is an irrelevant operator (at the pure system fixed point) with a scaling exponent $(\frac{\alpha}{\nu})_{\text{pure}}$. In this case the disordered system has the same exponents as the pure one, $\frac{\alpha}{\nu} = (\frac{\alpha}{\nu})_{\text{pure}}$. Therefore the correction to scaling we have obtained due to sample to sample fluctuations has the same exponent as the correction term connected with the irrelevant operator corresponding to randomness.

VI. COMPARISON OF THEORY WITH VARIANCE RESULTS OF THE RBAT MODELS

The derivation presented in the preceding section, as can be readily seen from Eqs. (27), (28), and (30), involved an expansion in the two parameters $\sqrt{[(\Delta t_c(x,l))^2]}$ and $\sqrt{[(\Delta t_c(x,l))^2]} l^{y_t}$. These scale as $K_r l^{-\frac{d}{2}}$ and $K_r l^{y_t - \frac{d}{2}} = K_r l^{\frac{\alpha}{2\nu}}$. Thus the derivation is valid for small K_r, meaning small disorder and small $\frac{\alpha}{\nu}$. For negative $\frac{\alpha}{\nu}$ the validity of the expansion improves as l increases, while a positive $\frac{\alpha}{\nu}$ is not possible [30].

In the case of the random-bond Ashkin-Teller model, we have asymptotically $C \approx b \ln l$ so that $\frac{\alpha}{\nu} = 0_+$. It seems that in this case the expansion is not justified. Practically though, for the accessible range of lattice sizes, things depend on the constant of proportionality b. If b is small, then for a finite but large interval of lattice sizes l the expansion is justified. Indeed, in the case of the highly random RBAT models $C_{0...4}$, b falls in the range $0.138(4) \leq b \leq 0.280(6)$. Upon inspection of Fig. 1 one may also see that the value of the specific heat of these models shows very little variation for lattice sizes $l \geq 16$. Thus the parameter $K_r l^{\frac{\alpha}{\nu}}$, which should scale with l as the specific heat does, increases very slowly with l. This implies that for the accessible range of lattice sizes l our expansion is valid. The specific heat of the weakly random model A_2 effectively diverges with a positive effective $\frac{\alpha}{\nu}$ but because of its weak degree of randomness there is good reason to believe that the expansion will be valid due to a small value of K_r expected for models with small randomness. The B_2 model with moderate disorder is expected to fall between the A_2 model and the $C_{0...4}$ models. Thus there is reason to hope that our theory is applicable to the variance results in the accessible range $4 \leq l \leq 256$. Indeed the agreement we now display between numerical data and theory is good.

For observables with $\rho > 0$ the two leading terms in (36) are the third and sixth terms. We use hyperscaling to write $l^{y_t} = l^{\frac{\alpha}{2\nu}+\frac{d}{2}}$ and substitute in (18) $l^{\frac{\alpha}{\nu}}$ by the behavior of the specific heat (8). Thus we propose for the RBAT models the leading behavior [34]

$$V(T_c, l) = a_v l^{2\rho} \ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)] + c_v l^{2\rho - \frac{d}{2}} \{\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]\}^{\frac{1}{2}} , \quad (41)$$

with $a_v \equiv E_0^2 K_r^2$ and $c_v \equiv -2D_1 E_0 K_r^2$ (note that for every thermodynamic quantity there are different coefficients E_0, D_1, etc.). The expression $\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]$ describes the singular behavior of the specific heat including the crossover from the pure model behavior, characterized by the pure model exponent $(\alpha/\nu)_{\text{pure}}$ [see Eq. (8)]. ρ is the critical exponent of the quantity whose variance is measured (e.g., $\rho = \frac{\gamma}{\nu}$ for χ and $\rho = \frac{\gamma^{(p)}}{\nu}$ for $\chi^{(p)}$).

In Fig. 8 we show the variance of χ, V_χ, of the seven critical RBAT models fitted by the function (41), where the parameters c_0, $(\alpha/\nu)_{\text{pure}}$, and $\frac{\gamma}{\nu}$ were taken from Tables I and II. For the sake of clarity (so that the data do not fall on top of each other) V_χ of the model C_i was multiplied by 2^{i+1}. The fitting parameters a_v, c_v are given in Table IV. The agreement with our scaling prediction is quite encouraging. The same analysis has been carried out for the variance of $\chi^{(p)}$, where $\frac{\gamma^{(p)}}{\nu}$ was taken from Table II, and the results are plotted in Fig. 13. Again the fitting parameters a_v, c_v are given in Table IV and the agreement between the numerical results and our scaling prediction is encouraging. We stress that the only fitting parameters of the fits in Figs. 8 and 13 are a_v, c_v; the other parameters of Eq. (41), $\rho, c_0, (\alpha/\nu)_p$, were obtained previously [11] from the specific heat results and from the results for χ and $\chi^{(p)}$.

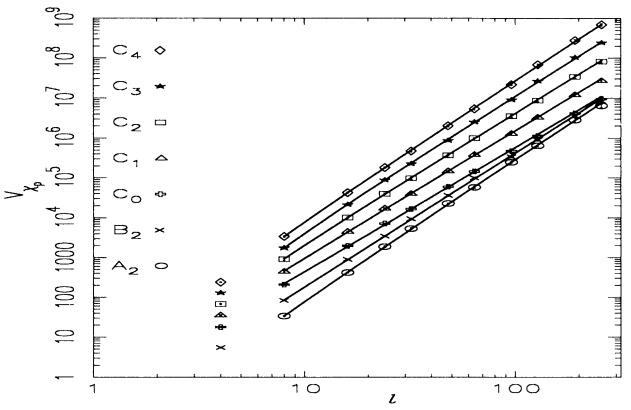

FIG. 13. The variance of $\chi^{(p)}$, $V_{\chi^{(p)}}$ as a function of $\log_{10} l$ for all critical models, $C_{0...4}$, and A_2, B_2 of the RBAT model. For the sake of clarity $V_{\chi^{(p)}}$ of the model C_i was multiplied by 2^{i+1}. The solid lines are fits to the form (41), yielding estimates for fitting parameters which are listed in Table IV.

Since the first term in (41) is the dominant one (by a factor of l^{y_t}, where $y_t \geq 1$), we test (41) again in another manner. In Fig. 14 we plot the scaled $V_\chi : V_\chi l^{-2\rho} / \ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]$. Indeed it seems that the data points approach a constant value, confirming the leading term in (41) which originates in the leading behavior of the variance (37). For the energy $\rho = (\alpha - 1)/\nu < 0$ so that the two leading terms in (36) are the third and the fifth ones. Again by using hyperscaling and substituting $l^{\frac{\alpha}{\nu}}$ by the behavior of the specific heat (8), we arrive at the scaling form for the variance of the energy

$$V_E(T_c, l) = (a_v \{\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]\}^2 + b_v \ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)])l^{-d} , \quad (42)$$

with $b_v = -2bE_0K_r^2$. In Fig. 11 we show the variance of the energy, V_E, of the seven critical RBAT models fitted by the function (42). For the sake of clarity V_E of the model C_i was multiplied by 2^{i+1}. The agreement

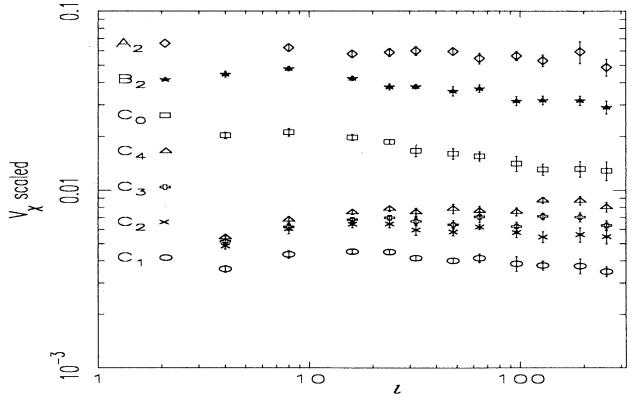

FIG. 14. The scaled variance of χ, $V_\chi l^{-2\frac{\gamma}{\nu}} / \ln[1 + c_0(l^{(\alpha/\nu)_p} - 1)]$, as a function of $\log_{10} l$ for all critical models, $C_{0...4}$, and A_2, B_2 of the RBAT model.

TABLE IV. Fitting parameters for the variances of $\chi, \chi^{(p)}, E$ for the critical models $C_{0...4}$, and A_2, B_2 according to Eqs. (41) and (42), using lattice sizes $l \geq 8$.

	V_χ		$V_{\chi^{(p)}}$		V_E	
	a_v	c_v	a_v	c_v	a_v	b_v
C_0(Ising)	0.0145(7)	0.11(2)	0.033(1)	0.07(4)	0.29(7)	0.8(2)
C_1	0.0039(1)	0.026(10)	0.0134(3)	-0.03(2)	0.128(14)	-1.45(19)
C_2	0.0059(1)	0.014(10)	0.0172(2)	-0.09(2)	0.135(13)	-1.19(16)
C_3	0.0069(2)	-0.01(1)	0.0147(3)	-0.12(2)	0.133(15)	-1.27(18)
C_4(Potts)	0.0082(2)	-0.04(1)	0.082(2)	-0.04(1)	0.121(13)	-1.25(17)
B_2	0.033(1)	0.13(2)	0.092(2)	0.19(4)	0.99(15)	1.2(2)
A_2	0.056(1)	0.028(12)	0.143(3)	0.05(3)	5.49(44)	1.28(23)

between theory and the numerical data is good and the fitting parameters a_v, b_v are given in Table IV.

For the magnetization and the specific heat $\rho = \frac{\alpha}{\nu}$ and $\rho = \frac{-\beta}{\nu}$, respectively. In these cases $|\rho|$ is small and the fifth and the sixth terms in Eq. (36) are of similar order in l . Thus one may not neglect one term with respect to the other as was done for the energy and the susceptibility. Thus we fit the variance of the specific heat to the form

$$V_C(T_c, l) = a_v\{\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]\}^3 + b_v l^{-\frac{d}{2}}\{\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]\}^{1.5} + c_v l^{-\frac{d}{2}}\{\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]\}^{2.5} . \quad (43)$$

In Fig. 15 the variance of the specific heat, V_C, of the seven critical RBAT models is fitted by the function (43), with the fitting coefficients given in Table V. The data for large lattice sizes are rather noisy and three parameter fits are not so reliable with only eleven data points, so that both the results and the fitting curves in Fig. 15 should be taken with a grain of salt. The obtained fitting coefficients are consistent with the coefficient E_0 being much smaller than b, D_1. A small value for E_0 is quite plausible if the specific heat as a function of the temperature is close to being symmetric around the critical point [see (30)]. This symmetry is supported by the symmetric form of the histograms of the specific heat as shown in Figs. 6 and 7. For the Ising model C_0 the errors of the coefficients a_v, b_v, and c_v are of the same order of magnitude as the coefficients themselves. However, for the other models the errors are reasonable and though a_v is small, we have $E_0^2 K_r^2 \equiv a_v > 0$, meaning that, asymptotically, the first term in (43) will dominate. This implies that the specific heat of the RBAT models is non-self-averaging, excluding possibly the random-bond Ising model. Possibly, the theory needs some changes in order to be applied to the specific heat C which diverges logarithmically (and as a double logarithm for the random-bond Ising model) and not with a simple power law.

In Fig. 16 the variance of the magnetization, V_M, of the seven critical RBAT models is fitted by the function

$$V_M = a_v l^{-2\frac{\beta}{\nu}} \ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)] + b_v l^{-\frac{\beta}{\nu}-\frac{d}{2}}\{\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]\}^{\frac{1}{2}} + c_v l^{-2\frac{\beta}{\nu}-\frac{d}{2}}\{\ln[1 + c_0(l^{(\alpha/\nu)_{\text{pure}}} - 1)]\}^{\frac{1}{2}} . \quad (44)$$

The fitting coefficients a_v, b_v, and c_v are given in Table V. The data are much more noisy than the data of the susceptibility (see Fig. 8).

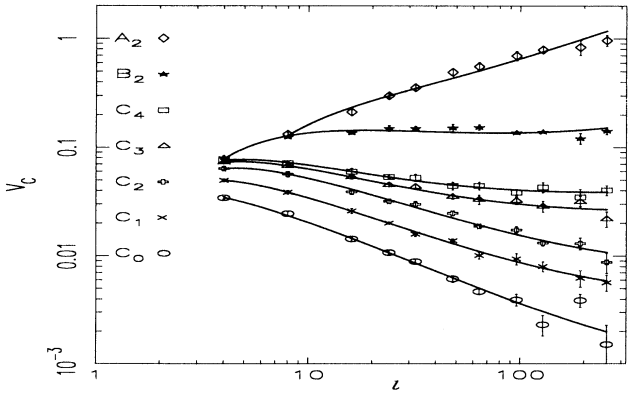

FIG. 15. The variance of the specific heat, V_C as a function of $\log_{10} l$ for all critical models, $C_{0...4}$, and A_2, B_2 of the RBAT model. The solid lines are fits to the form (43), yielding estimates for the fitting parameters a_v, b_v, c_v which are listed in Table V.

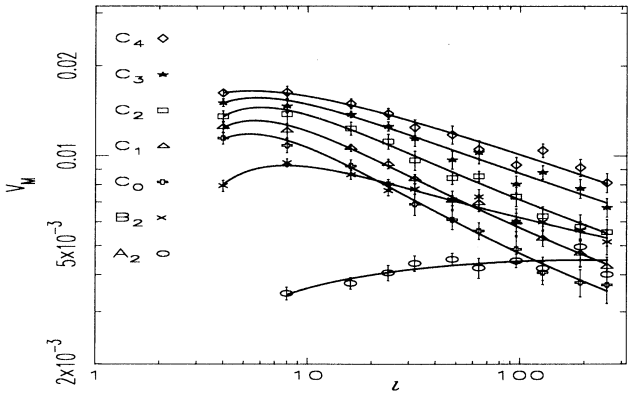

FIG. 16. The variance of the magnetization, V_M, as a function of $\log_{10} l$ for all critical models, $C_{0...4}$, and A_2, B_2 of the RBAT model. The solid lines are fits to the form (44), yielding estimates for the fitting parameters a_v, b_v, c_v which are listed in Table V.

TABLE V. Fitting parameters for the variances of the specific heat V_C and the magnetization V_M for the critical models $C_{0...4}$, and A_2, B_2 according to Eqs. (43) and (44), using lattice sizes $l \geq 4$.

	V_C			V_M		
	a_v	b_v	c_v	a_v	b_v	c_v
C_0(Ising)	0.000016(12)	0.028(10)	0.008(4)	0.0037(1)	0.25(2)	-0.28(2)
C_1	0.0000011(1)	-0.045(4)	0.0038(3)	0.00108(3)	0.102(9)	-0.12(1)
C_2	0.0000028(5)	-0.078(8)	0.0081(7)	0.00169(7)	0.10(2)	-0.12(2)
C_3	0.0000079(6)	-0.069(5)	0.0069(5)	0.00213(9)	0.054(20)	-0.068(24)
C_4(Potts)	0.0000099(4)	-0.062(5)	0.0057(4)	0.0027(1)	-0.004(23)	-0.004(28)
B_2	0.0079(6)	-0.44(9)	1.4(1)	0.0089(3)	0.27(3)	-0.30(4)
A_2	0.67(7)	-4.9(18)	36.5(54)	0.0167(8)	0.07(7)	-0.07(10)

To summarize, we have examined the sample to sample fluctuations in various thermodynamic quantities of some random-bond Ashkin-Teller models. These include the random-bond Ising and four-state Potts models. It was found that far from criticality all thermodynamic quantities examined are strongly self-averaging (that is, their variance scales as l^{-d}). At the critical point we found that the susceptibility χ, the susceptibility of the polarization $\chi^{(p)}$, and the magnetization M are non-self-averaging, while the energy E is weakly self-averaging. The data for the variance of the specific heat seem to imply weak self-averaging of the specific heat. Since the data are not accurate at the larger sizes used, this may well be a transient behavior, compatible with our theory which predicts that asymptotically the specific heat is non-self-averaging. A phenomenological finite size scaling theory was developed for the sample to sample fluctuations. Its main prediction is that when the specific heat exponent $\alpha < 0$ (α of the disordered model) then, for a quantity P which scales as l^{ρ} at criticality, its variance V_P will scale asymptotically as $l^{2\rho+\frac{\alpha}{\nu}}$. The theory is not applicable in the asymptotic limit ($l \to \infty$) to cases where $\frac{\alpha}{\nu} = 0_+$. Nonetheless in the accessible range of lattice sizes we found very good agreement between the theory and the data for $V_\chi, V_{\chi^{(p)}}$, and V_E. The data for V_χ are especially convincing. The theory also describes well the variance of models with weak disorder, exhibiting slow crossover to the randomness dominated behavior. The theory may also be compatible with the data for V_M and V_C, but evidence for this is less convincing. We note that if our assumption (20) is incorrect and should be replaced asymptotically by [29] $\delta T_c(l) \sim l^{-y_t}$, then our theory predicts that $V_P \sim l^{2\rho}$ independent of α. In this case all quantities (excluding the energy which has a nonvanishing nonsingular part) are non-self-averaging independent of α. In order to further test our theory we intend to study the sample to sample fluctuations in the site dilute three-dimensional Ising model where $\frac{\alpha}{\nu} < 0$ and our analysis holds.

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APPENDIX

In this section we draw some more conclusions based on the finite size scaling form (24), without making any assumptions on the explicit x dependence of the coefficients of $\tilde{Q}_x$.

What can we deduce about the coefficients of $\tilde{Q}_x$ from the Brout argument? Consider (24) in the limit $l \to \infty$, $\dot{t}_x$ finite. In this case ξ is finite, the Brout argument holds, and one expects $P_x(T)$ to be sample independent. This means that in this limit we expect the coefficients of $\tilde{Q}_x$ and its argument to converge to some x independent values. It follows that we can assume that these coefficients are distributed according to some unknown distribution function whose width $w(l)$ depends on l and tends to zero as $l \to \infty$.

Is there any limit in which one may recover the usual finite size scaling behavior, completely independent of the specific sample x? Consider the limit l large but finite and $T \approx T_c$. Let us now add the assumption that the width $w(l)$ tends to zero no slower than $l^{-d/2}$. Then for large enough l, according to Eqs. (20)–(23), $\dot{t}_x$ approaches $\dot{t}$ given by

$$\dot{t} = \frac{T - T_c(l)}{T_c} \tag{A1}$$

as $l^{-d/2}$, and

$$\tilde{Q}_x(\dot{t}_x l^{y_t}) \to \tilde{Q}(\dot{t} l^{y_t}) \ , \tag{A2}$$

so that we recover the usual [31] finite size scaling behavior.

The limit (A2) cannot account for the large sample to sample fluctuations that we have numerically observed at $T = T_c$ even for rather large values of l. Indeed, special care is needed in the case $T = T_c$, where ξ/l is not small and the Brout argument does not hold. It turns out that in this case the limit, where the x dependence of $\dot{t}_x$ can be ignored as in (A2), does not occur or is reached "slowly." When $T = T_c$, then $\dot{t}_x = -t_c(x,l) = \{t_c(l) - t_c(x,l)\} - t_c(l)$, so that according to (20) and (19) $\dot{t}_x$ is a difference of a fluctuating term of order $l^{-d/2}$ and

a term of a constant sign of order l^{-y_t}. Therefore

$$\frac{|\delta \dot{t}_x|}{|\dot{t}_x|} \sim \begin{cases} l^{y_t - d/2} = l^{\frac{\alpha}{2\nu}} = l^{-|\frac{\alpha}{2\nu}|} & \text{if } d/2 > y_t \quad (\alpha < 0) \\ 1 & \text{if } d/2 \leq y_t \quad (\alpha \geq 0). \end{cases} \tag{A3}$$

Thus for $T = T_c$ and positive α, $\dot{t}_x \not\to \dot{t}$ for large l and (A2) is not justified. In this case the relative fluctuations in the argument of $\tilde{Q}_x$ are of order 1 and their absolute magnitude scales as $l^{\frac{\alpha}{2\nu}}$ so that it increases with l. So for large l the argument of $\tilde{Q}_x$ is a constant plus a large fluctuating quantity which increases with l. It follows that $\tilde{Q}_x$ cannot be expanded as is done in Sec. V, and that the limit (A2) does not exist.

For negative α it follows that the fluctuations in the argument of $\tilde{Q}_x$ scale as $l^{-|\frac{\alpha}{2\nu}|}$. Since we have assumed that the fluctuations in the coefficients of $\tilde{Q}_x$, $w(l)$, scale as $l^{-d/2}$, then if $|\frac{\alpha}{2\nu}| < \frac{d}{2}$, then at $T = T_c$ $w(l)$ decreases faster than the fluctuations in the argument of $\tilde{Q}_x$. Thus one may consider the range of l for which

$$\tilde{Q}_x(\dot{t}_x l^{y_t}) \to \tilde{Q}(\dot{t}_x l^{y_t}) \ , \tag{A4}$$

where only the argument of $\tilde{Q}_x$ is x dependent. In this case the coefficients of $\tilde{Q}$ are some constants for which we need not assume anything about their x or l dependence. Consideration of the limit (A4) suffices to reach our main result (18) [but not corrections to (18)], independent of the assumptions made in Sec. V on the x dependence of the coefficients of $\tilde{Q}_x$.

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[26] These considerations have a large practical importance concerning the usual way of calculating the susceptibility in MC simulations. It is customary to define the susceptibility (per spin) as $\chi \equiv l^d\langle M^2\rangle$, which is well justified [9] when T_c is slightly higher than the temperature of the maximum of the true susceptibility of the sample. This definition cannot be used in the low temperature phase where one needs to subtract $\langle|M|\rangle^2$. But in the case of large fluctuations in $T_c(x,l)$, there exist samples in which T_c is smaller than the temperature of the maximum of the true susceptibility, and therefore $l^d\langle M^2\rangle > \chi = l^d(\langle M^2\rangle - \langle|M|\rangle^2)$. This effect is probably the reason for the asymmetry of the histograms of the susceptibility (Figs. 3–5). Those samples in which $T_c < T_c(x,l)$ are effectively in their low temperature phase and give an estimate of χ which is biased to higher values than the true χ.

[27] We do not consider $V_E/[E]^2$ since $[E] \to$ const as $l \to \infty$.

[28] M. Hennecke, Phys. Rev. B **48**, 6271 (1993).

[29] A. Aharony (private communication).

[30] J.T. Chayes, L. Chayes, D.S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).

[31] In the more commonly used form of finite size scaling the argument of $\tilde{Q}$ would be $t_x = \frac{T - T_c(x)}{T_c}$. But in the case of a disordered sample it is unclear how to define a critical temperature which depends on the sample x, $T_c(x)$, but does not depend on its size l. Hence we use scaling with respect to the reduced temperature $\dot{t}_x = \frac{T - T_c(x,l)}{T_c}$, where $\dot{t}_x$ is defined using the pseudocritical

temperature $T_c(x,l)$. In any case this version of finite size scaling is equivalent to the more common one [5].

[32] We assume the analytic part of P_x depends on $(t - \Delta t_c(x,l))$ and not on $\dot{t}_x$. If we assume that P_x has an analytic dependence on $\dot{t}_x$ we find that $[P^{\text{analytic}}]$ has terms that scale as l^{-y_t}. Such dependence of the analytic part on exponents connected with critical behavior does not seem plausible.

[33] Note that from the structure of (27) it is clear that $\Delta t_c(x,l)$ plays the role of a "critical temperature of the sample x."

[34] For $\frac{\alpha}{\nu} \geq 0$ it seems that one would need to include also the last term in (36) proportional to $l^{2\rho+2\frac{\alpha}{\nu}}$. But since this term is also proportional to K_r^4, this may not be necessary for small values of $\frac{\alpha}{\nu}$ or for small values of K_r. Indeed, attempts to include such a term in our fits were not successful, yielding a negative coefficient where only a positive one is consistent with the theory.